Universität Stuttgart

Schriftenreihe der Institute des Studiengangs Geodäsie und Geoinformatik

Technical Reports
Department of Geodesy and Geoinformatics

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Continuum Mechanics in Geophysics and Geodesy: Fundamental Principles

ISSN 0933-2839 Report Nr. 2003.2

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ISSN 0933-2839 Report Nr. 2003.2

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1 Introduction

1.1 Solids and fluids

Materials are usually classified either as *solids* or as *fluids*. Commonly, solids are considered to have a definite shape. Fluids, on the other hand, do not have this property and individual particles of a compositionally homogeneous fluid are envisaged as being capable of rearrangement without affecting the macrophysical properties of the fluid. Obviously, the distinguishing properties of solids and fluids are closely related to the ease by which the materials are deformed by applied forces. This suggests the following qualitative classification:

- A solid is a material for which changes in the applied forces cause simultaneous changes in the relative positions of the constituent particles.
- A fluid is a material for which steadily applied forces cause continuous changes in the relative positions of the constituent particles.

A more rigorous classification will be introduced below (Chap. 5).

Furthermore, fluids are commonly subdivided into *liquids* and *gases*. The characteristic difference between liquids and gases is the much larger compressibility of the latter. From a dynamical point of view, compressibility is, however, a less discriminatory property than fluidity. Therefore, liquids and gases are usually studied together.

1.2 Continuity principle

In the continuum theory of solids and fluids, a real material is represented by the concept of the continuum. The following definition will be used:

- A continuum is a fictitious material which completely fills a particular region of space at any time epoch and whose characteristic fields can be represented by continuous functions of space and time.

The adequacy of this concept in view of the discrete structure of real materials is obviously related to the enormous number of molecular particles contained in macrophysically small volumes and to the extreme shortness of molecular oscillations in comparison to the duration of macrophysical changes. To understand this more clearly, we envisage a measuring instrument

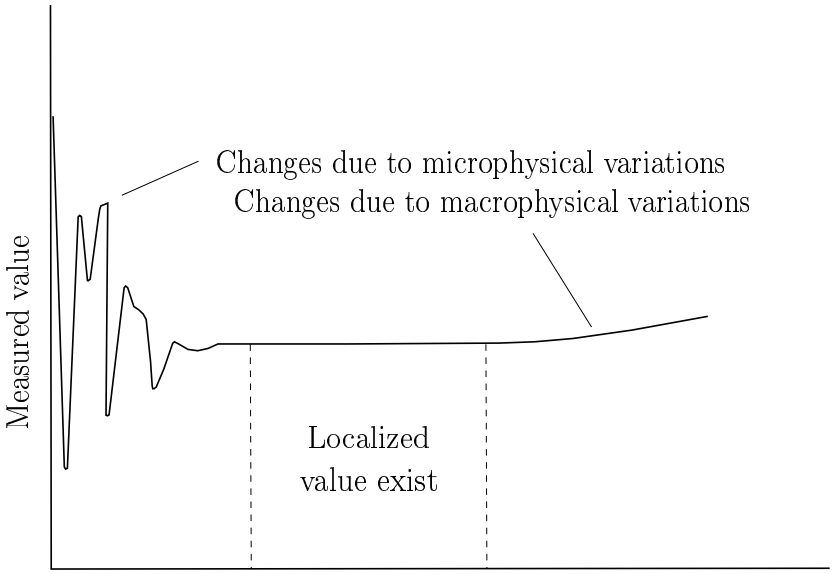


Figure 1.1: Dependence of the measured value of an arbitrary field associated with a real material on the effective volume or time interval considered.

which responds to some field associated with the material. The result of each measurement is an average value of this field over an effective volume during an effective time interval. Normally, the instrument is designed in such a way that further reductions, within limits, of the effective volume or the effective time interval would not change the value of the measurement. Then, the effective volume and time interval are *small* relative to the *macro*physical scale and the measurement is called *localized* in space and time. However, for the same measurement, the effective volume and time interval must be *large* relative to the *micro*physical scale, *i.e.* they must contain a sufficient number of particles and oscillations. This is in order that molecular fluctuations in space and time do not affect the value of the measurement. Experience has shown that it is usually possible to assign localized values to the fields associated with real materials in this sense (Fig. 1.1). We are thus led to the continuity principle:

- The macrophysical behaviour of a *real* material with given *discrete* structure is identical to that of a *fictitious* material with assumed *continuous* structure whose values of the fields are equal to the localized values of the respective fields, referred to a particular point in space and time, of the given real material.

The continuity principle justifies the description of the behaviour of real materials by continuous functions of space and time and the use of the mathematical methods of calculus for the closer study of this behaviour. Simultaneously, it provides a simple physical model in agreement with everyday experience. The principle therefore allows mathematical analysis to be guided by physical intuition.

1.3 Elements of continuum mechanics

In the following chapters, we will be concerned with continuum mechanics, which divides naturally into three major parts:

- General concepts and principles
- Constitutive equations
- Specialized field theories

The general concepts and principles apply to all types of continuum. They include the *kinematic* concepts of deformation and flow (Chap. 2), the *mechanical* concept of stress (Chap. 3), and the *dynamical* principles governing the conservation of mass, momentum and energy as well as the change of entropy (Chap. 4). Constitutive equations specify the particular type of continuum and, in particular, determine whether the continuum is a solid or a fluid (Chap. 5). The combination of the *general* concepts and principles with *special* constitutive equations results in the field theories of elastodynamics, viscodynamics and viscoelastodynamics (Chap. 6).

2 Deformation and flow

2.1 Introduction

In the following, we will refer to the continuum as the *body* and to any subset of it as a subbody. We will also introduce the 3-D Euclidean space and call any subset of it a *domain*. Hence, at any time epoch, any subbody fills some domain. Furthermore, we will distinguish between *particles* of the body and *points* of the 3-D Euclidean space. The identification of the particles of the body with the points of the domain occupied then defines the current state of the body at the time epoch considered.

Deformation studies distinguish between the undeformed initial state and the deformed current state. This view implies that the particular sequence of states by which the body has passed from the initial to the current state does not affect the latter. By contrast, *flow* studies do not distinguish an initial state, but specify the sequence of current states and, thus, emphasize the history of the body. Familiar examples of these concepts are *elastic* deformations and *viscous* flows, respectively.

We begin with a brief introduction into the two kinematic representations most widely used in continuum mechanics (Sec. 2.2). This is followed by a summary of the concepts of spatial and material time derivatives (Sec. 2.3). After this, infinitesimal strain and infinitesimal rotation (Sec. 2.4), finite deformation and finite strain (Sec. 2.5), and strain rate and vorticity (Sec. 2.6) will be discussed. The chapter concludes with an outline of the concepts of principal, spherical and deviatoric strains (Sec. 2.7).

2.2 Kinematic representations

Several types of kinematic representation of the motion (deformation or flow) of a body have been developed. Most widely used are the Eulerian and Lagrangian representations, which we introduce for arbitrary Cartesian tensor fields. We also present formulae for the gradients, differentials and integrals of fields in the two kinematic representations and compile several useful identities.

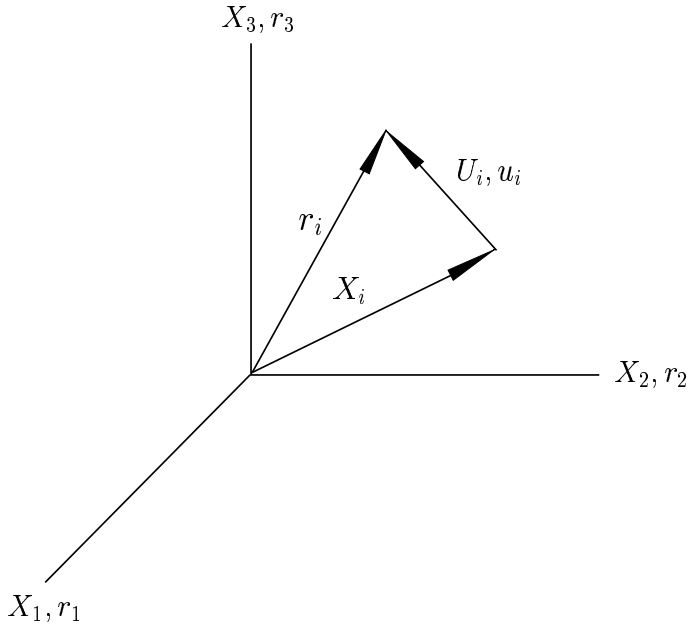


Figure 2.1: Material (initial) position, X_i, spatial (current) position, r_i and displacement, U_i or u_i, of some particle of the body.

2.2.1 Fields

A particular field is the position, which we use to refer particles or points to some Cartesian coordinate system. Distinguishing between the *initial* time epoch, $t = 0$, and the *current* time epoch, $t \geq 0$, we introduce the following kinematic representations (Fig. 2.1):

Eulerian representation: The independent variables are the *spatial* position, $\mathbf{r} \in \mathcal{R} \cup \partial\mathcal{R}$, with $\mathcal{R}$ the spatial 3-D domain (spatial volume) currently occupied by some subbody confined by the spatial 2-D domain (spatial boundary) $\partial\mathcal{R}$, and the *current* time epoch, $t \in [\,0, \infty)$. If X_i is the initial position of the particle currently at $\mathbf{r}$, the motion is given by

$$X_i = X_i(\mathbf{r}, t). \tag{2.1}$$

We assume the function to be single valued and continuously differentiable. If t is held fixed, (2.1) represents a mapping of the current state onto the initial state. For fixed $\mathbf{r}$, it allows us to identify those particles that successively occupy a particular spatial position.

Lagrangian representation: The independent variables are the *material* position, $\mathbf{X} \in \mathcal{X} \cup \partial\mathcal{X}$, with $\mathcal{X}$ the material 3-D domain (material volume) initially occupied by some subbody confined by the material 2-D domain (material boundary) $\partial\mathcal{X}$, and the *current* time epoch, $t \in [\,0, \infty)$.

If r_i is the current position of the particle initially at $\mathbf{X}$, the motion is given by

$$r_i = r_i(\mathbf{X}, t). \tag{2.2}$$

As before, we assume that the function is single valued and continuously differentiable. If t is held fixed, (2.2) represents a mapping of the initial state onto the current state. For fixed $\mathbf{X}$, it allows us to identify those spatial positions that are successively occupied by a particular particle.

With (2.1) and (2.2) single-valued and continuously differentiable mappings, they are unique inverses of each other. Necessary and sufficient for this is that the spatial and material Jacobian determinants of the respective 3×3 matrices of partial derivatives are positive:

$$J^{\mathbf{r}}(\mathbf{r}, t) := \det\left(\partial X_i(\mathbf{r}, t)/\partial r_j\right) > 0, \tag{2.3}$$

$$j^{\mathbf{X}}(\mathbf{X}, t) := \det\left[\partial r_i(\mathbf{X}, t)/\partial X_j\right) > 0. \tag{2.4}$$

A field related to the position is the displacement, whose Eulerian and Lagrangian representations are defined by (Fig. 2.1)

$$U_i(\mathbf{r}, t) := r_i - X_i(\mathbf{r}, t), \tag{2.5}$$

$$u_i(\mathbf{X}, t) := r_i(\mathbf{X}, t) - X_i. \tag{2.6}$$

More generally, any Cartesian tensor field associated with the body can be described using either kinematic representation. The appropriate mappings are

$$F_{ij\ldots} = F_{ij\ldots}(\mathbf{r}, t), \tag{2.7}$$

$$f_{ij\ldots} = f_{ij\ldots}(\mathbf{X}, t), \tag{2.8}$$

which are the Eulerian and Lagrangian representations, respectively, of the field. If the Eulerian representation is given, substitution of (2.2) into (2.7) yields the corresponding Lagrangian representation:

$$F_{ij\ldots}[\mathbf{r}(\mathbf{X}, t), t] \equiv f_{ij\ldots}(\mathbf{X}, t). \tag{2.9}$$

If, on the other hand, the Lagrangian representation is given, substitution of (2.1) into (2.8) gives the associated Eulerian representation:

$$f_{ij\ldots}[\mathbf{X}(\mathbf{r}, t), t] \equiv F_{ij\ldots}(\mathbf{r}, t). \tag{2.10}$$

Note that $F_{ij\ldots}$ specifies the current value of the field at the fixed spatial position $\mathbf{r}$, whereas $f_{ij\ldots}$ gives this value at the moving particle with the material position $\mathbf{X}$. As before, we suppose that both mappings are single valued and continuously differentiable. Sometimes, a field is more conveniently described in one kinematic representation than in the other. For example, in deformation studies, the undeformed state is distinguished from all other states. Displacements and

strains are therefore naturally defined using the Lagrangian representation, with the undeformed state taken as the initial state (Sec. 2.4). In contrast to this are flow studies, where, usually, no natural initial state can be distinguished. Thus, velocities and strain rates are more readily expressed using the Eulerian representation (Sec. 2.6).

We continue to use *upper-case indicial* symbols for the Eulerian representation of fields and *lower-case indicial* symbols for their Lagrangian representation. Note that the same symbol is employed both for the field value and for its functional dependence. In contrast to this, the *bold-face* symbols are used for the position when regarded as an independent variable. In the following, the independent variables spatial position, $\mathbf{r}$, material position, $\mathbf{X}$, and current time epoch, t, will usually be suppressed.

2.2.2 Gradients

When considering the gradients of fields, we define for brevity

$$F_{ij\ldots,k} := \frac{\partial F_{ij\ldots}}{\partial r_k}, \tag{2.11}$$

$$f_{ij\ldots,k} := \frac{\partial f_{ij\ldots}}{\partial X_k}, \tag{2.12}$$

which are the spatial and material gradients, respectively, of the field. Note that, by differentiation of (2.9) and (2.10), the following relationships apply:

$$F_{ij\ldots,l} r_{l,k} = f_{ij\ldots,k}, \tag{2.13}$$

$$f_{ij\ldots,l} X_{l,k} = F_{ij\ldots,k}. \tag{2.14}$$

2.2.3 Kronecker and Levi–Civita symbols, identities

Special quantities required in the following are the Kronecker symbol and the Levi–Civita symbol. The Kronecker symbol is a second-rank tensor with the following properties:

$$\delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}. \tag{2.15}$$

The Levi–Civita symbol is a third-rank tensor defined as follows:

$$\epsilon_{ijk} := \begin{cases} 1, & ijk \text{ even permutation of } 123 \\ -1, & ijk \text{ odd permutation of } 123 \\ 0, & ijk \text{ no permutation of } 123 \end{cases}. \tag{2.16}$$

Note that δ_{ij} and ϵ_{ijk} do not depend on the position, whence the Eulerian and Lagrangian representations are identical. For simplicity, lower-case symbols are always employed. For

particular derivations, the Jacobian identities are useful:

$$J^{\mathbf{r}}\epsilon_{lmn} \equiv \epsilon_{ijk} X_{i,l} X_{j,m} X_{k,n}, \tag{2.17}$$

$$j^{\mathbf{X}}\epsilon_{lmn} \equiv \epsilon_{ijk} r_{i,l} r_{j,m} r_{k,n}. \tag{2.18}$$

Other useful identities are

$$J^{\mathbf{r}} j^{\mathbf{X}} \equiv 1, \tag{2.19}$$

$$\frac{\partial X_i}{\partial X_j} \equiv \frac{\partial r_i}{\partial r_j} \equiv \delta_{ij}, \tag{2.20}$$

$$X_{i,k} r_{k,j} \equiv r_{i,k} X_{k,j} \equiv \delta_{ij}. \tag{2.21}$$

Obviously, $X_{i,j}$ and $r_{i,j}$ are mutually inverse tensors, whence we have

$$X^{-1}_{i,j} = r_{i,j}, \quad r^{-1}_{i,j} = X_{i,j}. \tag{2.22}$$

Finally, we list the Piola identities:

$$(J^{\mathbf{r}} X^{-1}_{j,i})_{,j} \equiv (j^{\mathbf{X}} r^{-1}_{j,i})_{,j} \equiv 0. \tag{2.23}$$

2.2.4 Differentials and integrals

In the following, we will be concerned with spatial and material 1-D differentials (differential lengths), dr_i and dX_i, respectively, spatial and material 2-D differentials (differential areas), $d^2 r_i$ and $d^2 X_i$, respectively, and spatial and material 3-D differentials (differential volumes), $d^3 r$ and $d^3 X$, respectively, for a fixed time epoch, t. The magnitudes of dr_i and dX_i, respectively, are defined by

$$dr := (dr_i \, dr_i)^{\frac{1}{2}}, \tag{2.24}$$

$$dX := (dX_i \, dX_i)^{\frac{1}{2}}. \tag{2.25}$$

Furthermore, we introduce the spatial unit vector collinear with dr_i and the material unit vector collinear with dX_i, respectively, by

$$p^{\mathbf{r}}_i := \frac{dr_i}{dr}, \tag{2.26}$$

$$P^{\mathbf{X}}_i := \frac{dX_i}{dX}. \tag{2.27}$$

Similarly, we define the magnitudes of $d^2 r_i$ and $d^2 X_i$, respectively, by

$$d^2 r := (d^2 r_i \, d^2 r_i)^{\frac{1}{2}}, \tag{2.28}$$

$$d^2 X := (d^2 X_i \, d^2 X_i)^{\frac{1}{2}} \tag{2.29}$$

and the spatial unit vector collinear with $d^2 r_i$ and the material unit vector collinear with $d^2 X_i$, respectively, by

$$n^{\mathbf{r}}_i := \frac{d^2 r_i}{d^2 r}, \tag{2.30}$$

$$N_i^{\mathbf{X}} := \frac{d^2 X_i}{d^2 X}. \tag{2.31}$$

The differentials and unit vectors introduced above may also be used as *fields* in the Eulerian and the Lagrangian representations. To restrict the number of symbols, no special upper- and lower-case symbols are introduced in these cases.

Differentials of arbitrary fields, $F_{ij\ldots}$ and $f_{ij\ldots}$, for a fixed time epoch, t, respectively, are introduced by

$$dF_{ij\ldots} := F_{ij\ldots,k}\, dr_k, \tag{2.32}$$

$$df_{ij\ldots} := f_{ij\ldots,k}\, dX_k. \tag{2.33}$$

The integrals of $F_{ij\ldots}$ over $\mathcal{R}$ and $f_{ij\ldots}$ over $\mathcal{X}$, respectively, are given by

$$\mathcal{F}_{ij\ldots} := \int_{\mathcal{R}} F_{ij\ldots}\, d^3 r, \tag{2.34}$$

$$\mathcal{F}_{ij\ldots} := \int_{\mathcal{X}} f_{ij\ldots}\, d^3 X. \tag{2.35}$$

2.2.5 Transformation formulae for differential length, area and volume

Eulerian representation: In view of (2.32), the Eulerian representation of the initial differential length, dX_i, associated with the spatial differential length, dr_i, is given by

$$dX_i = X_{i,j}\, dr_j. \tag{2.36}$$

To obtain the Eulerian representation of the initial differential area, $d^2 X_i$, in terms of the spatial differential area, $d^2 r_i$, we write it as the vector product of two differently directed initial differential lengths, $dX_i^{(1)}$ and $dX_i^{(2)}$ (Fig. 2.2):

$$d^2 X_i = \epsilon_{ijk}\, dX_j^{(1)}\, dX_k^{(2)}, \tag{2.37}$$

which, using (2.36), can be rewritten as

$$d^2 X_i = \epsilon_{ijk} X_{j,m}\, dr_m^{(1)}\, X_{k,n}\, dr_n^{(2)}. \tag{2.38}$$

Using the Jacobian identity (2.17), we obtain from (2.38) the expression

$$X_{j,i}\, d^2 X_j = J^{\mathbf{r}} \epsilon_{ijk}\, dr_j^{(1)}\, dr_k^{(2)}. \tag{2.39}$$

In view of the formula for the current differential area equivalent to (2.37):

$$d^2 r_i = \epsilon_{ijk}\, dr_j^{(1)}\, dr_k^{(2)}, \tag{2.40}$$

equation (2.39) reduces to

$$X_{j,i}\, d^2 X_j = J^{\mathbf{r}}\, d^2 r_i. \tag{2.41}$$

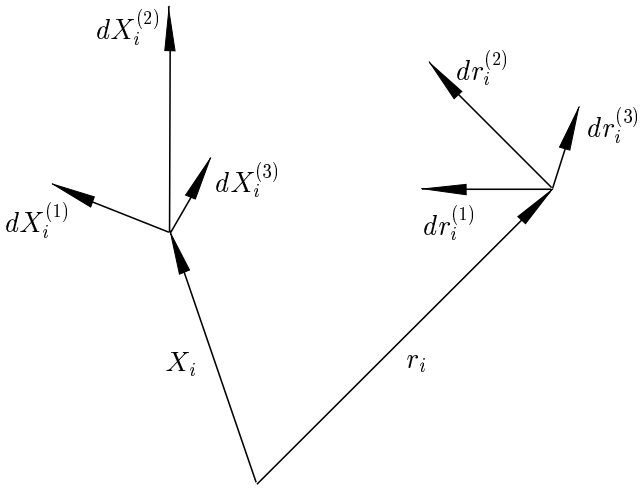


Figure 2.2: Material (initial) differential lenghts, $dX_i^{(k)}$, located at X_i and associated spatial (current) differential lenghts, $dr_i^{(k)}$, located at r_i.

Multiplication with $r_{i,k}$ and consideration of (2.21) and (2.22) finally provides

$$d^2X_i = J^{\mathbf{r}} X^{-1}_{j,i}\, d^2r_j. \tag{2.42}$$

The Eulerian representation of the initial differential volume, d^3X, in terms of the spatial differential volume, d^3r, can be obtained by writing it as a vector triple product (Fig. 2.2):

$$d^3X = \epsilon_{ijk}\, dX_i^{(1)}\, dX_j^{(2)}\, dX_k^{(3)}, \tag{2.43}$$

which, in view of (2.36), can be recast into

$$d^3X = \epsilon_{ijk} X_{i,l}\, dr_l^{(1)}\, X_{j,m}\, dr_m^{(2)}\, X_{k,n}\, dr_n^{(3)}. \tag{2.44}$$

Using (2.17) and the formula equivalent to (2.43):

$$d^3r = \epsilon_{ijk}\, dr_i^{(1)}\, dr_j^{(2)}\, dr_k^{(3)}, \tag{2.45}$$

we finally get

$$d^3X = J^{\mathbf{r}}\, d^3r. \tag{2.46}$$

Lagrangian representation: Since the Eulerian representation (2.1) is formally identical to the Lagrangian representation (2.2), the inverse relations corresponding to (2.36), (2.42) and (2.46) are obtained by replacing r_i and $J^{\mathbf{r}}$ by X_i and $j^{\mathbf{X}}$, respectively, and *vice versa*. This gives the following formulae:

$$dr_i \,=\, r_{i,j}\, dX_j, \tag{2.47}$$

$$d^2 r_i = j^{\mathbf{X}} r^{-1}_{j,i}\, d^2 X_j, \tag{2.48}$$

$$d^3 r = j^{\mathbf{X}}\, d^3 X. \tag{2.49}$$

2.3 Time derivatives

We first consider the time derivatives for arbitrary fields. Following this, we give expressions for the time derivatives of the differential length, the differential area and the differential volume.

2.3.1 Fields

The time rate of increase of any field with respect to a fixed spatial position is called the spatial time derivative. This needs to be distinguished from the material time derivative, which gives the time rate of increase with respect to a moving particle. The natural representation of the spatial time derivative is the Eulerian representation, whereas the Lagrangian representation is appropriate to the material time derivative.

Since the laws of dynamics apply to particles, it is the material time derivative that naturally appears in the fundamental principles. However, in flow studies, the *trajectories* of particles are in general unknown, whence $r_i(\mathbf{X}, t)$ cannot be specified. This excludes the knowledge of the material time derivative of r_i:

$$v_i := \left(\frac{\partial r_i}{\partial t}\right)_{\mathbf{X}}, \tag{2.50}$$

which is the Lagrangian representation of the velocity, and of the material time derivative of v_i:

$$\dot{v}_i := \left(\frac{\partial v_i}{\partial t}\right)_{\mathbf{X}}, \tag{2.51}$$

which is the Lagrangian representation of the acceleration. On the other hand, the Eulerian representation of the fields characterizing the flow can usually be determined. Reconsidering the velocity, we thus know

$$V_i = V_i(\mathbf{r}, t). \tag{2.52}$$

The spatial time derivative of V_i is

$$\dot{V}_i := \left(\frac{\partial V_i}{\partial t}\right)_{\mathbf{r}}, \tag{2.53}$$

which, in general, is different from $\dot{v}_i$. For example, in inhomogeneous and steady flow, the spatial time derivative of the velocity vanishes everywhere, whereas a particle may experience acceleration while moving to a neighbouring spatial position where the velocity is different.

Since the laws of dynamics involve the acceleration of particles and since the Lagrangian representation of the velocity is usually not possible, the acceleration must be calculated from the Eulerian representation of the velocity. To accomplish this, only the existence of the unknown

trajectories, $r_i = r_i(\mathbf{X}, t)$, must be assumed. Substitution into (2.52) gives

$$V_i = V_i[\mathbf{r}(\mathbf{X}, t), t] \tag{2.54}$$

and, by the chain rule of calculus,

$$\left(\frac{\partial V_i}{\partial t}\right)_{\mathbf{X}} = \left(\frac{\partial V_i}{\partial t}\right)_{\mathbf{r}} + V_{i,j}\left(\frac{\partial r_j}{\partial t}\right)_{\mathbf{X}}, \tag{2.55}$$

which, using (2.50) and $v_i \equiv V_i$, can be expressed as

$$\left(\frac{\partial V_i}{\partial t}\right)_{\mathbf{X}} = \left(\frac{\partial V_i}{\partial t}\right)_{\mathbf{r}} + V_{i,j}V_j. \tag{2.56}$$

This expresses the acceleration in terms of the Eulerian representation of the velocity and its derivatives. For convenience, the following notational simplifications are introduced:

$$\frac{dV_i}{dt} := \left(\frac{\partial V_i}{\partial t}\right)_{\mathbf{X}}, \tag{2.57}$$

$$\frac{DV_i}{Dt} := \left(\frac{\partial V_i}{\partial t}\right)_{\mathbf{r}}. \tag{2.58}$$

With these definitions, (2.56) takes the form

$$\frac{dV_i}{dt} = \frac{DV_i}{Dt} + V_{i,j}V_j. \tag{2.59}$$

The difference between the material and spatial time derivatives of V_i is given by the second term on the right-hand side of (2.59). Physically, this term describes the part of the material time derivative which is due the movement of the particle to a neighbouring spatial position where the velocity is in general different. For this reason, the term is called the *advective* time derivative.

The material time derivative of any other field can be calculated in the same way if its Eulerian representation is known. This allows the introduction of the material time-derivative operator:

$$\frac{d}{dt} := \left(\frac{D}{Dt} + V_i\frac{\partial}{\partial r_i}\right), \tag{2.60}$$

which can be applied to any field given in the Eulerian representation.

2.3.2 Differential length, area and volume

Also useful are expressions of the material time derivatives of the current differential length, the current differential area and the current differential volume. Before we derive these expressions, we calculate the material time derivative of $j^{\mathbf{X}}$. Expanding the determinant in (2.4), we obtain

$$j^{\mathbf{X}} = \epsilon_{ijk}r_{1,i}r_{2,j}r_{3,k}, \tag{2.61}$$

$$\frac{d\,j^{\mathbf{X}}}{dt} = \epsilon_{ijk}\left(\frac{dr_{1,i}}{dt}\,r_{2,j}r_{3,k} + r_{1,i}\,\frac{dr_{2,j}}{dt}\,r_{3,k} + r_{1,i}r_{2,j}\,\frac{dr_{3,k}}{dt}\right). \tag{2.62}$$

Interchanging the order of the derivatives and using $v_i := dr_i/dt$ and $V_i \equiv v_i$, we may write

$$\frac{dr_{i,j}}{dt} = v_{i,j} = V_{i,k} r_{k,j}. \tag{2.63}$$

Substituting this equation into (2.62) results in

$$\frac{d\, j^{\mathbf{X}}}{dt} = \epsilon_{ijk}(V_{1,l} r_{l,i} r_{2,j} r_{3,k} + r_{1,i} V_{2,l} r_{l,j} r_{3,k} + r_{1,i} r_{2,j} V_{3,l} r_{l,k}) \tag{2.64}$$

or, alternatively,

$$\frac{d\, j^{\mathbf{X}}}{dt} = V_{1,l}\epsilon_{ijk} r_{l,i} r_{2,j} r_{3,k} + V_{2,l}\epsilon_{ijk} r_{1,i} r_{l,j} r_{3,k} + V_{3,l}\epsilon_{ijk} r_{1,i} r_{2,j} r_{l,k}. \tag{2.65}$$

Six of the nine determinants vanish, because two rows of the associated matrices are identical in these cases. Keeping only the non-vanishing determinants, we have

$$\frac{d\, j^{\mathbf{X}}}{dt} = V_{1,1}\epsilon_{ijk} r_{1,i} r_{2,j} r_{3,k} + V_{2,2}\epsilon_{ijk} r_{1,i} r_{2,j} r_{3,k} + V_{3,3}\epsilon_{ijk} r_{1,i} r_{2,j} r_{3,k}, \tag{2.66}$$

which, using (2.61), reduces to

$$\frac{d\, j^{\mathbf{X}}}{dt} = V_{i,i}\, j^{\mathbf{X}}. \tag{2.67}$$

Differential length: With (2.47), we obtain for the differential length

$$\frac{d}{dt}\, dr_i = \frac{dr_{i,j}}{dt}\, dX_j, \tag{2.68}$$

which, upon interchanging the order of the derivatives on the right-hand side and considering $v_i := dr_i/dt$ and (2.13), yields

$$\frac{d}{dt}\, dr_i = V_{i,k} r_{k,j}\, dX_j. \tag{2.69}$$

Using (2.47) again, this equation takes the form

$$\frac{d}{dt}\, dr_i = V_{i,j}\, dr_j. \tag{2.70}$$

Differential area: According to (2.48), the differential area satisfies

$$d^2 r_i = j^{\mathbf{X}} X_{j,i}\, d^2 X_j, \tag{2.71}$$

so that the material time derivative takes the form

$$\frac{d}{dt}\, d^2 r_i = \left(\frac{d\, j^{\mathbf{X}}}{dt}\, X_{j,i} + j^{\mathbf{X}}\, \frac{dX_{j,i}}{dt} \right) d^2 X_j. \tag{2.72}$$

Using (2.67), the first term in the parentheses is given by

$$\frac{d\, j^{\mathbf{X}}}{dt}\, X_{j,i} = V_{k,k}\, j^{\mathbf{X}} X_{j,i}. \tag{2.73}$$

The second term can be evaluated by taking the material time derivative of (2.21), resulting in

$$\frac{dr_{i,k}}{dt}\,X_{k,j} + r_{i,k}\,\frac{dX_{k,j}}{dt} = 0. \tag{2.74}$$

Interchanging the order of the derivatives in the first term, rearranging the terms and using $v_i := dr_i/dt$ and (2.14) yields

$$r_{i,k}\frac{dX_{k,j}}{dt} = -V_{i,j} \tag{2.75}$$

or, after multiplication with $X_{l,i}$ and use of (2.21),

$$\frac{dX_{j,i}}{dt} = -V_{k,i}X_{j,k}. \tag{2.76}$$

If we substitute (2.73) and (2.76) into (2.72), we obtain

$$\frac{d}{dt}\,d^2r_i = j^{\mathbf{X}}(V_{k,k}X_{j,i} - V_{k,i}X_{j,k})\,d^2X_j, \tag{2.77}$$

which, using (2.48), finally reduces to

$$\frac{d}{dt}\,d^2r_i = V_{j,j}\,d^2r_i - V_{j,i}\,d^2r_j. \tag{2.78}$$

Differential volume: Using (2.49), we have, for the differential volume,

$$\frac{d}{dt}\,d^3r = \frac{d\,j^{\mathbf{X}}}{dt}\,d^3X. \tag{2.79}$$

Substituting (2.67) into this expression yields

$$\frac{d}{dt}\,d^3r = V_{i,i}\,j^{\mathbf{X}}\,d^3X \tag{2.80}$$

and, in view of (2.49), the formula

$$\frac{d}{dt}\,d^3r = V_{i,i}\,d^3r. \tag{2.81}$$

2.4 Infinitesimal strain and rotation

We consider neighbouring particles initially separated by the material differential length dX_i and experiencing the displacements u_i and $u_i + du_i$ (Fig. 2.3). In view of (2.33), the differential displacement, du_i, for a fixed time epoch, t, can then be expressed by

$$du_i = u_{i,j}\,dX_j, \tag{2.82}$$

where $u_{i,j}$ is the material displacement gradient. Normalizing du_i with respect to dX, we obtain with (2.27) for the differential displacement per material unit length the expression

$$\frac{du_i}{dX} = u_{i,j}P_j^{\mathbf{X}}. \tag{2.83}$$

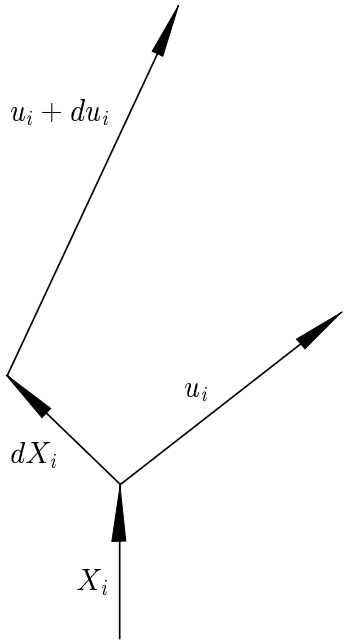


Figure 2.3: Displacements, u_i and $u_i + du_i$, for a fixed time epoch, t, of two particles initially at neighbouring material positions, X_i and $X_i + dX_i$, respectively.

This relation can be shown to represent a linear vector function mapping an arbitrarily directed material unit vector, $P_i^{\mathbf{X}}$, onto the relative displacement per material unit length, du_i/dX, so that $u_{i,j}$ is a second-rank tensor.

We now introduce the following decomposition:

$$u_{i,j} \equiv \tfrac{1}{2}(u_{i,j} + u_{j,i}) + \tfrac{1}{2}(u_{i,j} - u_{j,i}). \tag{2.84}$$

The first term on the right-hand side is *symmetric*:

$$e_{ij} := \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.85}$$

whereas the second term is *skew-symmetric*:

$$\alpha_{ij} := \tfrac{1}{2}(u_{i,j} - u_{j,i}). \tag{2.86}$$

To find simple interpretations of the symmetric and skew-symmetric parts of $u_{i,j}$, we assume that it is infinitesimal. Then, e_{ij} and α_{ij} are referred to as the infinitesimal strain tensor and the infinitesimal rotation tensor, respectively.

First, we set $\alpha_{ij} = 0$, so that (2.83) reduces to

$$\frac{du_i}{dX} = e_{ij} P_j^{\mathbf{X}}. \tag{2.87}$$

Defining the infinitesimal strain vector by

$$e_i := \frac{du_i}{dX}\,, \tag{2.88}$$

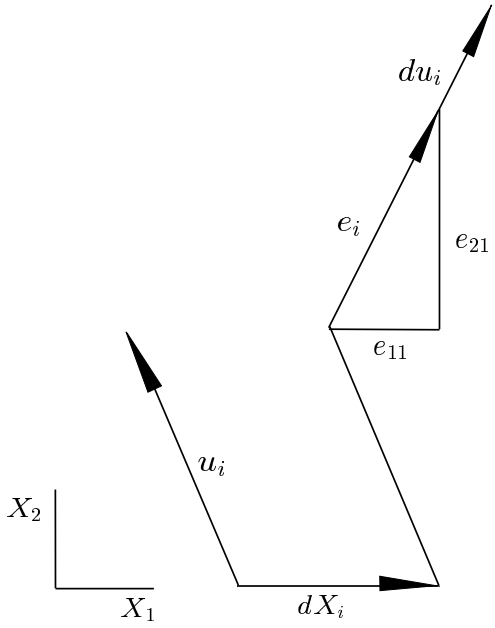


Figure 2.4: Longitudinal strain, e_{11}, and transverse strain, e_{21}, associated with some material differential length, dX_i, assumed to be infinitesimal.

equation (2.87) takes the form

$$e_i = e_{ij} P_j^{\mathbf{X}}. \tag{2.89}$$

The interpretation of the components of e_{ij} can be seen by considering

$$e_i Q_i^{\mathbf{X}} = e_{ij} Q_i^{\mathbf{X}} P_j^{\mathbf{X}}, \tag{2.90}$$

which represents the component of e_i in the direction of a second material unit vector, $Q_i^{\mathbf{X}}$. If, for example, $P_i^{\mathbf{X}} = (1, 0, 0)$ and $Q_i^{\mathbf{X}} = (1, 0, 0)$, the component describes the *longitudinal* strain (extension per material unit length) of the infinitesimal length initially oriented in the X_1 direction and is given by e_{11} (Fig. 2.4). If, on the other hand, $P_i^{\mathbf{X}} = (1, 0, 0)$ and $Q_i^{\mathbf{X}} = (0, 1, 0)$, the component describes the *transverse* strain (shear per material unit length) in the X_2 direction of the infinitesimal length initially oriented in the X_1 direction and is given by e_{21} (Fig. 2.4). More generally, we find that the main-diagonal components of e_{ij} are associated with *extension*, whereas the off-diagonal components of e_{ij} are associated with *shear*. However, since $e_{ij} = e_{ji}$, the rotation of the infinitesimal length initially oriented in the X_i direction is balanced by an equal, but opposite rotation of the infinitesimal length initially oriented in the X_j direction. Hence, for pure strain, any infinitesimal material surface with sides initially in these directions experiences no *net* rotation.

To illustrate α_{ij}, we set $e_{ij} = 0$, so that (2.82) becomes

$$du_i = \alpha_{ij}\, dX_j. \tag{2.91}$$

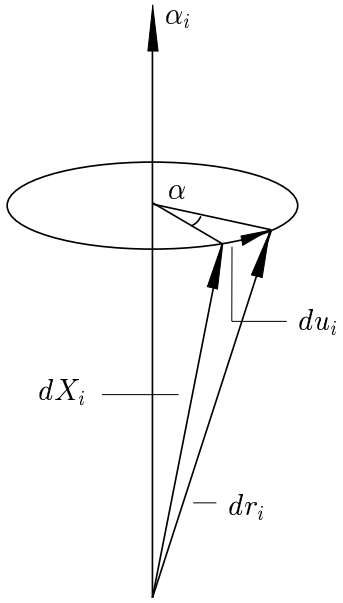


Figure 2.5: Rotation, α_i, associated with some material differential length, dX_i, assumed to be infinitesimal.

If we introduce the infinitesimal rotation vector by

$$\alpha_i := -\tfrac{1}{2}\epsilon_{ijk}\alpha_{jk}, \tag{2.92}$$

equation (2.91) is equivalent to

$$du_i = \epsilon_{ijk}\alpha_j \, dX_k. \tag{2.93}$$

This formula describes a relative rotation by the infinitesimal rotation angle α of two particles separated by the infinitesimal material length dX_i about an axis parallel to α_i (Fig. 2.5).

2.5 Finite deformation and strain

Before extending the concept of strain to finite deformations, it is necessary to formalize the concept of deformation.

2.5.1 Cauchy deformation and strain

We begin with the Eulerian representation and consider the initial differential length, dX_i, for a fixed time epoch, t. In view of (2.25) and (2.36), its squared magnitude can be written as

$$dX^2 = X_{i,j} \, dr_j \, X_{i,k} \, dr_k. \tag{2.94}$$

If we define the Cauchy deformation by

$$H^{\mathrm{C}}_{ij} := X_{k,i} X_{k,j}, \tag{2.95}$$

equation (2.94) becomes

$$dX^2 = H^{\mathrm{C}}_{ij}\, dr_i\, dr_j. \tag{2.96}$$

Using (2.25), we now consider the difference

$$\begin{aligned} dr^2 - dX^2 &= dr_i\, dr_i - H^{\mathrm{C}}_{ij}\, dr_i\, dr_j \\ &= (\delta_{ij} - X_{k,i}X_{k,j})\, dr_i\, dr_j. \end{aligned} \tag{2.97}$$

Introducing the Cauchy strain by

$$E^{\mathrm{C}}_{ij} := \tfrac{1}{2}(\delta_{ij} - X_{k,i}X_{k,j}), \tag{2.98}$$

equation (2.97) can be rewritten as

$$dr^2 - dX^2 = 2E^{\mathrm{C}}_{ij}\, dr_i\, dr_j. \tag{2.99}$$

We may use (2.5) to express E^{C}_{ij} in terms of U_i. Since, according to (2.5), $X_{i,j} = \delta_{ij} - U_{i,j}$, equation (2.98) takes the form

$$\begin{aligned} E^{\mathrm{C}}_{ij} &= \tfrac{1}{2}[\delta_{ij} - (\delta_{ki} - U_{k,i})(\delta_{kj} - U_{k,j})] \\ &= \tfrac{1}{2}(U_{i,j} + U_{j,i} - U_{k,i}U_{k,j}). \end{aligned} \tag{2.100}$$

2.5.2 Green deformation and strain

Alternatively, we consider the Lagrangian representation of the current differential length, dr_i, for a fixed time epoch, t. In view of (2.24) and (2.47), its squared magnitude becomes

$$dr^2 = r_{i,j}\, dX_j\, r_{i,k}\, dX_k. \tag{2.101}$$

If we introduce the Green deformation by

$$h^{\mathrm{G}}_{ij} := r_{k,i}r_{k,j}, \tag{2.102}$$

equation (2.101) becomes

$$dr^2 = h^{\mathrm{G}}_{ij}\, dX_i\, dX_j. \tag{2.103}$$

Next, we take the difference

$$\begin{aligned} dr^2 - dX^2 &= h^{\mathrm{G}}_{ij}\, dX_i\, dX_j - dX_i\, dX_i \\ &= (r_{k,i}r_{k,j} - \delta_{ij})\, dX_i\, dX_j. \end{aligned} \tag{2.104}$$

In terms of the Green strain defined by

$$e^{\mathrm{G}}_{ij} := \tfrac{1}{2}(r_{k,i}r_{k,j} - \delta_{ij}), \tag{2.105}$$

equation (2.104) can be recast into

$$dr^2 - dX^2 = 2e_{ij}^{\mathrm{G}}\, dX_i\, dX_j. \tag{2.106}$$

Using (2.6) to express e_{ij}^{G} in terms of u_i and considering $r_{i,j} = \delta_{ij} + u_{i,j}$ according to (2.6), we may rewrite (2.105) in the form

$$\begin{aligned} e_{ij}^{\mathrm{G}} &= \tfrac{1}{2}[(\delta_{ki} + u_{k,i})(\delta_{kj} + u_{k,j}) - \delta_{ij}] \\ &= \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}). \end{aligned} \tag{2.107}$$

2.5.3 Relation to infinitesimal strain

Assuming infinitesimal displacement gradients, the products in (2.100) and (2.107) may be neglected, so that

$$E_{ij}^{\mathrm{C}} = \tfrac{1}{2}(U_{i,j} + U_{j,i}), \tag{2.108}$$

$$e_{ij}^{\mathrm{G}} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.109}$$

Comparing these results with (2.85), we note that $e_{ij}^{\mathrm{G}} = e_{ij}$ in this special case. However, from (2.6) and (2.14), it follows that $U_{i,j} = u_{i,k}X_{k,j} = u_{i,k}(\delta_{kj} - U_{k,j})$, which, for $U_{i,j}$ and $u_{i,j}$ infinitesimal, reduces to $U_{i,j} = u_{i,j}$. Hence, $E_{ij}^{\mathrm{C}} = e_{ij}^{\mathrm{G}}$ applies on this assumption, *i.e.* the distinction between the Cauchy strain and the Green strain is not necessary.

2.6 Strain rate and vorticity

In contrast to the *trajectory* as the path followed by a particle during flow, the *streamline* is the curve whose tangent at any point has the direction of the velocity vector for the particle currently at this point. In the special case of *steady* flow, all fields are time-independent. Hence, the streamline pattern does not change with time and the trajectories coincide with the streamlines.

In deformation studies (Sec. 2.4), the actual trajectories are irrelevant and the deformation is profitably specified in terms of the displacement field with respect to the initial state, $u_i(\mathbf{X}, t)$, *i.e.* in the Langrangian representation. The geometric illustration of this displacement is a straight line directed from X_i to r_i, whether the actual movement does proceed along this line or not (Fig. 2.3).

In flow studies, it is obvious to specify the sequence of current states. In principle, this can be achieved by specifying the trajectory of each particle explicitly. However, as the trajectories are usually unknown, it is necessary to start from the observable spatial streamline pattern, $V_i = V_i(\mathbf{r}, t)$, *i.e.* to use the Eulerian representation.

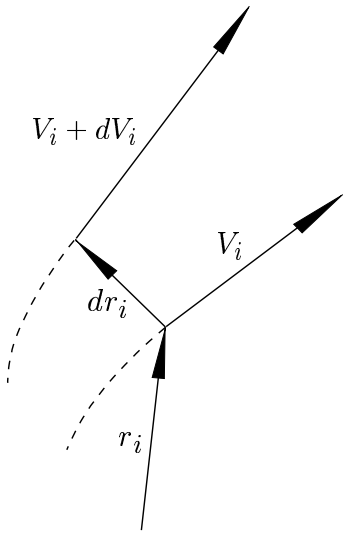


Figure 2.6: Velocities, V_i and $V_i + dV_i$, of two particles currently at neighbouring spatial positions, r_i and $r_i + dr_i$, respectively. The dashed lines indicate the unknown trajectories.

In Fig. 2.6, the dashed lines represent the unknown trajectories of two particles currently separated by the spatial differential length dr_i. Their current velocities are tangential to the trajectories at their respective positions and given by V_i and $V_i + dV_i$. In view of (2.32), the differential velocity, dV_i, for a fixed time epoch, t, can be written in terms of dr_i as

$$dV_i = V_{i,j}\, dr_j, \tag{2.110}$$

where $V_{i,j}$ is the spatial velocity gradient. Using (2.24) and (2.26), the differential velocity per spatial unit length is given by

$$\frac{dV_i}{dr} = V_{i,j} p_j^{\mathbf{r}}. \tag{2.111}$$

This represents a linear vector function, whence $V_{i,j}$ is a second-rank tensor.

For the interpretation of $V_{i,j}$, we decompose it into

$$V_{i,j} \equiv \tfrac{1}{2}(V_{i,j} + V_{j,i}) + \tfrac{1}{2}(V_{i,j} - V_{j,i}). \tag{2.112}$$

The first term on the right-hand side is *symmetric* and called the strain-rate tensor:

$$D_{ij} := \tfrac{1}{2}(V_{i,j} + V_{j,i}). \tag{2.113}$$

The second term is *skew-symmetric* and referred to as the vorticity tensor:

$$\Omega_{ij} := \tfrac{1}{2}(V_{i,j} - V_{j,i}). \tag{2.114}$$

The interpretation of D_{ij} is found by setting $\Omega_{ij} = 0$, in which case (2.111) becomes

$$\frac{dV_i}{dr} = D_{ij} p_j^{\mathbf{r}}. \tag{2.115}$$

Introducing the strain-rate vector by

$$D_i := \frac{dV_i}{dr}, \tag{2.116}$$

we can rewrite equation (2.115) as

$$D_i = D_{ij} p_j^{\mathbf{r}}. \tag{2.117}$$

For the interpretation of D_{ij}, we consider

$$D_i q_i^{\mathbf{r}} = D_{ij} q_i^{\mathbf{r}} p_j^{\mathbf{r}}, \tag{2.118}$$

which represents the component of D_i in the direction of a second spatial unit vector, $q_i^{\mathbf{r}}$. If, in particular, $p_i^{\mathbf{r}} = (1,0,0)$ and $q_i^{\mathbf{r}} = (1,0,0)$, the component D_{11} describes the *longitudinal* strain rate (extension rate per spatial unit length) of the infinitesimal length currently oriented in the r_1 direction. On the other hand, if $p_i^{\mathbf{r}} = (1,0,0)$ and $q_i^{\mathbf{r}} = (0,1,0)$, the component D_{21} describes the *transverse* strain rate (shear rate per spatial unit length) in the r_2 direction associated with the infinitesimal length currently oriented in the r_1 direction. More generally, we can state that the main-diagonal components of D_{ij} are associated with *extension*, whereas the off-diagonal components describe *shear*.

The interpretation of Ω_{ij} follows if we set $D_{ij} = 0$ and consider

$$dV_i = \Omega_{ij}\, dr_j. \tag{2.119}$$

With the vorticity vector given by

$$\Omega_i := -\tfrac{1}{2}\epsilon_{ijk}\Omega_{jk}, \tag{2.120}$$

equation (2.119) is equivalent to

$$dV_i = \epsilon_{ijk}\Omega_j\, dr_k. \tag{2.121}$$

This formula describes a relative rotation with the angular speed Ω of two particles separated by the infinitesimal spatial length dr_i about an axis parallel to Ω_i (Fig. 2.5).

2.7 Special measures of strain

2.7.1 Principal strains

As discussed in Sec. 2.4, the infinitesimal strain tensor, e_{ij}, assigns to an arbitrarily directed material unit vector, $P_i^{\mathbf{X}}$, the strain vector, e_i. Of particular interest are the principal directions, for which the two vectors are collinear and, therefore, the strains purely longitudinal. In general, these directions depend on $\mathbf{X}$. We therefore consider $P_i^{\mathbf{X}}$ as a *field* in the Lagrangian representation and write

$$e_i = \lambda P_i^{\mathbf{X}}. \tag{2.122}$$

Since, according to (2.89),

$$e_i = e_{ij} P_j^{\mathbf{X}}, \tag{2.123}$$

we obtain from these equations the linear relation

$$(e_{ij} - \lambda \delta_{ij}) P_j^{\mathbf{X}} = 0. \tag{2.124}$$

Non-trivial solutions to this equation result only if the coefficient determinant vanishes:

$$\det (e_{ij} - \lambda \delta_{ij}) = 0. \tag{2.125}$$

Expansion of the determinant leads to a cubic polynomial in λ:

$$\lambda^3 - a^{(1)} \lambda^2 + a^{(2)} \lambda - a^{(3)} = 0, \tag{2.126}$$

where

$$a^{(1)} := e_{ii}, \tag{2.127}$$

$$a^{(2)} := \tfrac{1}{2}(e_{ii} e_{jj} - e_{ij} e_{ij}), \tag{2.128}$$

$$a^{(3)} := \det e_{ij} \tag{2.129}$$

are the first, second and third invariants of e_{ij}, respectively. It can be shown that, with e_{ij} symmetric and real, the three roots of (2.126) are also real. They are called the principal values of e_{ij} and denoted by $e^{(1)}$, $e^{(2)}$ and $e^{(3)}$. For each $e^{(k)}$, the associated principal direction, $P_i^{\mathbf{X}(k)}$, is found by solving

$$(e_{ij} - e^{(k)} \delta_{ij}) P_j^{\mathbf{X}(k)} = 0 \tag{2.130}$$

subject to the condition $P_i^{\mathbf{X}(k)} P_i^{\mathbf{X}(k)} \equiv 1$. If $e^{(1)}$, $e^{(2)}$ and $e^{(3)}$ are distinct, the three orthogonal principal directions, $P_i^{\mathbf{X}(1)}$, $P_i^{\mathbf{X}(2)}$ and $P_i^{\mathbf{X}(3)}$, are uniquely determined. If, for example, $e^{(1)} = e^{(2)}$ and $e^{(1)} \neq e^{(3)}$, only $P_i^{\mathbf{X}(3)}$ is uniquely determined. The other principal directions are any pair of mutually orthogonal unit vectors in the plane normal to $P_i^{\mathbf{X}(3)}$, whence shearing cannot take place in this plane. If, even, $e^{(1)} = e^{(2)} = e^{(3)}$, any direction is a principal direction and shearing is not possible.

In any case, three mutually orthogonal principal directions, $P_i^{\mathbf{X}(1)}$, $P_i^{\mathbf{X}(2)}$ and $P_i^{\mathbf{X}(3)}$, can be selected as the basis of a new Cartesian coordinate system. Since, *per definitionem*, the strain is purely longitudinal in any of its principal directions, the strain matrix in this principal coordinate system must be diagonal, with the principal values of e_{ij} the main-diagonal elements of this matrix:

$$[e_{ij}] = \begin{bmatrix} e^{(1)} & 0 & 0 \\ 0 & e^{(2)} & 0 \\ 0 & 0 & e^{(3)} \end{bmatrix}. \tag{2.131}$$

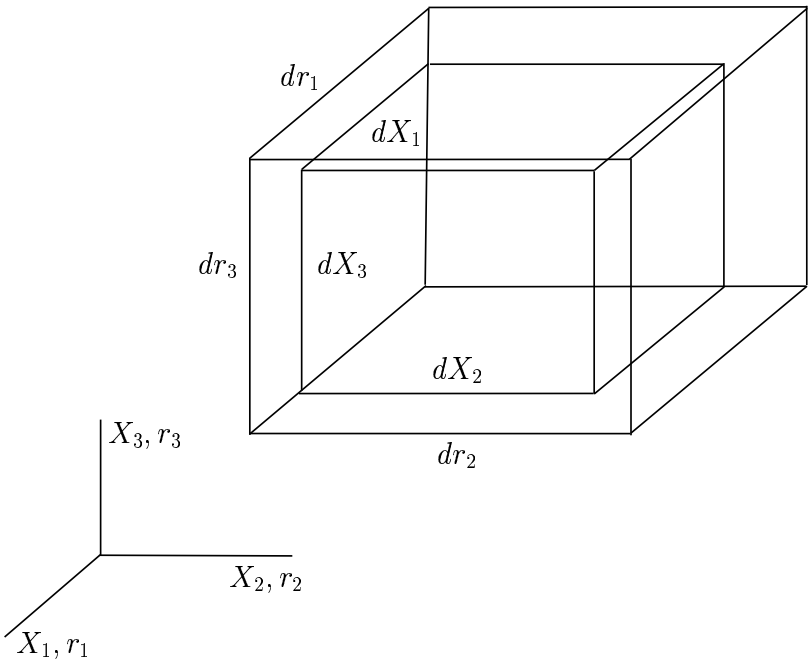


Figure 2.7: Material (initial) differential volume, $d^3X = dX_1\, dX_2\, dX_3$, and associated spatial (current) differential volume, $d^3r = dr_1\, dr_2\, dr_3$.

Obviously, three components fully determine the strain if expressed in a principal coordinate system. However, the description of the state of strain is only complete if the principal directions are also known. This requires the specification of three additional scalar quantities.

We note that our analysis of principal strains has only supposed that e_{ij} is symmetric and real. With the appropriate modifications, it therefore also applies to any other field with these properties, for example the Cauchy strain, E^{C}_{ij}, the Green strain, e^{G}_{ij}, and the strain rate, D_{ij}.

2.7.2 Spherical and deviatoric strains

Sometimes, it is useful to decompose e_{ij} into the sum of two particular fields:

$$e_{ij} \equiv \tfrac{1}{3} e_{kk} \delta_{ij} + e_{[ij]}. \tag{2.132}$$

The first term on the right-hand side is referred to as the spherical part of e_{ij}.:

$$e_{(ij)} := \tfrac{1}{3} e_{kk} \delta_{ij}. \tag{2.133}$$

Obviously, it is a diagonal tensor whose trace equals $a^{(1)}$. The second term, $e_{[ij]}$, is called the deviatoric part of e_{ij}.

The interpretations of e_{ii} and $e_{[ij]}$ can be found by considering an infinitesimal subbody initially occupying a rectangular parallelepiped with the material differential volume

$d^3X = dX_1\,dX_2\,dX_3$. Provided that the subbody is strained longitudinally along its edges, it occupies a rectangular parallelepiped with the current differential volume $d^3r = dr_1\,dr_2\,dr_3$ (Fig. 2.7). In view of the interpretation of the main-diagonal components of e_{ij} (Sec. 2.4), we can thus write

$$\begin{aligned}\frac{d^3r - d^3X}{d^3X} &= \frac{(1+e_{11})\,dX_1\,(1+e_{22})\,dX_2\,(1+e_{33})\,dX_3 - dX_1\,dX_2\,dX_3}{dX_1\,dX_2\,dX_3}\\ &= (1+e_{11})(1+e_{22})(1+e_{33}) - 1,\end{aligned} \tag{2.134}$$

which, taking into account that e_{ij} is infinitesimal, reduces to

$$\frac{d^3r - d^3X}{d^3X} = e_{11} + e_{22} + e_{33} = e_{ii}. \tag{2.135}$$

Since, by (2.132), $e_{[ii]} = 0$, the main-diagonal components of $e_{[ij]}$ do not involve volume changes. Moreover, with $e_{[ij]} = e_{[ji]}$, the off-diagonal components of $e_{[ij]}$ describe the skewing of the subbody initially occupying the rectangular parallelepiped. Since $e_{[ij]}$ is infinitesimal, this transformation can be shown not to be accompanied by volume changes. Therefore, e_{ii} measures the increase in volume per material unit volume, *i.e.* the *dilatation* of the material volume, whereas $e_{[ij]}$ describes its *distortion*.

We emphasize that the interpretations of e_{ii} as the dilatation and $e_{[ij]}$ as the distortion are contingent upon e_{ij} being infinitesimal and, therefore, cannot be extended to the Cauchy strain, E^{C}_{ij}, or the Green strain, e^{G}_{ij}.

3 Stress

3.1 Introduction

After reviewing the *kinematic* concepts of continuum mechanics in Chap. 2, this chapter discusses the *mechanical* concepts. We begin with the distinction between volume forces and surface forces (Sec. 3.2) and state the Cauchy traction principle for the latter (Sec. 3.3). Next, we will introduce the Cauchy stress and investigate its symmetry properties (Sec. 3.4). This is followed by the definitions of the Piola, Finger and Kirchhoff stresses (Sec. 3.5). Finally, the concepts of principal, spherical and deviatoric stresses will be outlined (Sec. 3.6). We note that, whereas the Cauchy stress is introduced on fairly general assumptions, the symmetry of the associated tensor is established only for the restricted case of equilibrium, absence of volume forces and homogeneous stress distribution (for a more general proof see Sec. 4.6).

3.2 Volume and surface forces

The forces acting upon some subbody can be classified by distinguishing between *long-range* and *short-range* interactions. Long-range interactions comprise *gravitational*, *electromagnetic* and *inertial* forces. These forces decrease very gradually with increasing distance between the interacting particles. As a result, long-range forces act uniformly upon the material contained within a sufficiently small volume, so that they are proportial to its size (Fig. 3.1a). In continuum mechanics, long-range forces are therefore called *volume* forces.

Short-range interactions comprise several types of *molecular* forces. Their common feature is that they decrease extremely rapidly with increasing distance between the interacting particles. Hence, short-range forces are appreciable only when this distance does not exceed molecular dimensions. A further consequence is that, if the material occupying a volume is acted upon by short-range forces originating from interactions with the material outside of this volume, these forces can only influence the particles on and immediately below the surface of this volume (Fig. 3.1b). In continuum mechanics, short-range forces are therefore classified as *surface* forces. They are specified more closely by constitutive equations (Chap. 5).

In the following, it will be supposed that volume and surface forces arise due to interactions that are *equal*, *opposite* and *collinear* (strong axiom of action and reaction). Because of this restriction, distributed volume and surface couples cannot arise.

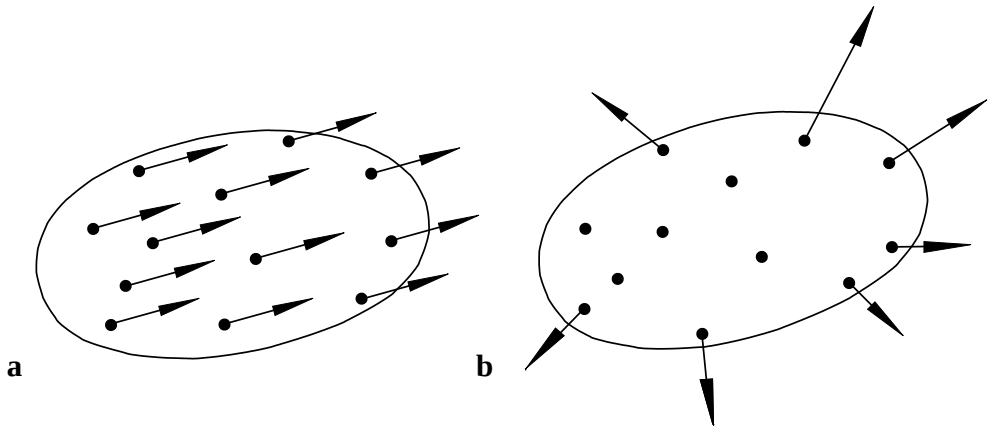


Figure 3.1: (a) Volume forces and (b) surface forces acting upon selected particles currently occupying some spatial volume.

3.2.1 Eulerian representation

We first define the integral volume force, $\mathcal{F}_i^{\mathrm{V}}$, acting upon the subbody currently occupying some spatial volume, $\mathcal{R}$. If F_i denotes the force per spatial unit volume, we then have

$$\mathcal{F}_i^{\mathrm{V}} := \int_{\mathcal{R}} F_i \, d^3 r. \tag{3.1}$$

Since gravitational and inertial forces are proportional to the mass, we may alternatively write (Fig. 3.2)

$$\mathcal{F}_i^{\mathrm{V}} = \int_{\mathcal{R}} \rho G_i \, d^3 r, \tag{3.2}$$

where G_i denotes the force per unit mass (gravity) and ρ the mass per spatial unit volume (volume-mass density). Next, let T_i denote the force per spatial unit area (Cauchy traction). With this, the integral surface force, $\mathcal{F}_i^{\mathrm{S}}$, currently exerted across the spatial boundary, $\partial\mathcal{R}$, is given by (Fig. 3.2)

$$\mathcal{F}_i^{\mathrm{S}} := \int_{\partial\mathcal{R}} T_i \, d^2 r. \tag{3.3}$$

Furthermore, we define by

$$\mathcal{F}_i := \mathcal{F}_i^{\mathrm{V}} + \mathcal{F}_i^{\mathrm{S}} \tag{3.4}$$

the integral force acting upon the subbody currently occupying $\mathcal{R}$. From (3.2) and (3.3), we then find

$$\mathcal{F}_i = \int_{\mathcal{R}} \rho G_i \, d^3 r + \int_{\partial\mathcal{R}} T_i \, d^2 r. \tag{3.5}$$

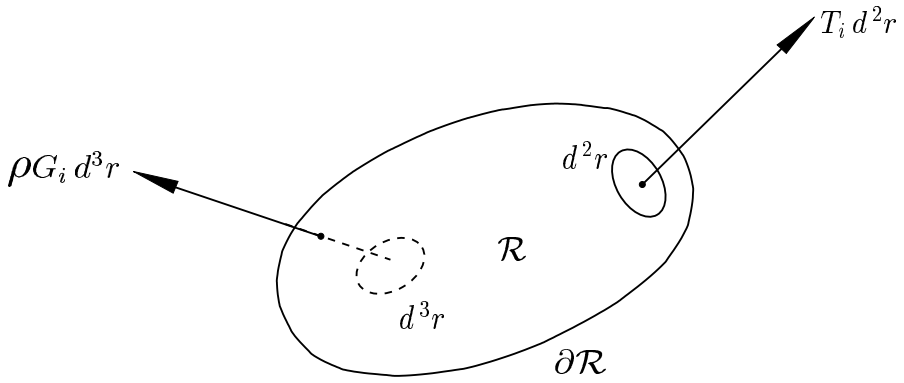


Figure 3.2: Differential volume force, $\rho G_i\, d^3r$, and differential surface force, $T_i\, d^2r$, currently acting upon the subbody occupying some spatial volume, $\mathcal{R}$, with the spatial boundary $\partial\mathcal{R}$.

3.2.2 Lagrangian representation

To find the Lagrangian representation of (3.5), we consider the square of (2.48):

$$d^2r_i\, d^2r_i = (j^{\mathbf{X}})^2\, r^{-1}_{j,i} r^{-1}_{k,i}\, d^2X_j\, d^2X_k. \tag{3.6}$$

Using (2.28) and (2.31), we obtain

$$d^2r = j^{\mathbf{X}} (r^{-1}_{j,i} r^{-1}_{k,i} N^{\mathbf{X}}_j N^{\mathbf{X}}_k)^{\frac{1}{2}}\, d^2X. \tag{3.7}$$

In view of (2.49) and this equation, (3.5) takes the form

$$\mathcal{F}_i = \int_{\mathcal{X}} j^{\mathbf{X}} \rho g_i\, d^3X + \int_{\partial\mathcal{X}} j^{\mathbf{X}} (r^{-1}_{k,j} r^{-1}_{l,j} N^{\mathbf{X}}_k N^{\mathbf{X}}_l)^{\frac{1}{2}}\, t_i\, d^2X. \tag{3.8}$$

where $\mathcal{X}$ is the material volume initially occupied by the subbody considered and $\partial\mathcal{X}$ is its material boundary. To simplify this equation, we introduce the Piola traction:

$$t^{\mathrm{P}}_i := j^{\mathbf{X}} (r^{-1}_{k,j} r^{-1}_{l,j} N^{\mathbf{X}}_k N^{\mathbf{X}}_l)^{\frac{1}{2}}\, t_i. \tag{3.9}$$

Considering (3.8) and (3.9), the Lagrangian representation corresponding to (3.5) is

$$\mathcal{F}_i = \int_{\mathcal{X}} j^{\mathbf{X}} \rho g_i\, d^3X + \int_{\partial\mathcal{X}} t^{\mathrm{P}}_i\, d^2X. \tag{3.10}$$

This equation shows that $j^{\mathbf{X}} \rho g_i$ and t^{P}_i can be interpreted as the force per material unit volume and the force per material unit area, respectively.

3.3 Cauchy traction principle

We proceed by defining the Cauchy traction more formally. For this purpose, we introduce as a special case of the continuity principle (Sec. 1.2) the Cauchy traction principle. To illustrate

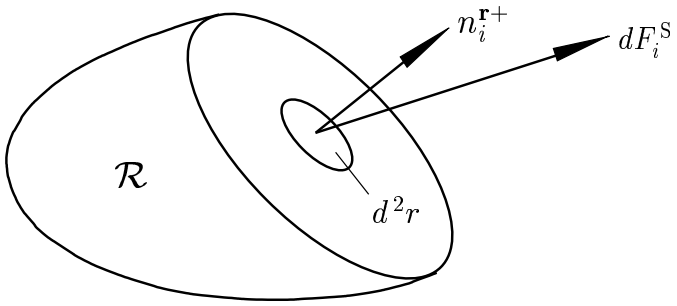


Figure 3.3: Differential surface force, dF_i^{S}, currently acting across some spatial differential area, d^2r, with the outward spatial unit vector normal to it, $n_i^{\mathbf{r}+}$, upon the subbody inside of some spatial volume, $\mathcal{R}$.

it, we consider a spatial surface, assumed to be planar and inside of some body, and isolate a spatial volume, $\mathcal{R}$, on one side of it. Furthermore, we denote by dF_i^{S} the differential surface force currently exerted across some spatial differential area, d^2r, with the outward spatial unit vector normal to it, $n_i^{\mathbf{r}+}$, upon the subbody inside of $\mathcal{R}$ (Fig. 3.3). The Cauchy traction principle then postulates the existence of the limit

$$T_i^{(\mathbf{n})} := \lim_{d^2r \to 0} \frac{dF_i^{\mathrm{S}}}{d^2r} \,. \tag{3.11}$$

Obviously, this limit is meaningful only if d^2r degenerates not to a curve, but to a point. Hence, we refer to $T_i^{(\mathbf{n})}$ as the Cauchy traction at this point. Normally, $T_i^{(\mathbf{n})}$ depends on the orientation of the subsurface, which is indicated by the superscript. Incidentally, a continuous distribution of forces acting across a surface is in general equivalent to a surface force and a surface couple. Since it can be shown that, at the limit $d^2r \to 0$, the couple per spatial unit area vanishes, this complication has been excluded from the preceding argument.

An elementary consequence of (3.11) and the strong axiom of action and reaction is

$$T_i^{(\mathbf{n})} \equiv -T_i^{(-\mathbf{n})}. \tag{3.12}$$

This means that the force exerted across d^2r by the subbody outside of $\mathcal{R}$ on the subbody inside of it is equal in magnitude, but opposite in direction to the force exerted across d^2r by the subbody inside of $\mathcal{R}$ on the subbody outside of it.

As shown in Sec. 3.4, the Cauchy tractions with respect to planes normal to Cartesian coordinate axes are particularly useful. Henceforth, we use the notation $T_i^{(j)}$ to denote the

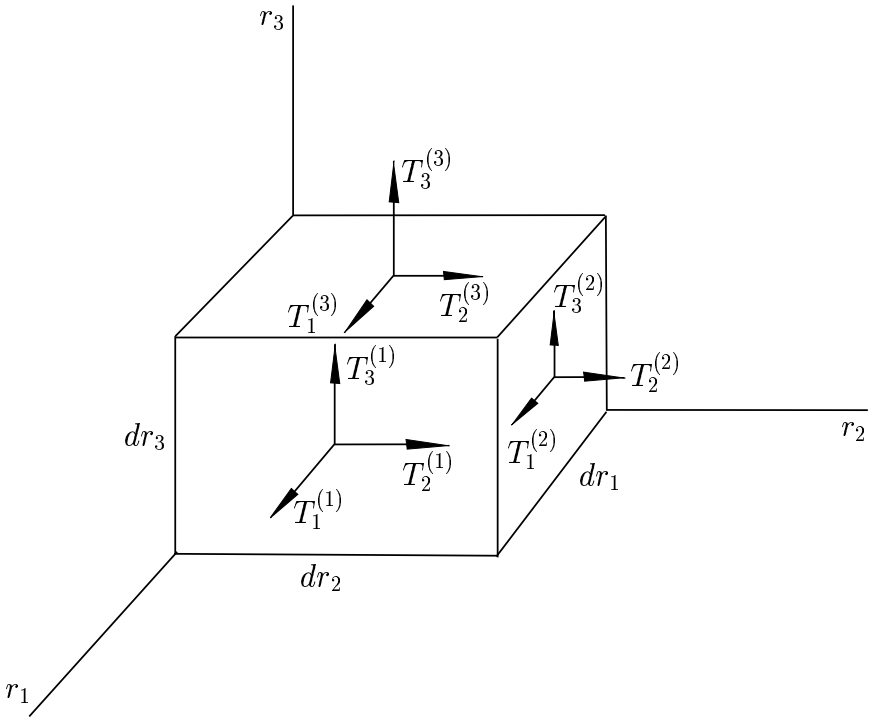


Figure 3.4: Traction components, $T_i^{(j)}$, acting on particles centred on orthogonal faces on the subbody currently inside of some cube.

Cauchy traction acting across a plane normal to the r_j coordinate. Figure 3.4 displays the Cartesian components of three such tractions, $T_i^{(1)}$, $T_i^{(2)}$ and $T_i^{(3)}$, acting on particles assumed to be centred on orthogonal faces of a cube. The figure also serves to illustrate the sign convention adopted. According to it, $T_i^{(j)}$ is positive if the ith component of the Cauchy traction exerted on the subbody inside of the cube across the face whose outward normal points in the r_j direction has the direction shown.

If the spatial differential lengths, dr_1, dr_2 and dr_3, of the cube approach zero, the points of action of the three Cauchy tractions converge. However, since they continue to act across three mutually orthogonal planes, they generally remain different at this limit. This agrees with the Cauchy traction principle and confirms that the state of traction at a given point cannot be described by a vector.

On account of (3.12), it follows that, if the cube is infinitesimal, the ith component of the Cauchy tractions acting on the subbody inside of the cube across the face with the outward normal in the r_j direction is equal and opposite to the ith component acting across the opposite face. Therefore, the three *normal* components are *tensile* tractions if positive or *compressive* tractions if negative. The six *tangential* components are not distinguished according to their signs and are collectively referred to as *shear* tractions.

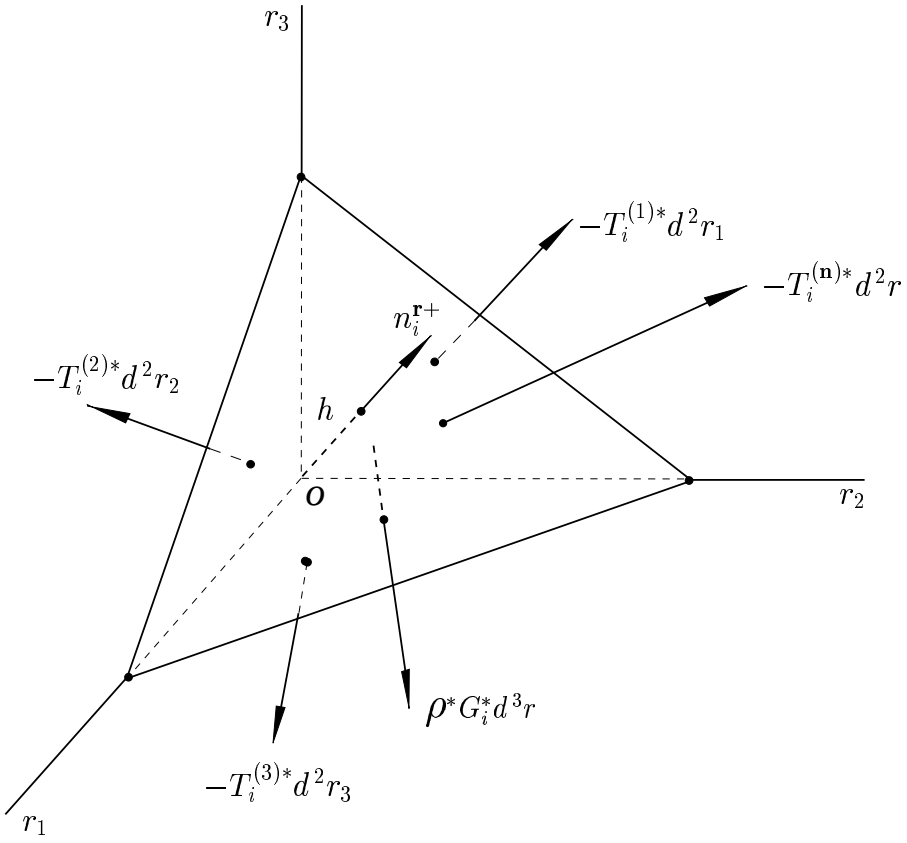


Figure 3.5: Differential volume and surface forces currently acting upon the subbody inside of some tetrahedron.

3.4 Cauchy stress

3.4.1 Definition

Figure 3.5 shows a tetrahedron with the spatial differential volume d^3r currently occupied by a subbody. Using the Eulerian representation of the equation of motion (Sec. 4.5.1), the time rate of increase of the differential linear momentum of the subbody currently occupying the tetrahedron equals the differential force currently acting upon it:

$$\rho^* \frac{dV_i^*}{dt} d^3r = \rho^* G_i^* d^3r + T_i^{(\mathbf{n})*} d^2r - T_i^{(1)*} d^2r_1 - T_i^{(2)*} d^2r_2 - T_i^{(3)*} d^2r_3, \tag{3.13}$$

where the asterisk indicates the appropriate spatial mean. In this equation, $\rho^* G_i^* d^3r$ is the differential volume force, $T_i^{(\mathbf{n})*} d^2r$ the differential surface force across the oblique face and $-T_i^{(k)*} d^2r_k$ the differential surface force across the face whose outward normal points opposite to the r_k direction. In view of (2.30) and $d^3r = \frac{1}{3} h\, d^2r$, where h is the altitude of the oblique face with respect to the origin, o, equation (3.13) reduces to

$$\frac{1}{3}\left(\frac{dV_i^*}{dt} - G_i^*\right)\rho^* h = T_i^{(\mathbf{n})*} - T_i^{(1)*} n_1^{\mathbf{r}+} - T_i^{(2)*} n_2^{\mathbf{r}+} - T_i^{(3)*} n_3^{\mathbf{r}+}. \tag{3.14}$$

Next, we let $h \to 0$ without changing the orientation of h or the position of o. Then, (3.14) becomes

$$T_i^{(\mathbf{n})} = T_i^{(1)} n_1^{\mathbf{r}+} + T_i^{(2)} n_2^{\mathbf{r}+} + T_i^{(3)} n_3^{\mathbf{r}+}, \tag{3.15}$$

where all tractions now apply to the origin, o, as indicated by dropping the asterisks. This equation allows us to determine the Cauchy traction at a given point acting across an arbitrarily inclined plane provided that the Cauchy tractions acting across three mutually orthogonal planes through this point are known. If we define

$$T_{ij} := T_i^{(j)} \tag{3.16}$$

and apply the summation convention, (3.15) is transformed into the Cauchy stress formula:

$$T_i^{(\mathbf{n})} = T_{ij} n_j^{\mathbf{r}+}. \tag{3.17}$$

Accordingly, the nine components of $T_i^{(j)}$ form the components of a second-rank tensor, T_{ij}, which is called the Cauchy stress. In other words, T_{ij} represents a linear vector function which maps an arbitrary spatial unit vector, $n_i^{\mathbf{r}+}$, onto the Cauchy traction, $T_i^{(\mathbf{n})}$, acting across the surface whose outward spatial unit vector normal to it is $n_i^{\mathbf{r}+}$.

3.4.2 Symmetry

It is sometimes convenient to arrange the components of the Cauchy stress as the elements of a matrix:

$$[T_{ij}] := \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}, \tag{3.18}$$

where the first subscript indicates the traction component and the second identifies the normal to the plane considered. If the Cauchy stress tensor is symmetric, $T_{ij} = T_{ji}$, *i.e.* the pairs of elements placed symmetrically with respect to the main diagonal of the matrix are equal:

$$T_{21} = T_{12}, \quad T_{31} = T_{13}, \quad T_{32} = T_{23}. \tag{3.19}$$

Here, the symmetry of the stress tensor is established only for the special case of *equilibrium*, *absence of volume forces* and *homogeneous stress distribution*. A proof on more general assumptions will be given below (Sec. 4.6).

To prove the first relation of (3.19), we consider the differential surface-torque component in the r_3 direction acting upon the subbody currently occupying a cube with the spatial differential lengths dr_1, dr_2 and dr_3 (Fig. 3.4). Because of the assumed homogeneity of the stress field, the differential surface-force components normal to the r_3 direction become

$$T_{11}\, dr_2\, dr_3, \quad T_{21}\, dr_2\, dr_3 \quad \text{(face with outward normal in } r_1 \text{ direction)},$$

$$T_{12}\,dr_1\,dr_3, \quad T_{22}\,dr_1\,dr_3 \quad \text{(face with outward normal in } r_2 \text{ direction)},$$

$$T_{13}\,dr_1\,dr_2, \quad T_{23}\,dr_1\,dr_2 \quad \text{(face with outward normal in } r_3 \text{ direction)}$$

and act at the centres of the respective faces. With $T_{11}\,dr_2\,dr_3$ and $T_{22}\,dr_1\,dr_3$ balanced by equal, opposite and collinear forces acting across the opposite faces, torques due to the normal forces cannot arise. Similarly, the torques due to $T_{13}\,dr_1\,dr_2$ and $T_{23}\,dr_1\,dr_2$ are balanced by those due to equal and opposite forces acting across the opposite faces with the same moment arms, $dr_2/2$ and $dr_1/2$, respectively. The differential surface-torque component in the r_3 direction is thus given by $dr_1(T_{21}\,dr_2\,dr_3) - dr_2(T_{12}\,dr_1\,dr_3)$. However, since equilibrium applies and volume forces are absent, this component vanishes, leaving $T_{12} = T_{21}$. The second and third relations of (3.19) can be proved in a similar way by considering the components of the surface-torque balance in the r_2 and r_1 directions, respectively. On the restrictive conditions stated above, we have therefore established

$$T_{ij} = T_{ji}. \tag{3.20}$$

3.5 Non-Cauchy stresses

Any surface forces acting in the body are in general associated with deformation. When introducing the concept of traction, it is thus natural to normalize the forces acting in the deformed state across some spatial surface with respect to the unit area of this surface. This view has led to the definition of the Cauchy stress, which is normally given in the Eulerian representation (Sec. 3.3). If an undeformed state of the body can be distinguished, the use of the Lagrangian representation with this state serving as the initial state is more natural (Sec. 2.2). This allows three other types of stress to be introduced (Fig. 3.6).

3.5.1 Piola stress

We consider the subbody occupying the volumes $\mathcal{X}$ and $\mathcal{R}$ in the undeformed initial state and the deformed current state, respectively. In view of (2.30), (3.3) and (3.17), the differential surface force, dF_i^{S}, currently acting across some spatial differential area, d^2r_i, on this subbody is expressible as follows:

$$dF_i^{\mathrm{S}} = T_{ij}\,d^2r_j. \tag{3.21}$$

Using (2.48), $dF_i^{\mathrm{S}} = df_i^{\mathrm{S}}$ and $T_{ij} = t_{ij}$, this equation can be rewritten as

$$df_i^{\mathrm{S}} = j\,{}^{\mathbf{X}}r_{k,j}^{-1}t_{ij}\,d^2X_k. \tag{3.22}$$

If we define the Piola stress by

$$t_{ij}^{\mathrm{P}} := j\,{}^{\mathbf{X}}r_{j,k}^{-1}t_{ik}, \tag{3.23}$$

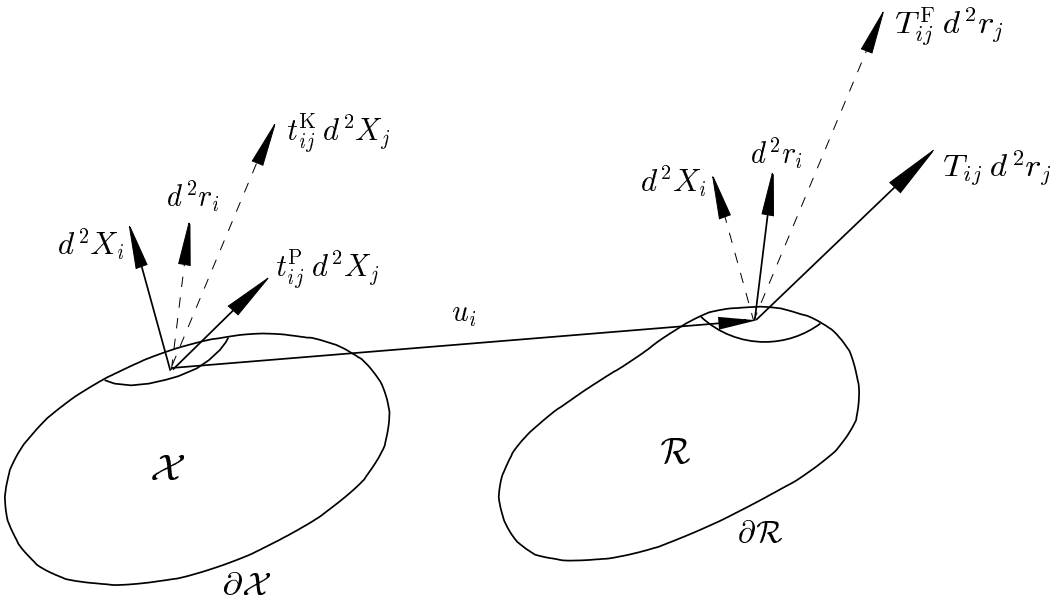


Figure 3.6: Differential areas and differential tractions related to the Cauchy, T_{ij}, Piola, t^{P}_{ij}, Finger, T^{F}_{ij}, and Kirchhoff, t^{K}_{ij}, stresses.

equation (3.22) can be recast into

$$df^{\mathrm{S}}_i = t^{\mathrm{P}}_{ij}\, d^2 X_j. \tag{3.24}$$

Comparing (3.21) and (3.24) shows that, in contrast to the Cauchy stress, t_{ij}, the Piola stress, t^{P}_{ij}, is obtained by normalizing the currently acting differential surface force, df^{S}_i, in terms of the material differential area, $d^2 X_i$. Inspection of (3.23) shows that, in general, t^{P}_{ij} is not symmetric. The inverse relationship corresponding to (3.23) is found to be

$$T_{ij} = J^{\mathbf{r}} X^{-1}_{j,k} T^{\mathrm{P}}_{ik}. \tag{3.25}$$

3.5.2 Finger stress

For the definition of the Finger stress, T^{F}_{ij}, we consider instead of the *actual* differential surface force, dF^{S}_i, currently acting across the spatial differential area, $d^2 r_i$, the *fictitious* differential surface force, $d\hat{F}^{\mathrm{S}}_i$, obtained from dF^{S}_i in the same way that the material differential length, dX_i, is related the spatial differential length, dr_i. In view of (2.36), we thus define

$$d\hat{F}^{\mathrm{S}}_i := X_{i,j}\, dF^{\mathrm{S}}_j, \tag{3.26}$$

which, using (3.21), becomes

$$d\hat{F}^{\mathrm{S}}_i = X_{i,j} T_{jk}\, d^2 r_k. \tag{3.27}$$

If we introduce the Finger stress by

$$T^{\mathrm{F}}_{ij} := X_{i,k} T_{kj}, \tag{3.28}$$

equation (3.27) can be rewritten as

$$d\hat{F}_i^S = T_{ij}^{\mathrm{F}}\, d^2 r_j. \tag{3.29}$$

Equation (3.28) shows that T_{ij}^{F} is usually not symmetric. The inverse relation associated with (3.28) is found to be

$$T_{ij} = X_{i,k}^{-1} T_{kj}^{\mathrm{F}}. \tag{3.30}$$

3.5.3 Kirchhoff stress

The Kirchhoff stress, t_{ij}^{K}, is related to the Piola stress, t_{ij}^{P}, by considering instead of the actual differential surface force, df_i^{S}, the *fictitious* differential surface force, $d\hat{f}_i^{\mathrm{S}}$. Using (2.22) and (3.26), we obtain

$$d\hat{f}_i^{\mathrm{S}} = r_{i,j}^{-1}\, df_j^{\mathrm{S}} \tag{3.31}$$

or, with (3.24), the equation

$$d\hat{f}_i^{\mathrm{S}} = r_{i,j}^{-1} t_{jk}^{\mathrm{P}}\, d^2 X_k. \tag{3.32}$$

If we define the Kirchhoff stress by

$$t_{ij}^{\mathrm{K}} := r_{i,k}^{-1} t_{kj}^{\mathrm{P}}, \tag{3.33}$$

equation (3.32) can be rewritten as

$$d\hat{f}_i^{\mathrm{S}} = t_{ij}^{\mathrm{K}}\, d^2 X_j. \tag{3.34}$$

Substitution of (3.23) into (3.33) alternatively yields

$$t_{ij}^{\mathrm{K}} := j^{\mathbf{X}} r_{i,k}^{-1} r_{j,l}^{-1} t_{kl}, \tag{3.35}$$

whose inverse relation is

$$T_{ij} = J^{\mathbf{r}} X_{i,k}^{-1} X_{j,l}^{-1} T_{kl}^{\mathrm{K}}. \tag{3.36}$$

Inspection of (3.35) shows that t_{ij}^{K} is symmetric provided that t_{ij} is also symmetric.

3.6 Special measures of stress

3.6.1 Principal stresses

At any point, the Cauchy stress tensor, T_{ij}, assigns to an arbitrarily directed spatial unit vector, $n_i^{\mathbf{r}+}$, the Cauchy traction, T_i, acting across the plane whose outward spatial unit vector normal to it is $n_i^{\mathbf{r}+}$. As in Sec. 2.7 for the strain, we are interested in the principal directions, for which the two vectors are collinear. Since, in general, these directions depend on $\mathbf{r}$, we consider $n_i^{\mathbf{r}}$ as a *field* in the Eulerian representation and write

$$T_i = \lambda n_i^{\mathbf{r}}. \tag{3.37}$$

Since, according to (3.17),

$$T_i = T_{ij} n_j^{\mathbf{r}}, \tag{3.38}$$

we obtain from (3.37) and (3.38) the linear expression

$$(T_{ij} - \lambda \delta_{ij}) n_j^{\mathbf{r}} = 0. \tag{3.39}$$

Necessary and sufficient for non-trivial solutions to this equation is that the coefficient determinant vanishes:

$$\det (T_{ij} - \lambda \delta_{ij}) = 0. \tag{3.40}$$

Expansion of the determinant yields a cubic polynomial in λ:

$$\lambda^3 - B^{(1)} \lambda^2 + B^{(2)} \lambda - B^{(3)} = 0, \tag{3.41}$$

with

$$B^{(1)} := T_{ii}, \tag{3.42}$$

$$B^{(2)} := \tfrac{1}{2}(T_{ii} T_{jj} - T_{ij} T_{ij}), \tag{3.43}$$

$$B^{(3)} := \det T_{ij} \tag{3.44}$$

as the first, second and third invariants of the stress tensor, respectively. Assuming that T_{ij} is symmetric and real, it can be shown that the three roots of (3.41) are also real. They constitute the principal values of T_{ij}, in the following denoted by $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$. For each $T^{(k)}$, the associated principal direction, $n_i^{\mathbf{r}(k)}$, is found by solving

$$(T_{ij} - T^{(k)} \delta_{ij}) n_j^{\mathbf{r}(k)} = 0 \tag{3.45}$$

subject to the condition $n_i^{\mathbf{r}(k)} n_i^{\mathbf{r}(k)} \equiv 1$. If $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ are distinct, three uniquely determined orthogonal directions, $n_i^{\mathbf{r}(1)}$, $n_i^{\mathbf{r}(2)}$ and $n_i^{\mathbf{r}(3)}$, result. If, for example, $T^{(1)} = T^{(2)}$ and $T^{(1)} \neq T^{(3)}$, only $n_i^{\mathbf{r}(3)}$ is uniquely determined, whereas $n_i^{\mathbf{r}(1)}$ and $n_i^{\mathbf{r}(2)}$ are any two directions in the plane normal to $n_i^{\mathbf{r}(3)}$. Then, there is no shear-stress component in this plane and the state of stress is called *cylindrical*. If $T^{(1)} = T^{(2)} = T^{(3)}$, any three directions identify principal directions. As a consequence, there is no shear-stress component in any plane and the state of stress is called *spherical*. Since this is the only state of stress in a fluid at rest (Sec. 5.3.1), this state is also called *hydrostatic*.

In any case, three mutually orthogonal principal directions, $n_i^{\mathbf{r}(1)}$, $n_i^{\mathbf{r}(2)}$ and $n_i^{\mathbf{r}(3)}$, exist and may serve as the basis of a new Cartesian coordinate system. If referred to it, the stress matrix is diagonal:

$$[T_{ij}] = \begin{bmatrix} T^{(1)} & 0 & 0 \\ 0 & T^{(2)} & 0 \\ 0 & 0 & T^{(3)} \end{bmatrix}. \tag{3.46}$$

Hence, all shear-stress components vanish when the stress tensor is referred to the principal coordinate system.

We note in conclusion that, since the condition for $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ to be real requires T_{ij} to be symmetric and real, our discussion of principal stresses also applies to the Kirchhoff stress, t^{K}_{ij}.

3.6.2 Spherical and deviatoric stresses

We decompose T_{ij} into two tensors as follows:

$$T_{ij} \equiv \tfrac{1}{3} T_{kk} \delta_{ij} + T_{[ij]}. \tag{3.47}$$

The first term on the right-hand side is referred to as the spherical part of T_{ij}:

$$T_{(ij)} := \tfrac{1}{3} T_{kk} \delta_{ij}. \tag{3.48}$$

However, normally, the mechanical pressure defined by

$$P := -\tfrac{1}{3} T_{ii} \tag{3.49}$$

is used. Obviously, it is the negative of the mean of the normal components of T_{ij}. The second term, $T_{[ij]}$, is called the deviatoric part of $T_{(ij)}$. Further below, it will be shown that, if the body is isotropic, a *spherical* state of stress produces *dilatation*, whereas a *deviatoric* state of stress is associated with *distortion* (Secs. 5.2.5, 5.3.3 and 5.4.3).

4 Dynamical principles

4.1 Introduction

With the general concepts of deformation (Chap. 2) and stress (Chap. 3) available, this chapter is concerned with the dynamical principles governing the relationship between these concepts. After presenting the generalized Gauss theorem (Sec. 4.2) and the Reynolds transport theorem (Sec. 4.3), we will state the continuity equation (Sec. 4.4) and the equation of motion (Sec. 4.5). We will then briefly return to the properties of the Cauchy stress tensor and use the angular-momentum principle to establish its symmetry on more general assumptions than considered above (Sec. 4.6). After this, we will use the first law of thermodynamics to formulate the energy equation (Sec. 4.7) and the second law of thermodynamics to formulate the entropy inequality (Sec. 4.8). This will be followed by an investigation of reversible adiabatic and isothermal deformations (Sec. 4.9) and by a study of the dissipation function (Sec. 4.10). Finally, we will deduce from the integral forms of the general principles of continuum mechanics the interface conditions governing the relevant fields near possible discontinuity surfaces of the material parameters (Sec. 4.11). We continue to suppose the absence of distributed volume and surface couples and, in addition, of distributed spin angular momentum, which are sufficient conditions for establishing the symmetry of the Cauchy stress tensor.

4.2 Generalized Gauss theorem

In the following sections, the Eulerian and Lagrangian representations of the continuity equation, the equation of motion and the energy equation will, in the first instance, be given in their *integral* forms. From these, the respective *differential* forms will then be derived. This requires the use of the generalized Gauss theorem, whose Eulerian and Langrangian representations are given here.

4.2.1 Eulerian representation

For a continuously differentiable field, $F_{ij\ldots}$ applying to some property taken per spatial unit area, the generalized Gauss theorem has the form

$$\int_{\partial\mathcal{R}} F_{ij\ldots k}\, d^2 r_k = \int_{\mathcal{R}} F_{ij\ldots k,k}\, d^3 r, \tag{4.1}$$

where $\partial\mathcal{R}$ is the spatial boundary of $\mathcal{R}$ and $d^2 r_i$ points outward.

4.2.2 Lagrangian representation

To obtain the Lagrangian representation of the generalized Gauss theorem, we consider (2.14), (2.48) and (2.49) and rewrite (4.1) as

$$\int_{\partial\mathcal{X}} j\,{}^{\mathbf{X}}r^{-1}_{l,k} f_{ij\dots k}\, d^2X_l = \int_{\mathcal{X}} j\,{}^{\mathbf{X}}r^{-1}_{l,k} f_{ij\dots k,l}\, d^3X, \tag{4.2}$$

with $\partial\mathcal{X}$ the material boundary of $\mathcal{X}$ and d^2X_i pointing outward. Next, we introduce the tensor field of arbitrary rank, $f^{\mathrm{P}}_{ij\dots}$, applying to the same property per material unit area (Piola field):

$$f^{\mathrm{P}}_{ij\dots l} := j\,{}^{\mathbf{X}}r^{-1}_{k,l} f_{ij\dots k}. \tag{4.3}$$

If we take the derivative, we obtain

$$f^{\mathrm{P}}_{ij\dots l,l} = (j\,{}^{\mathbf{X}}r^{-1}_{l,k} f_{ij\dots k})_{,l}. \tag{4.4}$$

The right-hand side of this equation can be simplified by means of (2.23), giving

$$f^{\mathrm{P}}_{ij\dots l,l} = j\,{}^{\mathbf{X}}r^{-1}_{k,l} f_{ij\dots k,l}. \tag{4.5}$$

Comparing (4.2), (4.3) and (4.5) gives

$$\int_{\partial\mathcal{X}} f^{\mathrm{P}}_{ij\dots k}\, d^2X_k = \int_{\mathcal{X}} f^{\mathrm{P}}_{ij\dots k,k}\, d^3X, \tag{4.6}$$

which is the Lagrangian representation of the generalized Gauss theorem. Note that, implicitly, we have also established the following relations:

$$\int_{\partial\mathcal{R}} F_{ij\dots k}\, d^2r_k = \int_{\partial\mathcal{X}} f^{\mathrm{P}}_{ij\dots k}\, d^2X_k, \tag{4.7}$$

$$\int_{\mathcal{R}} F_{ij\dots k,k}\, d^3r = \int_{\mathcal{X}} f^{\mathrm{P}}_{ij\dots k,k}\, d^3X, \tag{4.8}$$

where, in (4.7), the surfaces are not necessarily closed.

4.3 Reynolds transport theorem

The physical properties of a body may refer to its particles or to its subbodies. Examples of the second case are the integrals of distributions of volume and surface forces acting upon subbodies (Sec. 3.2). As for particles, we call the time rate of increase of the integral of any physical property with respect to some subbody the material time derivative of this integral. This is to be distinguished from the time rate of increase of the integral of this property with respect to a given spatial domain, which is the spatial time derivative of this integral. Henceforth, we will use the following abbreviations for the material and spatial time derivatives, respectively:

$$d_t := \frac{d}{dt}, \quad D_t := \frac{D}{Dt}. \tag{4.9}$$

To find the relationship between the derivatives, we consider the subbody currently occupying some spatial volume, $\mathcal{R}$, and denote by $F_{ij\ldots}$ an arbitrary tensor field per spatial unit volume. Considering the integral, $\mathcal{F}_{ij\ldots}$, of $F_{ij\ldots}$ over $\mathcal{R}$ defined by (2.34), its material time derivative has the form

$$\begin{aligned} d_t \mathcal{F}_{ij\ldots} &= d_t \int_{\mathcal{R}} F_{ij\ldots}\, d^3 r \\ &= \int_{\mathcal{R}} (d_t F_{ij\ldots} + F_{ij\ldots}\, d_t)\, d^3 r. \end{aligned} \tag{4.10}$$

In view of (2.81), this equation becomes

$$d_t \mathcal{F}_{ij\ldots} = \int_{\mathcal{R}} (d_t F_{ij\ldots} + F_{ij\ldots} V_{k,k})\, d^3 r. \tag{4.11}$$

Using (2.60), we furthermore obtain

$$d_t \mathcal{F}_{ij\ldots} = \int_{\mathcal{R}} D_t F_{ij\ldots}\, d^3 r + \int_{\mathcal{R}} (F_{ij\ldots} V_k)_{,k}\, d^3 r, \tag{4.12}$$

which, by (4.1), can be recast into

$$d_t \mathcal{F}_{ij\ldots} = \int_{\mathcal{R}} D_t F_{ij\ldots}\, d^3 r + \int_{\partial\mathcal{R}} F_{ij\ldots} V_k\, d^2 r_k. \tag{4.13}$$

This equation states that the time rate of increase of the intergral of a field carried by the subbody currently occupying some spatial volume, $\mathcal{R}$, equals the integral of the time rate of increase of this field inside of $\mathcal{R}$ plus its outward flux through its spatial boundary, $\partial\mathcal{R}$. This relationship is known as Reynolds transport theorem.

4.4 Continuity equation

4.4.1 Eulerian representation

The continuity equation is the form of the mass-conservation principle suitable to continuum mechanics. To derive its Eulerian representation, we consider the subbody currently occupying some spatial volume, $\mathcal{R}$. The mass carried by it equals the mass, $\mathcal{M}$, currently contained in $\mathcal{R}$:

$$\mathcal{M} := \int_{\mathcal{R}} \rho\, d^3 r. \tag{4.14}$$

If no mass is created or destroyed, the mass carried by the subbody does not change:

$$d_t \mathcal{M} = 0. \tag{4.15}$$

Comparing (4.12), (4.14) and (4.15), we thus obtain

$$\int_{\mathcal{R}} [D_t \rho + (\rho V_i)_{,i}]\, d^3 r = 0. \tag{4.16}$$

Since $\mathcal{R}$ is arbitrary, it follows that

$$D_t\rho + (\rho V_i)_{,i} = 0, \tag{4.17}$$

which is the continuity equation. A different form is obtained by rewriting it as

$$D_t\rho + \rho_{,i}V_i + \rho V_{i,i} = 0, \tag{4.18}$$

which, in view of (2.60), is equivalent to

$$d_t\rho + \rho V_{i,i} = 0. \tag{4.19}$$

If the body is incompressible, it is necessary that the volume-mass density in the neighbourhood of individual particles does not change:

$$d_t\rho = 0. \tag{4.20}$$

Then, (4.19) reduces to

$$V_{i,i} = 0, \tag{4.21}$$

which is the incompressibility condition.

Equation (4.17) can be used to derive an alternative form of the Reynolds transport theorem. For this purpose, we assume

$$\mathcal{F}_{ij\ldots} = \int_{\mathcal{R}} \rho G_{ij\ldots}\, d^3r, \tag{4.22}$$

where $G_{ij\ldots}$ is the field $F_{ij\ldots}$ per unit mass. With this equation, (4.12) becomes

$$\begin{aligned} d_t \int_{\mathcal{R}} \rho G_{ij\ldots}\, d^3r &= \int_{\mathcal{R}} D_t(\rho G_{ij\ldots})\, d^3r + \int_{\mathcal{R}} (\rho G_{ij\ldots} V_k)_{,k}\, d^3r \\ &= \int_{\mathcal{R}} \rho(D_t G_{ij\ldots} + V_k G_{ij\ldots,k})\, d^3r + \int_{\mathcal{R}} G_{ij\ldots}[D_t\rho + (\rho V_k)_{,k}]\, d^3r. \end{aligned} \tag{4.23}$$

Considering (2.60) and (4.17), we finally get

$$d_t \int_{\mathcal{R}} \rho G_{ij\ldots}\, d^3r = \int_{\mathcal{R}} \rho\, d_t G_{ij\ldots}\, d^3r. \tag{4.24}$$

4.4.2 Lagrangian representation

For specifying the continuity equation in the Lagrangian representation, we consider the subbody which, at the initial and current time epochs, occupies the spatial volumes $\mathcal{R}(t=0)$ and $\mathcal{R}(t)$, respectively. Conservation of mass requires

$$\int_{\mathcal{R}^{(0)}} \rho^{(0)} (d^3r)^{(0)} = \int_{\mathcal{R}} \rho\, d^3r, \tag{4.25}$$

where the abbreviations $F^{(0)}_{ij\ldots} := F_{ij\ldots}(t=0)$ and $f^{(0)}_{ij\ldots} := f_{ij\ldots}(t=0)$ for the initial parts of $F_{ij\ldots}$ and $f_{ij\ldots}$, respectively, have been used. In view of (2.49), the associated Lagrangian representation takes the form

$$\int_{\mathcal{X}} j^{\mathbf{X}(0)} \rho^{(0)}\, d^3X = \int_{\mathcal{X}} j^{\mathbf{X}} \rho\, d^3X. \tag{4.26}$$

Since $r^{(0)}_i = X_i$, it follows that, $r^{(0)}_{i,j} = \delta_{ij}$. Hence, by (2.4), $(j^{\mathbf{X}})^{(0)} = 1$ and (4.26) reduces to

$$\int_{\mathcal{X}} (\rho^{(0)} - j^{\mathbf{X}} \rho)\, d^3X = 0. \tag{4.27}$$

With $\mathcal{X}$ arbitrary, it follows that

$$j^{\mathbf{X}} \rho = \rho^{(0)}, \tag{4.28}$$

which is equivalent to

$$d_t(j^{\mathbf{X}} \rho) = 0. \tag{4.29}$$

Using (2.49) and $d^3X = (d^3r)^{(0)}$, we obtain $j^{\mathbf{X}} = d^3r/(d^3r)^{(0)}$ and (4.28) can be rewritten as

$$\rho\, d^3r = \rho^{(0)}\, (d^3r)^{(0)}, \tag{4.30}$$

from which we get

$$d_t(\rho\, d^3r) = 0. \tag{4.31}$$

Equations (4.28)–(4.31) constitute alternative expressions of the continuity equation in the Lagrangian representation.

If the body is incompressible, it is necessary that the volume-mass density in the neighbourhood of individual particles does not change:

$$\rho = \rho^{(0)}. \tag{4.32}$$

Then, (4.28) reduces to

$$j^{\mathbf{X}} = 1, \tag{4.33}$$

which is the incompressibility condition. If, in addition, the strain is infinitesimal, the expansion of $j^{\mathbf{X}} := \det r_{i,j}$, with $r_{i,j} = \delta_{ij} + u_{i,j}$ according to (2.6), takes the form

$$j^{\mathbf{X}} = 1 + u_{i,i}. \tag{4.34}$$

Comparing (4.33) and (4.34), we obtain

$$u_{i,i} = 0. \tag{4.35}$$

4.5 Equation of motion

4.5.1 Eulerian representation

We assume internal forces that are equal and opposite (weak axiom of action and reaction) and consider the subbody currently occupying some spatial volume, $\mathcal{R}$, with the spatial boundary $\partial\mathcal{R}$. The time rate of increase of the integral linear momentum, $\mathcal{P}_i$, of the subbody equals the integral force, $\mathcal{F}$, acting upon it:

$$d_t\mathcal{P}_i = \mathcal{F}_i. \tag{4.36}$$

With (3.5), (3.17) and

$$d_t\mathcal{P}_i := d_t\int_{\mathcal{R}} \rho V_i \, d^3r, \tag{4.37}$$

we obtain

$$\begin{aligned} d_t\int_{\mathcal{R}} \rho V_i \, d^3r &= \int_{\mathcal{R}} \rho G_i \, d^3r + \int_{\partial\mathcal{R}} T_i \, d^2r \\ &= \int_{\mathcal{R}} \rho G_i \, d^3r + \int_{\partial\mathcal{R}} T_{ij} \, d^2r_j. \end{aligned} \tag{4.38}$$

Considering (4.1) and (4.24), this equation becomes

$$\int_{\mathcal{R}} (T_{ij,j} + \rho G_i - \rho \, d_t V_i) \, d^3r = 0. \tag{4.39}$$

Since $\mathcal{R}$ is arbitrary, it follows that

$$T_{ij,j} + \rho G_i = \rho \, d_t V_i, \tag{4.40}$$

which is also referred to as the Cauchy equation of motion.

In the case of static equilibrium, $d_t V_i = 0$ and (4.40) reduces to

$$T_{ij,j} + \rho G_i = 0, \tag{4.41}$$

which is the equilibrium equation. Obviously, the three components of (4.41) are insufficient to determine the six independent components of T_{ij} even if the three components of G_i are known. Hence, the problem of determining a static distribution of stress cannot be solved without consideration of the corresponding distribution of strain. The equations to be included take the form of stress-strain relations, which are also known as constitutive equations (Chap. 5).

4.5.2 Lagrangian representation

We consider (2.49), (4.8) and (4.28) in (4.39) and obtain

$$\int_{\mathcal{X}} (t^{\mathrm{P}}_{ij,j} + \rho^{(0)} g_i - \rho^{(0)} \, d_t v_i) \, d^3X = 0 \tag{4.42}$$

or, with $v_i := d_t u_i$ and $\mathcal{X}$ arbitrary,

$$t^{\mathrm{P}}_{ij,j} + \rho^{(0)} g_i = \rho^{(0)}\, d_t v_i. \tag{4.43}$$

This is the Lagrangian representation of the equation of motion in terms of the Piola stress, t^{P}_{ij}, and the initial volume-mass density, $\rho^{(0)}$. Note that it formally agrees with the Eulerian representation of the equation of motion, (4.40), in terms of the Cauchy stress, T_{ij}, and the current volume-mass density, ρ.

4.6 Symmetry of Cauchy stress tensor

So far, the symmetry of the Cauchy stress tensor has been proved only for the restrictive case of equilibrium, absence of volume forces and homogeneous stress distribution (Sec. 3.5). We now remove these constraints and prove the symmetry of the Cauchy stress tensor on more general assumptions.

For this purpose, we assume internal forces that are equal, opposite and collinear (strong axiom of action and reaction) and consider the subbody currently occupying some spatial volume, $\mathcal{R}$, with the spatial boundary $\partial\mathcal{R}$. The time rate of increase of its integral angular momentum equals the integral torque associated with the volume and surface forces acting upon it:

$$d_t \int_{\mathcal{R}} \epsilon_{ijk} r_j \rho V_k \, d^3 r = \int_{\mathcal{R}} \epsilon_{ijk} r_j \rho G_k \, d^3 r + \int_{\partial\mathcal{R}} \epsilon_{ijk} r_j T_k \, d^2 r. \tag{4.44}$$

Using (2.20), (3.17), (4.1), (4.24), $v_i := d_t r_i$ and $V_i = v_i$, this equation becomes

$$\int_{\mathcal{R}} \epsilon_{ijk} (V_j V_k + r_j \, d_t V_k) \rho \, d^3 r = \int_{\mathcal{R}} \epsilon_{ijk} [r_j (\rho G_k + T_{kl,l}) + \delta_{jl} T_{kl}] \, d^3 r, \tag{4.45}$$

which, by (4.40) and $\epsilon_{ijk} V_j V_k \equiv 0$, reduces to

$$\int_{\mathcal{R}} \epsilon_{ijk} T_{kj} \, d^3 r = 0 \tag{4.46}$$

or, since $\mathcal{R}$ is arbitrary, to

$$\epsilon_{ijk} T_{kj} = 0. \tag{4.47}$$

With $T_{ij} \neq 0$ in general and ϵ_{ijk} skew-symmetric with respect to any pair of indices, this equation is satisfied only if

$$T_{ij} = T_{ji}. \tag{4.48}$$

This proves the symmetry of T_{ij} also for the case of disequilibrium, presence of volume forces and inhomogeneous stress distribution. The symmetry properties of the Piola, Finger and Kirchhoff stress tensors have been considered above (Sec. 3.5).

4.7 Energy equation

The energy equation to be derived in this section is a consequence of the equation of motion and the first law of thermodynamics. As an additional field, the energy equation involves the internal energy, which, in classical thermodynamics, is a state function and related to the other thermal state functions by the caloric state equation. We here assume that the changes of state are *thermomechanical*, *i.e.* only due to *work* or *heat*. Since work and heat are not state functions, their time derivatives for given changes of state are not uniquely determined. For clarity, we use δ_t for the material time-derivative operator applied to work or heat.

4.7.1 Eulerian representation

We begin by considering the time rate at which work is performed on the subbody currently occupying some spatial volume, $\mathcal{R}$, with the spatial boundary $\partial\mathcal{R}$. We call this the integral work-input rate and denote it by $\delta_t\mathcal{E}^{\,\mathrm{W}}$. Obviously, $\delta_t\mathcal{E}^{\,\mathrm{W}}$ must equal the time rates at which the integral volume and surface forces perform work on the subbody currently inside of $\mathcal{R}$:

$$\delta_t\mathcal{E}^{\,\mathrm{W}} = \int_{\mathcal{R}} \rho G_i V_i \, d^3r + \int_{\partial\mathcal{R}} T_i V_i \, d^2r. \tag{4.49}$$

Using (3.17) and (4.1), we can alternatively write

$$\delta_t\mathcal{E}^{\,\mathrm{W}} = \int_{\mathcal{R}} [(\rho G_i + T_{ij,j})V_i + T_{ij}V_{i,j}] \, d^3r. \tag{4.50}$$

In view of (4.40), this equation is equivalent to

$$\delta_t\mathcal{E}^{\,\mathrm{W}} = \int_{\mathcal{R}} (\rho V_i \, d_t V_i + T_{ij}V_{i,j}) \, d^3r, \tag{4.51}$$

which, using (4.24) and $V_i \, d_t V_i = \frac{1}{2} \, d_t(V_i V_i)$, can be recast into

$$\delta_t\mathcal{E}^{\,\mathrm{W}} = d_t \int_{\mathcal{R}} \tfrac{1}{2}\rho V_i V_i \, d^3r + \int_{\mathcal{R}} T_{ij}V_{i,j} \, d^3r. \tag{4.52}$$

In view of (2.112)–(2.114) and the symmetry of T_{ij}, we finally obtain

$$\delta_t\mathcal{E}^{\,\mathrm{W}} = d_t \int_{\mathcal{R}} \tfrac{1}{2}\rho V_i V_i \, d^3r + \int_{\mathcal{R}} T_{ij}D_{ij} \, d^3r, \tag{4.53}$$

where D_{ij} the symmetric part of $V_{i,j}$ defined by (2.113). The first term on the right-hand side is recognized as the material time derivative of the integral kinetic energy. The second term is the integral stress power contributing to the integral work-input rate.

Since we are concerned with thermomechanical changes of state, the integral heat-input rate, $\delta_t\mathcal{E}^{\,\mathrm{Q}}$, must be added to the integral work-input rate, $\delta_t\mathcal{E}^{\,\mathrm{W}}$. With C the heat-production rate per unit mass and Q_i the heat flux per spatial unit area (heat-flux density), we thus get

$$\delta_t\mathcal{E}^{\,\mathrm{Q}} = \int_{\mathcal{R}} \rho C \, d^3r - \int_{\partial\mathcal{R}} Q_i \, d^2r, \tag{4.54}$$

which, by (4.1), can be recast into

$$\delta_t \mathcal{E}^{\,\mathrm{Q}} = \int_{\mathcal{R}} (\rho C - Q_{i,i})\, d^3 r. \tag{4.55}$$

The first law of thermodynamics states that a state function, $\mathcal{E}$, exists, so that

$$d_t \mathcal{E} = \delta_t \mathcal{E}^{\,\mathrm{W}} + \delta_t \mathcal{E}^{\,\mathrm{Q}}. \tag{4.56}$$

The quantity $\mathcal{E}$ is called the total energy and characterizes the thermomechanical state of the subbody considered. Alternatively, $\mathcal{E}$ can be decomposed into the integral kinetic energy, $\mathcal{E}^{\,\mathrm{K}}$, and the integral internal energy, $\mathcal{E}^{\,\mathrm{U}}$:

$$\mathcal{E} = \mathcal{E}^{\,\mathrm{K}} + \mathcal{E}^{\,\mathrm{U}}. \tag{4.57}$$

Bearing in mind that $\frac{1}{2}\rho V_i V_i$ is the kinetic energy per spatial unit volume and U the internal energy per unit mass and taking the material time derivative, it follows that

$$d_t \mathcal{E} = d_t \int_{\mathcal{R}} \tfrac{1}{2}\rho V_i V_i \, d^3 r + d_t \int_{\mathcal{R}} \rho U \, d^3 r. \tag{4.58}$$

Equating (4.56) and (4.58) and considering (4.53) and (4.55) gives

$$d_t \int_{\mathcal{R}} \rho U \, d^3 r = \int_{\mathcal{R}} T_{ij} D_{ij} \, d^3 r + \int_{\mathcal{R}} (\rho C - Q_{i,i})\, d^3 r. \tag{4.59}$$

By (4.24), this equation reduces to

$$\int_{\mathcal{R}} (\rho \, d_t U - T_{ij} D_{ij} - \rho C + Q_{i,i})\, d^3 r = 0 \tag{4.60}$$

or, since $\mathcal{R}$ is arbitrary, to

$$T_{ij} D_{ij} + \rho C - Q_{i,i} = \rho \, d_t U, \tag{4.61}$$

which is the energy equation for thermomechanical changes of state of the body in the Eulerian representation.

4.7.2 Lagrangian representation

To state the energy equation in the Lagrangian representation, we start from (4.60). We also use (2.19), (2.22), (3.36) and the symmetry of the Kirchhoff stress, giving

$$T_{ij} = (j^{\,\mathbf{X}})^{-1} r_{i,l} r_{j,k} t^{\mathrm{K}}_{kl}. \tag{4.62}$$

Furthermore, we take the material time derivative of (2.105) and consider $v_i := d_t r_i$, giving

$$d_t e^{\mathrm{G}}_{ij} = \tfrac{1}{2}(v_{k,i} r_{k,j} + v_{k,j} r_{k,i}). \tag{4.63}$$

In view of (2.14), we can rewrite (2.113) in the form

$$D_{ij} = \tfrac{1}{2}(v_{i,k} X_{k,j} + v_{j,k} X_{k,i}), \tag{4.64}$$

which, using (2.21), can be rewritten as

$$d_t e^{\mathrm{G}}_{kl} = r_{i,l}\, r_{j,k} D_{ij}. \tag{4.65}$$

Considering also (4.8) and (4.28), the Lagrangian representation of (4.60) is found to be

$$\int_{\mathcal{X}} (\rho^{(0)}\, d_t u - t^{\mathrm{K}}_{ij}\, d_t e^{\mathrm{G}}_{ij} - \rho^{(0)} c + q^{\mathrm{P}}_{i,i})\, d^3 X = 0, \tag{4.66}$$

where q^{P}_i is the heat flux per material unit area (Piola heat-flux density). Since $\mathcal{X}$ is arbitrary, we also have

$$t^{\mathrm{K}}_{ij}\, d_t e^{\mathrm{G}}_{ij} + \rho^{(0)} c - q^{\mathrm{P}}_{i,i} = \rho^{(0)}\, d_t u, \tag{4.67}$$

which is the energy equation for thermomechanical changes of state of the body in the Lagrangian representation. We note that the equation formally agrees with (4.61), *i.e.* its Eulerian representation.

4.8 Entropy inequality

The first law of thermodynamics states the mutual convertibility of heat and work, but imposes no restrictions on the direction and extent of the conversion. This restriction is provided by the second law of thermodynamics. Its quantitative form is based on the existence of two further state functions, *i.e.* the entropy and the thermodynamic temperature.

4.8.1 Eulerian representation

We continue to consider the subbody currently occupying some spatial volume, $\mathcal{R}$, with the spatial boundary $\partial\mathcal{R}$. The integral entropy-input rate, $\mathcal{G}$, into this subbody is given by

$$\mathcal{G} := \int_{\mathcal{R}} \rho B\, d^3 r - \int_{\partial\mathcal{R}} H_i\, d^2 r_i, \tag{4.68}$$

where B is the entropy-production rate per unit mass and H_i the entropy flux per spatial unit area (entropy-flux density). Furthermore, we introduce the integral entropy-increase rate, $\mathcal{S}$, by

$$\mathcal{S} := d_t \int_{\mathcal{R}} \rho S\, d^3 r, \tag{4.69}$$

where S is the entropy per unit mass. The second law of thermodynamics maintains that

$$\mathcal{S} \geq \mathcal{G}. \tag{4.70}$$

Thus,

$$d_t \int_{\mathcal{R}} \rho S\, d^3 r \geq \int_{\mathcal{R}} \rho B\, d^3 r - \int_{\partial\mathcal{R}} H_i\, d^2 r_i, \tag{4.71}$$

which, using (4.1) and (4.24), becomes

$$\int_{\mathcal{R}} (\rho\, d_t S - \rho B + H_{i,i})\, d^3 r \geq 0. \tag{4.72}$$

Since $\mathcal{R}$ is arbitrary, it follows that

$$B - \frac{H_{i,i}}{\rho} \leq d_t S. \tag{4.73}$$

Assuming that the changes of state are thermomechanical, B and H can be written as

$$B = \frac{C}{\Theta}, \quad H_i = \frac{Q_i}{\Theta}, \tag{4.74}$$

where $\Theta > 0$ is the thermodynamic temperature. Using these eqations, (4.73) takes the form

$$\frac{C}{\Theta} - \frac{1}{\rho}\left(\frac{Q_i}{\Theta}\right)_{,i} \leq d_t S \tag{4.75}$$

or, equivalently,

$$\frac{C}{\Theta} - \frac{Q_{i,i}}{\rho\Theta} + \frac{Q_i \Theta_{,i}}{\rho\Theta^2} \leq d_t S. \tag{4.76}$$

The last two relations are forms of the Clausius-Duhem inequality and must be satisfied by any thermomechanical change of state. They express that the rate of increase of entropy is never less than the rate of increase of this quantity due to heat input alone. Changes during which the body remains asymptotically in equilibrium are called *reversible* or *conservative* and the equality applies. However, real changes always imply disequilibrium and involve the conversion of work into heat. Such changes are therefore called *irreversible* or *dissipative* and the inequality is appropriate.

4.8.2 Lagrangian representation

To state the entropy inequality in the Lagrangian representation, we use (2.49), (4.8) and (4.28), which allow us to rewrite (4.72) as

$$\int_{\mathcal{X}} (\rho^{(0)}\, d_t s - \rho^{(0)} b + h^{\mathrm{P}}_{i,i})\, d^3 X \geq 0, \tag{4.77}$$

with h^{P}_i the entropy flux per material unit area (Piola entropy-flux density). Since $\mathcal{X}$ is arbitrary, the inequality also applies to the integrand, giving

$$b - \frac{h^{\mathrm{P}}_{i,i}}{\rho^{(0)}} \leq d_t s. \tag{4.78}$$

Assuming thermomechanical changes, we have

$$b = \frac{c}{\theta}, \quad h^{\mathrm{P}}_i = \frac{q^{\mathrm{P}}_i}{\theta}, \tag{4.79}$$

which makes (4.78) equivalent to the relations

$$\frac{c}{\theta} - \frac{1}{\rho^{(0)}}\left(\frac{q^{\mathrm{P}}_i}{\theta}\right)_{,i} \leq d_t s, \tag{4.80}$$

$$\frac{c}{\theta} - \frac{q^{\mathrm{P}}_{i,i}}{\rho^{(0)}\theta} + \frac{q^{\mathrm{P}}_i \theta_{,i}}{\rho^{(0)}\theta^2} \leq d_t s. \tag{4.81}$$

4.9 Adiabatic and isothermal deformations

Of special interest are thermomechanical changes of state for which the energy equation can be written in terms of purely mechanical quantities. These are *adiabatic* and *isothermal* deformations. We here assume that the changes are also reversible.

4.9.1 Eulerian representation

For an adiabatic deformation, we must have

$$C = 0, \quad Q_i = 0, \tag{4.82}$$

so that (4.61) and (4.76) reduce to

$$T_{ij} D_{ij} \;=\; \rho \, d_t U, \tag{4.83}$$

$$d_t S \;=\; 0. \tag{4.84}$$

The last equation shows that reversible adiabatic changes are also *isentropic*.

If the deformation is isothermal, we replace the internal energy per unit mass, U, by the Helmholtz free energy per unit mass, Ψ:

$$\Psi := U - S\Theta. \tag{4.85}$$

Since, for isothermal changes, it is necessary that

$$\Theta_{,i} = 0, \quad d_t \Theta = 0, \tag{4.86}$$

equations (4.61) and (4.76) take the forms

$$T_{ij} D_{ij} + \rho C - Q_{i,i} - \rho \Theta \, d_t S = \rho \, d_t \Psi, \tag{4.87}$$

$$\frac{C}{\Theta} - \frac{Q_{i,i}}{\rho \Theta} = d_t S. \tag{4.88}$$

Consideration of (4.88) in (4.87) finally yields

$$T_{ij} D_{ij} = \rho \, d_t \Psi. \tag{4.89}$$

4.9.2 Lagrangian representation

The derivations correspond to those for the Eulerian representation. Thus, for an adiabatic deformation, we have

$$c = 0, \quad q_i^{\mathrm{P}} = 0, \tag{4.90}$$

whence (4.67) and (4.80) reduce to

$$t^{\mathrm{K}}_{ij}\, d_t e^{\mathrm{G}}_{ij} \;=\; \rho^{(0)}\, d_t u, \tag{4.91}$$

$$d_t s \;=\; 0. \tag{4.92}$$

For an isothermal deformation, we use

$$\psi := u - s\theta \tag{4.93}$$

and take into account that

$$\theta_{,i} = 0, \quad d_t\theta = 0. \tag{4.94}$$

Then, (4.67) and (4.80) take the forms

$$t^{\mathrm{K}}_{ij}\, d_t e^{\mathrm{G}}_{ij} + \rho^{(0)} c - q^{\mathrm{P}}_{i,i} - \rho^{(0)}\theta\, d_t s = \rho^{(0)}\, d_t\psi, \tag{4.95}$$

$$\frac{c}{\theta} - \frac{q^{\mathrm{P}}_{i,i}}{\rho^{(0)}\theta} = d_t s. \tag{4.96}$$

Considering (4.96) in (4.95) finally gives

$$t^{\mathrm{K}}_{ij}\, d_t e^{\mathrm{G}}_{ij} = \rho^{(0)}\, d_t\psi. \tag{4.97}$$

4.10 Dissipation function

In the case of irreversible processes, it may be assumed that the Cauchy stress can be decomposed into two parts according to

$$T_{ij} = T^{\mathrm{C}}_{ij} + T^{\mathrm{D}}_{ij} \tag{4.98}$$

in the Eulerian representation, where T^{C}_{ij} and T^{D}_{ij} are the *conservative* Cauchy stress and the *dissipative* Cauchy stress, respectively. Then, (4.61) can be rewritten as

$$T^{\mathrm{C}}_{ij} D_{ij} + T^{\mathrm{D}}_{ij} D_{ij} + \rho C - Q_{i,i} = \rho\, d_t U. \tag{4.99}$$

We now introduce the dissipation function by

$$D := T^{\mathrm{D}}_{ij} D_{ij}. \tag{4.100}$$

For reversible processes, we have

$$D = 0 \tag{4.101}$$

and, according to (4.76), also

$$\frac{C}{\Theta} - \frac{Q_{i,i}}{\rho\Theta} + \frac{Q_i \Theta_{,i}}{\rho\Theta^2} = d_t S. \tag{4.102}$$

Combination of (4.99)–(4.102) yields

$$T_{ij}^{\mathrm{C}} D_{ij} + \rho\Theta\, d_t S - \frac{Q_i \Theta_{,i}}{\Theta} = \rho\, d_t U. \tag{4.103}$$

Since U is a state function, (4.103) also applies to irreversible processes. The equation can therefore be used to eliminate T_{ij}^{C} in (4.99), giving

$$\frac{C}{\Theta} + \frac{D - Q_{i,i}}{\rho\Theta} + \frac{Q_i \Theta_{,i}}{\rho\Theta^2} = d_t S. \tag{4.104}$$

For an adiabatic change, we must have

$$C = 0, \quad Q_i = 0 \tag{4.105}$$

and, according to (4.76), also

$$d_t S \geq 0. \tag{4.106}$$

In view of $\rho > 0$ and $\Theta > 0$, these conditions reduce (4.104) to

$$D \geq 0, \tag{4.107}$$

which shows that the dissipation function is non-negative. Further below, this property will be used to impose constraints on the constitutive equation of Newtonian–viscous fluids (Sec. 5.3.6).

4.11 Interface conditions

When deriving the differential forms of the general principles of continuum mechanics from the respective integral forms in the preceding sections, the use of the generalized Gauss theorem has been essential. The main assumption for its validity is that the fields involved are continuously differentiable.

However, of interest are also cases in which this assumption is violated at particular interfaces within the body. Common examples are jump discontinuities of the material properties or infinitesimally thin sheets of mass. In these situations, the generalized Gauss theorem may still be applied in the domains adjacent to the interface, but the fields must connected via interface conditions across the discontinuity.

In the following, the stress interface condition will be derived in the Eulerian and Lagrangian representations.

4.11.1 Eulerian representation

To formulate the interface condition concisely, we consider internal and external spatial volumes, $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$, respectively, currently separated by a smooth spatial interface, $\partial\mathcal{R}^{(1)}$, and locally

assign to it the outward spatial unit vector, $n_i^{\mathbf{r}+}$, normal to it and directed into $\mathcal{R}^{(2)}$ (Fig. 4.1a). Wherever necessary, we distinguish by $F^-_{ij\ldots}$ and $F^+_{ij\ldots}$ an arbitrary tensor field in $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$, respectively. Furthemore, we define the internal limit, the external limit and the jump of $F_{ij\ldots}$, respectively, by

$$[F_{ij\ldots}]^- := \lim_{\epsilon\to 0+} F^-_{ij\ldots}(\mathbf{r} - \epsilon\mathbf{n}^{\mathbf{r}+}), \tag{4.108}$$

$$[F_{ij\ldots}]^+ := \lim_{\epsilon\to 0+} F^+_{ij\ldots}(\mathbf{r} + \epsilon\mathbf{n}^{\mathbf{r}+}), \tag{4.109}$$

$$[F_{ij\ldots}]^+_- := [F_{ij\ldots}]^+ - [F_{ij\ldots}]^-, \tag{4.110}$$

where $\mathbf{r} \in \partial\mathcal{R}^{(1)}$ and $\epsilon > 0$ have been assumed. The field may be continuated onto the interface using

$$F^\pm_{ij\ldots} := \tfrac{1}{2}\{[F_{ij\ldots}]^- + [F_{ij\ldots}]^+\}. \tag{4.111}$$

To ensure coupling of the body on the interface, we require that the subbody is welded across it, *i.e.* neither slip nor cavitation is allowed. In the Eulerian representation, this can formally be expressed by the velocity interface condition:

$$[V_i]^+_- = 0. \tag{4.112}$$

If the body is currently confined to $\mathcal{R}^{(1)}$, then $\partial\mathcal{R}^{(1)}$ is its spatial boundary and we may assume that $[V_i]^-$ is prescribed. Then, (4.112) reduces to the velocity boundary condition:

$$[V_i]^- = C_i. \tag{4.113}$$

To derive the stress interface condition, we recall (4.38):

$$d_t \int_{\mathcal{R}^{(1)}\cup\mathcal{R}^{(2)}} \rho V_i \, d^3r = \int_{\mathcal{R}^{(1)}\cup\mathcal{R}^{(2)}} \rho G_i \, d^3r + \int_{\partial\mathcal{R}^{(1)}} T_{ij} \, d^2r_j. \tag{4.114}$$

Considering (4.24) as well as a thin disk with the current volume $\mathcal{V}$, the current area $\mathcal{A}$ and the current thickness h straddling $\partial\mathcal{R}^{(1)}$ (Fig. 4.1a), this equation can be rewritten as

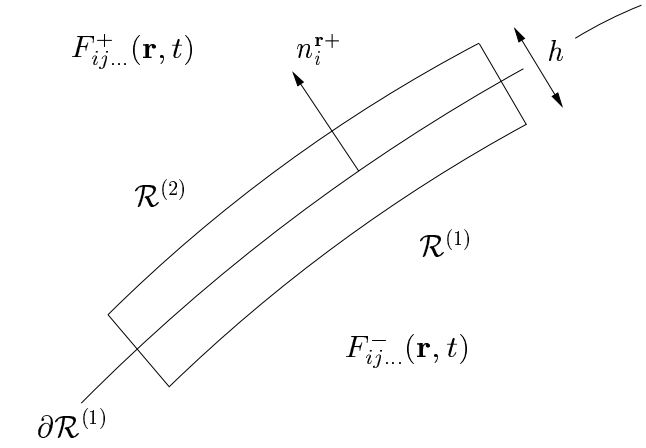
$$\int_{\mathcal{V}} \rho \, d_t V_i \, d^2r \, dh' = \int_{\mathcal{V}} \rho G_i \, d^2r \, dh' + \int_{\mathcal{A}} (T^+_{ij} - T^-_{ij}) n_j^{\mathbf{r}+} \, d^2r + O(h), \tag{4.115}$$

where h' is the coordinate normal to $\partial\mathcal{R}^{(1)}$ and $O(h)$ is the contribution to the surface integral from the mantle of the disk. We now choose a coordinate origin of h' on $\partial\mathcal{R}^{(1)}$ and allow an infinitesimally thin sheet of mass on it. Then, ρ is given by

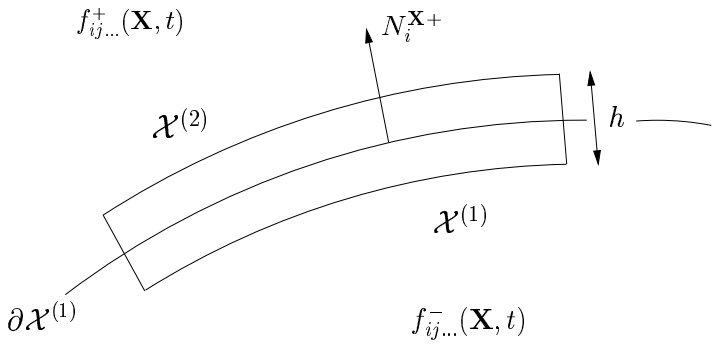
$$\rho = \rho' + \Sigma \, \delta(h'), \tag{4.116}$$

where ρ' is the possibly discontinuous volume-mass density, Σ the mass per spatial unit area (interface-mass density) and δ the Dirac delta function. With this equation, (4.115) reduces to

$$\int_{\mathcal{V}} \rho' \, d_t V_i \, d^2r \, dh' + \int_{\mathcal{A}} \Sigma \, d_t V_i \, d^2r = \int_{\mathcal{V}} \rho' G_i \, d^2r \, dh' + \int_{\mathcal{A}} \Sigma \, G_i \, d^2r + \int_{\mathcal{A}} (T^+_{ij} - T^-_{ij}) n_j^{\mathbf{r}+} \, d^2r + O(h). \tag{4.117}$$



a



b

Figure 4.1: Thin disk of thickness h straddling an interface of the body (a) in the current state and (b) in the initial state. The symbols are explained in the text.

If $h \to 0$ such that the disk continues to straddle $\partial\mathcal{R}^{(1)}$, we obtain

$$\int_{\mathcal{A}} \{\Sigma(d_t V_i - G_i^{\pm}) - [T_{ij}]^+_- \, n_j^{\mathbf{r}+}\} \, d^2 r = 0, \tag{4.118}$$

where (4.108)–(4.112) have been used. Since $\mathcal{A}$ is arbitrary, the integrand must vanish:

$$[T_{ij}]^+_- \, n_j^{\mathbf{r}+} = \Sigma(d_t V_i - G_i^{\pm}), \tag{4.119}$$

which is the stress interface condition. If the body is currently confined to $\mathcal{R}^{(1)}$, $\partial\mathcal{R}^{(1)}$ is its spatial boundary and $[T_{ij}]^+ \, n_j^{\mathbf{r}+} = 0$, so that (4.119) reduces to the stress boundary condition:

$$[T_{ij}]^- \, n_j^{\mathbf{r}+} = -\Sigma(d_t V_i - G_i^{\pm}). \tag{4.120}$$

4.11.2 Lagrangian representation

To formulate interface conditions in the Lagrangian representation, we consider internal and external material volumes, $\mathcal{X}^{(1)}$ and $\mathcal{X}^{(2)}$, respectively, separated by a smooth material interface,

$\partial\mathcal{X}^{(1)}$, and locally assign to it the outward material unit vector, $N_i^{\mathbf{X}+}$, normal to it and directed into $\mathcal{X}^{(2)}$ (Fig. 4.1b). Wherever necessary, we distinguish by $f^-_{ij\ldots}$ and $f^+_{ij\ldots}$ an arbitrary tensor field in $\mathcal{X}^{(1)}$ and $\mathcal{X}^{(2)}$, respectively. Assuming $\mathbf{X} \in \partial\mathcal{X}^{(1)}$ and $\epsilon > 0$, we define

$$[f_{ij\ldots}]^- := \lim_{\epsilon\to 0+} f^-_{ij\ldots}(\mathbf{X} - \epsilon \mathbf{N}^{\mathbf{X}+}), \tag{4.121}$$

$$[f_{ij\ldots}]^+ := \lim_{\epsilon\to 0+} f^+_{ij\ldots}(\mathbf{X} + \epsilon \mathbf{N}^{\mathbf{X}+}), \tag{4.122}$$

$$[f_{ij\ldots}]^+_- := [f_{ij\ldots}]^+ - [f_{ij\ldots}]^-. \tag{4.123}$$

Similar to the Eulerian representation, we may continue the field onto the material interface using

$$f^\pm_{ij\ldots} := \tfrac{1}{2}\{[f_{ij\ldots}]^- + [f_{ij\ldots}]^+\}. \tag{4.124}$$

The assumption that the subbody is welded across the material interface is expressed by the position interface condition:

$$[r_i]^+_- = 0. \tag{4.125}$$

If $\partial\mathcal{X}^{(1)}$ is a material boundary, we may assume that $[r_i]^-$ is prescribed. Then, (4.125) reduces to the position boundary condition

$$[r_i]^- = d_i. \tag{4.126}$$

To obtain the stress interface condition, we reconsider (4.118). In view of (2.48), $v_i := d_t r_i$, $d^2 r_i^+ := n_i^{\mathbf{r}+}\, d^2 r$ and $\Sigma_i := \Sigma n_i^{\mathbf{r}+}$, it can be rewritten as

$$\int_{\mathcal{A}^{(0)}} j^{\mathbf{X}} r^{-1}_{k,j}\{\sigma_j(d_t^2 r_i - g_i^\pm) - [t_{ij}]^+_-\} d^2 X_k^+ = 0, \tag{4.127}$$

where $\mathcal{A}^{(0)}$ is the initial area of the thin disk (Fig. 4.1b) and (4.121)–(4.124) have been implied. Using (4.3), equation (4.127) simplifies to

$$\int_{\mathcal{A}^{(0)}} \{\sigma_j^{\mathrm{P}}(d_t^2 r_i - g_i^\pm) - [t^{\mathrm{P}}_{ij}]^+_-\}\, d^2 X_j^+ = 0 \tag{4.128}$$

or, with $\sigma^{\mathrm{P}} := \sigma_i^{\mathrm{P}} N_i^{\mathbf{X}+}$ and $d^2 X := N_i^{\mathbf{X}+}\, d^2 X_i^+$, to

$$\int_{\mathcal{A}^{(0)}} \{\sigma^{\mathrm{P}}(d_t^2 r_i - g_i^\pm) - [t^{\mathrm{P}}_{ij}]^+_- N_j^{\mathbf{X}+}\}\, d^2 X = 0, \tag{4.129}$$

where σ^{P} is the mass per material unit area (Piola interface-mass density).

Since $\mathcal{A}^{(0)}$ is arbitrary, we finally obtain

$$[t^{\mathrm{P}}_{ij}]^+_- N_j^{\mathbf{X}+} = \sigma^{\mathrm{P}}(d_t^2 r_i - g_i^\pm) \tag{4.130}$$

as the stress interface condition. With $\partial\mathcal{X}^{(1)}$ a material boundary, we have $[t^{\mathrm{P}}_{ij}]^+ N_j^{\mathbf{X}+} = 0$. Then, (4.130) reduces to the stress boundary condition:

$$[t^{\mathrm{P}}_{ij}]^- N_j^{\mathbf{X}+} = -\sigma^{\mathrm{P}}(d_t^2 r_i - g_i^\pm). \tag{4.131}$$

5 Constitutive equations

5.1 Introduction

In the preceding chapters, we have been concerned with the development of the kinematic concepts of deformation and flow (Chap. 2), the mechanical concept of stress (Chap. 3) and the continuum forms of the principles of dynamics and thermodynamics (Chap. 4). A common feature of these concepts and principles is that they apply to *arbitrary* continuous bodies.

In the present chapter, we are concerned with the response of *particular* continuous bodies to applied forces. Since this response is a consequence of the internal constitution of the real material considered, the governing relations are called constitutive equations. We will begin with the constitutive equations for two simple cases: *elastic solids* (Sec. 5.2) and *viscous fluids* (Sec. 5.3). Since real materials respond in highly complex ways when the full ranges of applied force and temperature are taken into account, the elastic and viscous constitutive equations can only be approximately satisfied by particular materials for limited ranges of force and temperature. Some of these restrictions will be removed when studying *viscoelastic bodies* (Sec. 5.4).

Since the unstressed state serves as a natural initial state for elastic solids, we employ for them the Lagrangian representation. On the other hand, particles usually cannot be traced in viscous fluids, which suggests for their description the Eulerian representation. When studying viscoelasticity, we restrict our analysis to perturbations of an initially unstressed state, whence the Lagrangian representation is used.

5.2 Elastic solids

A continuous body is a Hookean elastic solid if each of the components of the stress acting upon any of its particles at the current time epoch is a homogeneous linear function of the components of the strain experienced by that particle simultaneously. Direct consequences of this definition are that an elastic solid does not show creep at constant stress or stress relaxation at constant strain. The constitutive equation expressing this behaviour, here called the *generalized* Hooke law, can be written as

$$t_{ij}^{\mathrm{K}} = m_{ijkl} e_{kl}^{\mathrm{G}}, \tag{5.1}$$

where m_{ijkl} is the elasticity tensor. This tensor equation represents nine scalar equations involving the $9 \times 9 = 81$ components of m_{ijkl} as coefficients. The number of independent components can be successively reduced if particular assumptions on the stress, the strain and the energy equation are exploited.

5.2.1 Symmetry of stress

With t_{ij}^{K} symmetric (Sec. 3.7), we have $t_{ij}^{\mathrm{K}} = t_{ji}^{\mathrm{K}}$ and, in view of (5.1), also

$$(m_{ijkl} - m_{jikl}) e_{kl}^{\mathrm{G}} = 0. \tag{5.2}$$

Since e_{ij}^{G} is arbitrary, it is necessary that

$$m_{ijkl} = m_{jikl}. \tag{5.3}$$

Hence, m_{ijkl} is symmetric with respect to i and j and the number of independent components of m_{ijkl} is reduced to $6 \times 9 = 54$.

5.2.2 Symmetry of strain

To reduce the number of independent components further, we decompose m_{ijkl} into its symmetric and skew-symmetric parts with respect to k and l:

$$m_{ijkl} \equiv \tfrac{1}{2}(m_{ijkl} + m_{ijlk}) - \tfrac{1}{2}(m_{ijkl} - m_{ijlk}). \tag{5.4}$$

Since e_{ij}^{G} is symmetric *per definitionem* (Sec. 2.5.2), $e_{ij}^{\mathrm{G}} = e_{ji}^{\mathrm{G}}$ and it follows that

$$(m_{ijkl} - m_{ijlk}) e_{kl}^{\mathrm{G}} = 0. \tag{5.5}$$

Combining (5.1), (5.4) and (5.5), we notice that the skew-symmetric part of m_{ijkl} with respect to k and l does not contribute to t_{ij}^{K}. Without loss of generality, we may therefore replace m_{ijkl} by its symmetric part with respect to k and l. Continuing to denote this part by m_{ijkl}, we thus have

$$m_{ijkl} = m_{ijlk}, \tag{5.6}$$

which reduces the number of independent components of m_{ijkl} to $6 \times 6 = 36$.

5.2.3 Perfect elasticity

We now assume that the elastic solid undergoes a reversible deformation that is either adiabatic or isothermal. According to (4.91), the energy equation is given in the first case by

$$t_{ij}^{\mathrm{K}} \, d_t e_{ij}^{\mathrm{G}} = \rho^{(0)} \, d_t u, \tag{5.7}$$

whereas, according to (4.97), it is given in the second case by

$$t_{ij}^{\mathrm{K}}\, d_t e_{ij}^{\mathrm{G}} = \rho^{(0)}\, d_t \psi. \tag{5.8}$$

If we define

$$w := \begin{cases} \rho^{(0)} u, & \text{adiabatic deformation} \\ \rho^{(0)} \psi, & \text{isothermal deformation} \end{cases}, \tag{5.9}$$

equations (5.7) and (5.8) can be merged into a single equation:

$$t_{ij}^{\mathrm{K}}\, d_t e_{ij}^{\mathrm{G}} = d_t w. \tag{5.10}$$

We may interpret w as the strain energy per material unit volume. Since the deformation has been assumed to be reversible, w is conservative.

To employ (5.10) for the further reduction of the number of independent components of m_{ijkl}, we suppose that $u_{i,j}$ is infinitesimal, that all components of t_{ij}^{K} are of the same order of magnitude and that only the lowest-order terms are retained. It then follows from the definitions of e_{ij}, e_{ij}^{G}, t_{ij} and t_{ij}^{K} that

$$e_{ij}^{\mathrm{G}} = e_{ij}, \tag{5.11}$$

$$t_{ij}^{\mathrm{K}} = t_{ij}, \tag{5.12}$$

which are correct to the first and zeroth order in $u_{i,j}$, respectively. In view of these equations, (5.1) and (5.10), respectively, reduce to

$$t_{ij} = m_{ijkl} e_{kl}, \tag{5.13}$$

$$dw = t_{ij}\, de_{ij}. \tag{5.14}$$

Since e_{ij} is infinitesimal, the average stress acting along the path of some particle from its initial position to its current position is half the current value of t_{ij}. Assuming that w vanishes in the initial state, we thus obtain from (5.14) by integration

$$w = \tfrac{1}{2} t_{ij} e_{ij} \tag{5.15}$$

or, upon substitution of (5.1),

$$w = \tfrac{1}{2} m_{ijkl} e_{ij} e_{kl}, \tag{5.16}$$

which is correct to the second order in e_{ij}. Solids for which w is given by this equation are called *perfectly* elastic solids.

To reduce the number of independent elastic moduli further, we now decompose m_{ijkl} into its symmetric and skew-symmetric parts with respect to ij and kl:

$$m_{ijkl} \equiv \tfrac{1}{2}(m_{ijkl} + m_{klij}) + \tfrac{1}{2}(m_{ijkl} - m_{klij}). \tag{5.17}$$

Since e_{ij} is symmetric *per definitionem* (Sec. 2.4), $e_{ij} = e_{ji}$ and it follows that

$$(m_{ijkl} - m_{klij})e_{ij}e_{kl} = 0. \tag{5.18}$$

Combining (5.16)–(5.18), we notice that the skew-symmetric part of m_{ijkl} with respect to ij and kl does not contribute to w. Then, without loss of generality, m_{ijkl} can be replaced by its symmetric part with respect to ij and kl. Denoting this part also by m_{ijkl}, we thus have

$$m_{ijkl} = m_{klij}, \tag{5.19}$$

which represents 15 independent conditions and reduces the number of independent components of m_{ijkl} to 21.

5.2.4 Isotropy

Since the 21 independent quantities characterizing a perfectly elastic solid constitute the components of the elasticity tensor, m_{ijkl}, their values change under transformations of the coordinate system according to the transformation formula for fourth-rank tensors. This directionality of elastic solids is called *anisotropy*. The existence of particular transformations of the coordinate system which do not change the values of these components is called *aelotropy*.

A simple example of aelotropy is the invariance of the components of m_{ijkl} at a particular point under a reflection of the coordinate system with respect to some plane through that point. Such a plane is called a plane of elastic symmetry. Note that, in general, a plane of elastic symmetry is not parallel to a plane of geometric symmetry of the elastic solid or a plane of symmetry of the strain or stress field. On the other hand, a plane of crystal symmetry in a single crystal is parallel to a plane of elastic symmetry. It can be shown that the existence of a plane of elastic symmetry reduces the number of independent components of m_{ijkl} from 21 to 13 at that point. If three orthogonal planes of elastic symmetry exist, the solid is called *orthotropic* and the number of independent components of m_{ijkl} reduces to nine.

The highest symmetry is reached if m_{ijkl} is invariant with respect to any orthogonal transformation of the coordinate system. A tensor showing this type of invariance is referred to as an *isotropic* tensor. Similarly, an elastic solid for which m_{ijkl} is isotropic at any point is called an isotropic elastic solid. Real materials that approximate elastic isotropy are polycrystalline materials with randomly oriented crystal grains. Here, the effective volume (Sec. 1.2) must contain a sufficiently large number of crystal grains in order that isotropy holds statistically.

The trivial case of an isotropic tensor is the zero tensor, $0_{ij\ldots}$, of any rank. Clearly, all zeroth-rank tensors (scalars) are isotropic, but there is only the trivial isotropic first-rank tensor (vector). We note without proof that all isotropic second-rank tensors are scalar multiples of δ_{ij}, that all isotropic third-rank tensors are scalar multiples of ϵ_{ijk} and that the isotropic fourth-rank

tensor is of the form

$$f_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{5.20}$$

with arbitrary scalars λ, μ and ν.

If f_{ijkl} equals m_{ijkl}, it is symmetric with respect to i and j, and with respect to k and l. For either type of symmetry, (5.20) reduces to

$$m_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{5.21}$$

where, now, λ and μ are the first and second Lamé parameters, respectively. Substituting (5.21) into (5.1), we obtain the *isotropic* Hooke law:

$$t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}. \tag{5.22}$$

5.2.5 Spherical and deviatoric parts

A different form of the isotropic Hooke law employs the spherical and deviatoric parts of e_{ij} and t_{ij}. According to (2.132) and (3.47), we have

$$e_{ij} \equiv \tfrac{1}{3}e_{kk}\delta_{ij} + e_{[ij]}, \tag{5.23}$$

$$t_{ij} \equiv \tfrac{1}{3}t_{kk}\delta_{ij} + t_{[ij]}. \tag{5.24}$$

To obtain the *spherical* Hooke law, we contract (5.22):

$$t_{ii} = 3ke_{ii}, \tag{5.25}$$

with the elastic bulk modulus given by

$$k := \lambda + \tfrac{2}{3}\mu. \tag{5.26}$$

To relate the deviatoric parts, we combine (5.22)–(5.24):

$$\tfrac{1}{3}t_{kk}\delta_{ij} + t_{[ij]} = \lambda e_{kk}\delta_{ij} + 2\mu(\tfrac{1}{3}e_{kk}\delta_{ij} + e_{[ij]}). \tag{5.27}$$

Using (5.25) and (5.26), this equation reduces to the *deviatoric* Hooke law:

$$t_{[ij]} = 2\mu e_{[ij]}, \tag{5.28}$$

whence μ is also called the elastic shear modulus.

5.2.6 Incompressibility

If the elastic solid is incompressible, the Hooke law further simplifies. To see this, we consider (4.35), *i.e.* the Lagrangian representation of the incompressibility condition for infinitesimal strain:

$$u_{i,i} = 0. \tag{5.29}$$

In general, t_{ii} and μ remain finite even if the solid is incompressible. In view of (5.25), (5.26) and $e_{ii} = u_{i,i}$, we must therefore also require

$$\lambda \to \infty, \quad k \to \infty. \tag{5.30}$$

Note that, on the assumption of incompressibility, (5.25) is no longer valid, *i.e.* the spherical stress is no longer related to the strain components. On the other hand, the definition of the mechanical pressure continues to apply. Hence, using $e_{ii} = 0$ and $t_{ii} = -3p$, equations (5.23) and (5.24), respectively, reduce to

$$e_{ij} = e_{[ij]}, \tag{5.31}$$

$$t_{ij} = -p\delta_{ij} + t_{[ij]}. \tag{5.32}$$

Combining (5.28), (5.31) and (5.32), we obtain

$$t_{ij} = -p\delta_{ij} + 2\mu e_{ij}, \tag{5.33}$$

which is the *incompressible* Hooke law.

5.2.7 Strain energy

Finally, we derive the strain energy per material unit volume for an isotropic elastic solid. For this purpose, we substitute (5.22) into (5.15), giving

$$w = \tfrac{1}{2}\lambda e_{ii}e_{jj} + \mu e_{ij}e_{ij}. \tag{5.34}$$

A consequence of (5.23) is

$$e_{ij}e_{ij} = \tfrac{1}{3}e_{ii}e_{jj} + e_{[ij]}e_{[ij]}. \tag{5.35}$$

Using (5.26) and this equation, (5.34) can alternatively be written as

$$w = \tfrac{1}{2}k e_{ii}e_{jj} + \mu e_{[ij]}e_{[ij]}. \tag{5.36}$$

Since w has been assumed to vanish in the undeformed state, it must be positive in the deformed state. With the spherical and deviatoric components of e_{ij} independently variable, necessary and sufficient for $w \geq 0$ are

$$k \geq 0, \quad \mu \geq 0 \tag{5.37}$$

and, in view of (5.26), $\lambda \geq -\frac{2}{3}\mu$. However, in real materials, negative values of λ are not realized, so that

$$\lambda \geq 0. \tag{5.38}$$

In the case of incompressibility, we have

$$e_{ii} = 0, \quad \lambda \to \infty, \quad k \to \infty, \tag{5.39}$$

which reduce (5.34) and (5.36), respectively, to

$$w = \mu e_{ij} e_{ij}, \tag{5.40}$$

$$w = \mu e_{[ij]} e_{[ij]}. \tag{5.41}$$

5.3 Viscous fluids

5.3.1 Thermodynamic pressure

Fluids at rest or in uniform flow cannot support deviatoric stress. Hence, the stress is purely spherical:

$$T_{ij}^{(0)} = -P^{(0)} \delta_{ij}, \tag{5.42}$$

with the superscript 0 indicating the state of rest or uniform flow. In thermodynamics, $P^{(0)}$ is related to $\rho^{(0)}$ and possibly other state functions by a kinetic state equation:

$$P^{(0)} = P^{(0)}(\rho^{(0)}, \ldots). \tag{5.43}$$

In fluid dynamics, we are usually concerned with non-uniformly flowing fluids. We then introduce the thermodynamic pressure, Π, by the same function of ρ and possibly other state functions that holds for $P^{(0)}$ at rest or uniform flow:

$$\Pi = \Pi(\rho, \ldots). \tag{5.44}$$

Since $P^{(0)} = \Pi^{(0)}$, it follows that $\Pi^{(0)} = -T_{ii}^{(0)}/3$. However, in non-uniformly flowing fluids, Π does not equal the negative of the mean normal stress in general. A special form of (5.44) is the kinetic state equation for a *barotropic* fluid:

$$\Pi = \Pi(\rho). \tag{5.45}$$

Examples of barotropy are fluids subject to adiabatic or isothermal changes of state. An *incompressible* fluid is governed by a barotropic state equation of the form

$$\rho = \text{constant}. \tag{5.46}$$

Note that, in an incompressible fluid, Π is no longer a function of ρ.

5.3.2 Newtonian viscosity

The characteristic feature of viscous fluids is that they can support deviatoric stress. We here assume that the difference between the stress in non-uniform flow and the stress at rest or in uniform flow is a homogeneous linear function of the strain rate. Hence,

$$T_{ij} = -\Pi\delta_{ij} + N_{ijkl}D_{kl}, \tag{5.47}$$

which is the *generalized* Navier–Poisson law, with N_{ijkl} the viscosity tensor. A viscous fluid whose constitutive equation is given by (5.47) is referred to as a *Newtonian–viscous* fluid. We note that, in view of the symmetries of D_{ij} and T_{ij}, the number of independent components of N_{ijkl} is $6 \times 6 = 36$. In the following, we will only consider the case that N_{ijkl} is isotropic. Since D_{ij} and T_{ij} are symmetric, N_{ijkl} has the form of (5.21):

$$N_{ijkl} = \chi\delta_{ij}\delta_{kl} + \eta\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{5.48}$$

where χ and η are the first and second viscosity parameters, respectively. Substituting this equation into (5.47) yields

$$T_{ij} = -\Pi\delta_{ij} + \chi D_{kk}\delta_{ij} + 2\eta D_{ij}, \tag{5.49}$$

which is the *isotropic* Navier–Poisson law.

5.3.3 Spherical and deviatoric parts

As for the isotropic Hooke law (Sec. 5.2), equation (5.49) can be decomposed into spherical and deviatoric parts. According to (2.132) and (3.47), we have

$$D_{ij} \equiv \tfrac{1}{3}D_{kk}\delta_{ij} + D_{[ij]}, \tag{5.50}$$

$$T_{ij} \equiv \tfrac{1}{3}T_{kk}\delta_{ij} + T_{[ij]}. \tag{5.51}$$

To obtain the *spherical* Navier–Poisson law, we contract (5.49):

$$T_{ii} = -3\Pi + 3\kappa D_{ii}, \tag{5.52}$$

with the bulk viscosity given by

$$\kappa := \chi + \tfrac{2}{3}\eta. \tag{5.53}$$

To relate the deviatoric parts, we combine (5.49)–(5.51):

$$\tfrac{1}{3}T_{kk}\delta_{ij} + T_{[ij]} = -\Pi\delta_{ij} + \chi D_{kk}\delta_{ij} + 2\eta\,(\tfrac{1}{3}D_{kk}\delta_{ij} + D_{[ij]}). \tag{5.54}$$

In view of (5.52) and (5.53), this equation reduces to the *deviatoric* Navier–Poisson law:

$$T_{[ij]} = 2\eta D_{[ij]}, \tag{5.55}$$

whence η is also called the shear viscosity.

5.3.4 Stokes condition and incompressibility

Since $T_{ii} = -3P$, equation (5.52) can be rewritten as

$$P = \Pi - \kappa D_{ii}. \tag{5.56}$$

This equation shows that $P = \Pi$ holds only if one of the following conditions is satisfied:

$$\kappa = 0, \tag{5.57}$$

$$D_{ii} = 0, \tag{5.58}$$

which are the Stokes condition and, in view of (4.21) and $D_{ii} = V_{i,i}$, the incompressibility condition, respectively. If the Stokes condition applies, it follows from (5.53) that $\chi = -\frac{2}{3}\eta$, whence (5.49) reduces to

$$T_{ij} = -\Pi\delta_{ij} - \tfrac{2}{3}\eta\, D_{kk}\delta_{ij} + 2\eta D_{ij}. \tag{5.59}$$

Inspection of this equation shows that, although $D_{ii} \neq 0$ in general, no contribution to the spherical stress arises from the corresponding term, *i.e.* $T_{ij} + \Pi\delta_{ij}$ is purely deviatoric. If the incompressibility condition applies, (5.49) becomes

$$T_{ij} = -\Pi\delta_{ij} + 2\eta D_{ij}, \tag{5.60}$$

with $T_{ij} + \Pi\delta_{ij}$ purely deviatoric. Note that, with $\Pi = -T_{ii}/3$ in (5.59) and (5.60), Π may be replaced by P. Furthemore, in both equations, only the shear viscosity, η, is left as the characteristic parameter.

5.3.5 Inviscidity

A fluid is *inviscid* if it cannot support deviatoric stress even in non-uniform flow. Its constitutive equation therefore is

$$T_{ij} = -\delta_{ij}\Pi. \tag{5.61}$$

As is obvious from this equation, $\Pi = -T_{ii}/3$ always holds in an inviscid fluid. If, in addition, Π satisfies (5.45), the fluid is called *elastic*.

5.3.6 Stress power

Further insight is gained by considering the stress power per spatial unit volume:

$$Z := T_{ij}D_{ij}. \tag{5.62}$$

In view of (5.49), (5.50) and (5.53), we obtain

$$Z = -\Pi D_{ii} + \kappa D_{ii}D_{jj} + 2\eta D_{[ij]}D_{[ij]}. \tag{5.63}$$

The term $-\Pi D_{ii}$ can be positive or negative and, therefore, represents the *conservative* contribution to the stress power. The term $\kappa D_{ii}D_{jj}+2\eta D_{[ij]}D_{[ij]}$ represents the *dissipative* contribution. In view of (4.100) and (4.107), this contribution is also given by the dissipation function, D, whence we have

$$\kappa D_{ii}D_{jj} + 2\eta D_{[ij]}D_{[ij]} \geq 0. \tag{5.64}$$

Since the spherical and deviatoric components of D_{ij} are independently variable, necessary and sufficient for the inequality to hold are

$$\kappa \geq 0, \quad \eta \geq 0 \tag{5.65}$$

and, by (5.54), also

$$\chi \geq -\frac{2}{3}\eta. \tag{5.66}$$

If, in particular, $\kappa = 0$, the term $\kappa D_{ii}D_{jj}$ vanishes. Then, spherical strain changes are completely conservative, *i.e.* the entire dissipation is due to deviatoric strain changes. If, alternatively, $D_{ii} = 0$, both $-\Pi D_{ii}$ and $\kappa D_{ii}D_{jj}$ vanish, *i.e.* the stress power is fully dissipative.

5.4 Viscoelastic bodies

Elastic solids and viscous fluids represent end members of a suite of materials with varied response characteristics. Intermediate members of this suite incorporating characteristics of both end members are called viscoelastic bodies. This nomenclature suggests some difficulty in deciding whether a viscoelastic body be considered as a solid or a fluid. However, usually, the distinction between a solid and a fluid can be made. For our purposes, the following improvements of our preliminary definitions of solidity and fluidity (Sec. 1.1) are appropriate.

Solids: Following the application of a time-independent deviatoric stress, the strain ultimately converges to a finite value. Conversely, following the application of a time-independent deviatoric strain, the stress ultimately converges to a finite value.

Fluids: Following the application of a time-independent deviatoric stress, the strain rate ultimately converges to a finite value. Conversely, following the application of a time-independent deviatoric strain, the stress ultimately decays to zero.

5.4.1 Linear viscoelasticity

A characteristic feature of elastic solids is that the current stress is a function of the current strain. In viscous fluids, the current stress is a function of the current strain rate. In contrast to this, viscoelastic bodies exhibit a feature which is known as the memory hypothesis. This

implies that the stress at the current time epoch is not a *function* of the value of the strain or strain rate at this epoch, but a *functional* of the values at all past epochs.

In formalizing this statement, we restrict our analysis to the case that the functional is linear. Then,

$$t_{ij}^{\mathrm{K}}(t) = \int_0^\infty e_{kl}^{\mathrm{G}}(t-\tau)\, dm_{ijkl}(\tau), \tag{5.67}$$

with t the current time epoch, τ the lapse time interval, $m_{ijkl}(\tau)$ the relaxation tensor and the integral a Stieltjes convolution integral. Upon transforming it into a Riemann convolution integral, we obtain

$$t_{ij}^{\mathrm{K}}(t) = m_{ijkl}(0)\, e_{kl}^{\mathrm{G}}(t) + \int_0^\infty e_{kl}^{\mathrm{G}}(t-\tau)\, d_\tau m_{ijkl}(\tau)\, d\tau. \tag{5.68}$$

Integration by parts gives

$$t_{ij}^{\mathrm{K}}(t) = \int_t^0 m_{ijkl}(\tau)\, d_\tau e_{kl}^{\mathrm{G}}(t-\tau)\, d\tau, \tag{5.69}$$

where $e_{ij}^{\mathrm{G}}(t-\tau) = 0$ for $\tau \geq t$ has been assumed as the initial condition. After a change of variable, we obtain

$$t_{ij}^{\mathrm{K}}(t) = \int_0^t m_{ijkl}(t-t')\, d_{t'} e_{kl}^{\mathrm{G}}(t')\, dt', \tag{5.70}$$

with $t' = t - \tau$ the excitation time epoch. The roles of e_{ij}^{G} and t_{ij}^{K} in the preceding derivation may be interchanged, which leads to a constitutive equation of the form

$$e_{ij}^{\mathrm{G}}(t) = \int_0^t c_{ijkl}(t-t')\, d_{t'} t_{kl}^{\mathrm{K}}(t')\, dt', \tag{5.71}$$

with $c_{ijkl}(t-t')$ the creep tensor.

We now assume that $u_{i,j}$ is infinitesimal, that the components of t_{ij}^{K} are of the same order of magnitude and that only the lowest-order terms are retained. As in Sec. 5.2.3, it then follows from the definitions of e_{ij}, e_{ij}^{G}, t_{ij} and t_{ij}^{K} that

$$e_{ij}^{\mathrm{G}} = e_{ij}, \tag{5.72}$$

$$t_{ij}^{\mathrm{K}} = t_{ij}, \tag{5.73}$$

which are correct to the first and zeroth orders in $u_{i,j}$, respectively. In view of these equations, (5.70) and (5.71) take the forms

$$t_{ij}(t) = \int_0^t m_{ijkl}(t-t')\, d_{t'} e_{kl}(t')\, dt', \tag{5.74}$$

$$e_{ij}(t) = \int_0^t c_{ijkl}(t-t')\, d_{t'} t_{kl}(t')\, dt', \tag{5.75}$$

which are equivalent forms of the *generalized* constitutive equation of linear viscoelasticity. We note that, in view of the symmetry of e_{ij} and t_{ij}, the number of independent components of $c_{ijkl}(t-t')$ and $m_{ijkl}(t-t')$, respectively, is $6 \times 6 = 36$ (Secs. 5.2.1 and 5.2.2).

5.4.2 Isotropy

The following analysis will be restricted to *isotropic* viscoelastic bodies. Since t_{ij} and e_{ij} are symmetric, the isotropic forms of m_{ijkl} and c_{ijkl} are given by equations that formally agree with (5.21). Hence, choosing scalars m_1, m_2 and c_1, c_2, so that

$$m_{ijkl} = m_1\delta_{ij}\delta_{kl} + m_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{5.76}$$

$$c_{ijkl} = c_1\delta_{ij}\delta_{kl} + c_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{5.77}$$

substitution into (5.74) and (5.75), respectively, yields

$$t_{ij} = \int_0^t m_1(t-t')\, d_{t'}e_{kk}(t')\, \delta_{ij}\, dt' + 2\int_0^t m_2(t-t')\, d_{t'}e_{ij}(t')\, dt', \tag{5.78}$$

$$e_{ij} = \int_0^t c_1(t-t')\, d_{t'}t_{kk}(t')\, \delta_{ij} dt' + 2\int_0^t c_2(t-t')\, d_{t'}t_{ij}(t')\, dt', \tag{5.79}$$

where m_1 and m_2 are the first and second relaxation functions, respectively, and c_1 and c_2 are the first and second creep functions, respectively.

5.4.3 Spherical and deviatoric parts

To decompose (5.78) and (5.79) into spherical and deviatoric parts, we reconsider the decompositions (2.132) and (3.47):

$$e_{ij} \equiv \tfrac{1}{3}\delta_{ij}e_{kk} + e_{[ij]}, \tag{5.80}$$

$$t_{ij} \equiv \tfrac{1}{3}\delta_{ij}t_{kk} + t_{[ij]}. \tag{5.81}$$

Introducing

$$m_3 := m_1 + \tfrac{2}{3}m_2, \tag{5.82}$$

$$c_3 := c_1 + \tfrac{2}{3}c_2, \tag{5.83}$$

we obtain in analogy to Sec. 5.2.5 the relations

$$t_{ii}(t) = 3\int_0^t m_3(t-t')\, d_{t'}e_{ii}(t')\, dt', \tag{5.84}$$

$$t_{[ij]}(t) = 2\int_0^t m_2(t-t')\, d_{t'}e_{[ij]}(t')\, dt', \tag{5.85}$$

with m_2 and m_3 the shear-relaxation and bulk-relaxation functions, respectively, and

$$e_{ii}(t) = 3\int_0^t c_3(t-t')\, d_{t'}t_{ii}(t')\, dt', \tag{5.86}$$

$$e_{[ij]}(t) = 2\int_0^t c_2(t-t')\, d_{t'}t_{[ij]}(t')\, dt', \tag{5.87}$$

with c_2 and c_3 the shear-creep and bulk-creep functions, respectively.

5.4.4 Incompressibility

If the viscoelastic body is incompressible, the forms of the constitutive equation further simplify. With $u_{i,j}$ assumed to be infinitesimal, we have, according to (4.35),

$$u_{i,i} = 0. \tag{5.88}$$

However, in general, c_2, m_2 and t_{ii} remain finite even for incompressibility. Considering (5.82)–(5.84), (5.86) and $e_{ii} = u_{i,i}$, we must therefore also require

$$m_1 \to \infty, \quad m_3 \to \infty, \tag{5.89}$$

$$c_1 = -\tfrac{2}{3}c_2, \quad c_3 = 0. \tag{5.90}$$

Note that, on the assumption of incompressibility, (5.84) and (5.86) no longer apply, *i.e.* the spherical stress and the spherical strain are no longer related. On the other hand, the definition of the mechanical pressure continues to apply. Hence, using $e_{ii} = 0$ and $t_{ii} = -3p$, equations (5.80) and (5.81), respectively, reduce to

$$e_{ij} = e_{[ij]}, \tag{5.91}$$

$$t_{ij} = -p\delta_{ij} + t_{[ij]}. \tag{5.92}$$

Combining (5.85), (5.87), (5.91) and (5.92), we obtain

$$t_{ij}(t) \;=\; -p\delta_{ij} + 2\int_0^t m_2(t-t')\, d_{t'} e_{ij}(t')\, dt', \tag{5.93}$$

$$e_{ij}(t) \;=\; 2\int_0^t c_2(t-t')\, d_{t'}[t_{ij}(t') + p(t')\delta_{ij}]\, dt', \tag{5.94}$$

which are forms of the *incompressible* constitutive equation of linear viscoelasticity.

5.4.5 Relaxation experiment

The significance of the relaxation functions is illustrated by studying the response following the sudden application of a constant strain at $t = 0+$. This is referred to as the relaxation experiment. Considering

$$e_{ij}(t) = e_{ij}\, H_+(t) \tag{5.95}$$

in (5.84) and (5.85), where e_{ij} is time independent and H_+ the right-handed Heaviside step function defined by $H_+(t-t') := H[t-(t'+0)]$, the equations, respectively, become

$$t_{ii}(t) \;=\; 3m_3(t)\, e_{ii}\, H_+(t), \tag{5.96}$$

$$t_{[ij]}(t) \;=\; 2m_2(t)\, e_{[ij]}\, H_+(t). \tag{5.97}$$

Typical examples of $m_2(t)$ are shown in Fig. 5.1.

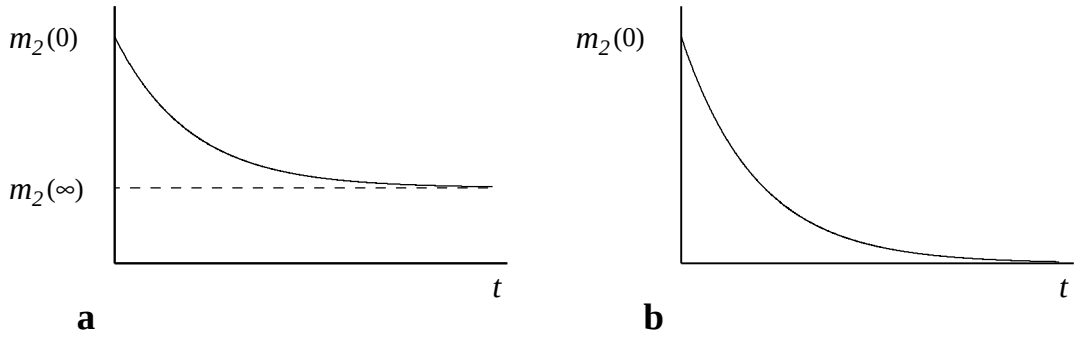


Figure 5.1: Schematic representation of the shear-relaxation function, $m_2(t)$, for (a) a viscoelastic solid and (b) a viscoelastic fluid.

5.4.6 Creep experiment

Alternatively, the response following the sudden application of a constant stress at $t = 0+$ may be investigated, which is called the creep experiment. Considering

$$t_{ij}(t) = t_{ij}\, H_+(t) \tag{5.98}$$

in (5.86) and (5.87), with t_{ij} time-independent, the equations, respectively, become

$$e_{ii}(t) \;=\; 3c_3(t)\, t_{ii}\, H_+(t), \tag{5.99}$$

$$e_{[ij]}(t) \;=\; 2c_2(t)\, t_{[ij]}\, H_+(t). \tag{5.100}$$

Typical examples of $c_2(t)$ are shown in Fig. 5.2.

5.4.7 Solids and fluids

We may use the above results to specify further our definitions of solidity and fluidity.

Solids: For a viscoelastic body to be a solid it is necessary and sufficient that, upon the sudden application of a time-independent deviatoric strain, the deviatoric stress and, thus, $m_2(t)$ in (5.97) converge to a finite value as $t \to \infty$ (Fig. 5.1a). Then, this equation formally agrees with (5.28), *i.e.* the deviatoric constitutive equation of an elastic solid, which suggests the following definition the elastic shear modulus of a viscoelastic solid:

$$\mu := \lim_{t\to\infty} m_2(t). \tag{5.101}$$

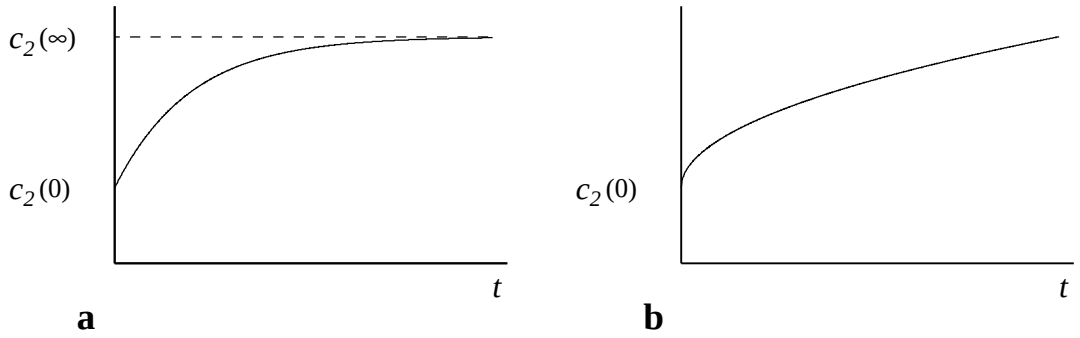


Figure 5.2: Schematic representation of the shear-creep function, $c_2(t)$, for (a) a viscoelastic solid and (b) a viscoelastic fluid.

Fluids: For a viscoelastic body to be a fluid it is necessary and sufficient that, upon the sudden application of a time-independent deviatoric stress, the strain rate converges to a finite value as $t \to \infty$ (Fig. 5.2b). Assuming

$$d_{t'} e_{[ij]}(t') = H(t')\, d_{[ij]}, \tag{5.102}$$

where $d_{[ij]}$ is time-independent, (5.85) becomes

$$t_{[ij]}(t) = 2d_{[ij]} \int_0^t m_2(t - t')\, dt'. \tag{5.103}$$

The steady-flow condition is obviously equivalent to the requirement that the integral converges as $t \to \infty$, so that the equation formally agrees with (5.55), *i.e.* the deviatoric constitutive equation of a viscous fluid. This suggests the following definition of the shear viscosity of a viscoelastic fluid:

$$\eta := \lim_{t\to\infty} \int_0^t m_2(t - t')\, dt'. \tag{5.104}$$

5.4.8 Strain energy

To calculate the strain energy per material unit volume stored in the viscoelastic body, we assume infinitesimal strain and consider, similarly to (5.15),

$$w(t) = \tfrac{1}{2} t_{ij}(t)\, e_{ij}(t), \tag{5.105}$$

where $w = 0$ in the undeformed state and $w > 0$ in the deformed state. Substituting the strain history (5.95) into (5.78), the associated stress history is

$$t_{ij}(t) = [m_1(t)\, e_{kk}\delta_{ij} + 2m_2(t)\, e_{ij}]H_+(t) \tag{5.106}$$

and (5.105) takes the form

$$w(t) = [\tfrac{1}{2} m_1(t)\, e_{ii} e_{jj} + m_2(t)\, e_{ij} e_{ij}] H_+(t). \tag{5.107}$$

Using (5.35) and (5.82), this can be rewritten as

$$w(t) = [\tfrac{1}{2} m_3(t)\, e_{ii} e_{jj} + m_2(t)\, e_{[ij]} e_{[ij]}] H_+(t). \tag{5.108}$$

Since the spherical and deviatoric components of e_{ij} are independent, necessary and sufficient for $w(t) \geq 0$ are

$$m_2(t) \geq 0, \quad m_3(t) \geq 0. \tag{5.109}$$

In view of (5.82), we furthermore obtain $m_1(t) \geq -\frac{2}{3} m_2(t)$. However, as for elastic materials, $m_1(t)$ is found to be non-negative, so that also

$$m_1(t) \geq 0. \tag{5.110}$$

We note that, on the assumption of a stress history of the form

$$t_{ij}(t) = t_{ij}\, H_+(t), \tag{5.111}$$

where t_{ij} is time-independent, the constraints

$$c_1(t) \geq 0, \quad c_2(t) \geq 0, \quad c_3(t) \geq 0 \tag{5.112}$$

on the creep functions are obtained in complete analogy to the deduction given above.

6 Field theories

6.1 Introduction

This chapter is concerned with the specialized field theories corresponding to the particular types of constitutive behaviour discussed in Chap. 5. Hence, the chapter will summarize the field equations and interface conditions of elastodynamics (Sec. 6.2), viscodynamics (Sec. 6.3) and viscoelastodynamics (Sec. 6.4). In each case, the specialized field theory will be restricted to mechanical changes of state and first given in non-linear form for inhomogeneous and, possibly, anisotropic materials. Following this, the equations will be linearized and the special cases of homogeneity and isotropy be considered. Finally, the uniqueness of the solution to the equations will be proved for each type of constitutive behaviour.

6.2 Elastodynamics

In this section, we consider a Hookean–elastic solid and describe its deformation by means of the Lagrangian representation, with the undeformed state taken as the initial state. We denote the material volume of the solid by $\mathcal{X}$ and its material boundary by $\partial\mathcal{X}^{(2)}$. Furthermore, $\mathcal{X}$ is divided by a material interface, $\partial\mathcal{X}^{(1)}$, into the internal material volume, $\mathcal{X}^{(1)}$, and the external material volume, $\mathcal{X}^{(2)}$. The outward material unit vectors normal to $\partial\mathcal{X}^{(1)}$ and $\partial\mathcal{X}^{(2)}$ are $N_i^{\mathbf{X}+}$ (Fig. 6.1). We admit prescribed jump discontinuities of the material parameters on $\partial\mathcal{X}^{(1)}$, but assume continuity elsewhere. Hence, the field equations are defined in $\mathcal{X}^{(1)} \cup \mathcal{X}^{(2)}$, the interface conditions are specified on $\partial\mathcal{X}^{(1)}$ and the boundary conditions on $\partial\mathcal{X}^{(2)}$. These equations must be supplemented by appropriate initial conditions.

6.2.1 Non-linear theory

From (2.4), (2.105), (3.23), (3.35), (4.28), (4.43) and (5.1), respectively, the field equations of elastodynamics in the Lagrangian representation are

$$j^{\mathbf{X}} := \det r_{i,j}, \tag{6.1}$$

$$e_{ij}^{\mathrm{G}} := \tfrac{1}{2}(r_{k,i}r_{k,j} - \delta_{ij}), \tag{6.2}$$

$$t_{ij}^{\mathrm{P}} := j^{\mathbf{X}} r_{j,k}^{-1} t_{ik}, \tag{6.3}$$

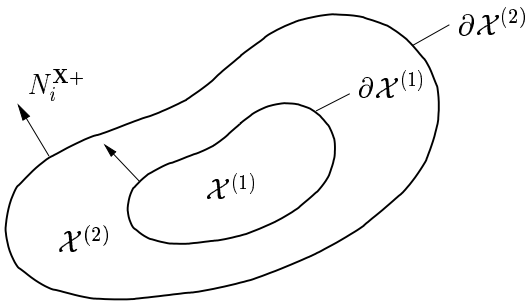


Figure 6.1: Domains employed in the Lagrangian representation of elastodynamics or viscoelastodynamics. The symbols are explained in the text.

$$t_{ij}^{\mathrm{K}} := j^{\mathbf{X}} r_{i,k}^{-1} r_{j,l}^{-1} t_{kl}, \tag{6.4}$$

$$j^{\mathbf{X}} \rho = \rho^{(0)}, \tag{6.5}$$

$$t_{ij,j}^{\mathrm{P}} + \rho^{(0)} g_i = \rho^{(0)}\, d_t^2 r_i, \tag{6.6}$$

$$t_{ij}^{\mathrm{K}} = m_{ijkl} e_{kl}^{\mathrm{G}}, \tag{6.7}$$

where $X_i \in \mathcal{X}^{(1)} \cup \mathcal{X}^{(2)}$. With g_i, m_{ijkl} and $\rho^{(0)}$ prescribed fields, the above equations represent 26 scalar equations for 26 scalar unknowns (nine components of t_{ij}^{P}, six components of t_{ij}^{K}, six components of e_{ij}^{G}, three components of r_i, $j^{\mathbf{X}}$ and ρ).

From (4.3), we have

$$\sigma_i^{\mathrm{P}} = j^{\mathbf{X}} r_{i,j}^{-1} \sigma_j \tag{6.8}$$

and, in view of (4.125), (4.126), (4.130) and (4.131), the associated interface and boundary conditions applying to $X_i \in \partial\mathcal{X}^{(1)}$ and $X_i \in \partial\mathcal{X}^{(2)}$, respectively, are

$$[r_i]_-^+ = 0 \quad \text{and} \quad [t_{ij}^{\mathrm{P}}]_-^+ N_j^{\mathbf{X}+} = \sigma^{\mathrm{P}} (d_t^2 r_i - g_i^{\pm}), \tag{6.9}$$

$$[r_i]^- = c_i \quad \text{or} \quad [t_{ij}^{\mathrm{P}}]^- N_j^{\mathbf{X}+} = -\sigma^{\mathrm{P}} (d_t^2 r_i - g_i^{\pm}), \tag{6.10}$$

where $\sigma^{\mathrm{P}} := \sigma_i^{\mathrm{P}} N_i^{\mathbf{X}+}$ and c_i are assumed to be prescribed.

6.2.2 Linearized theory

We use $r_i = X_i + u_i$ and assume that the magnitudes of the components of $u_{i,j}$ are sufficiently small in comparison to unity that only the lowest-order terms be retained. Considering (2.85) and supposing the magnitudes of the components of t_{ij} are of the same order of magnitude, we obtain from (6.1)–(6.5) the equations

$$j^{\mathbf{X}} = 1, \tag{6.11}$$

$$e_{ij}^{\mathrm{G}} = e_{ij}, \tag{6.12}$$

$$t_{ij}^{\mathrm{P}} = t_{ij}^{\mathrm{K}} = t_{ij}, \tag{6.13}$$

$$\rho = \rho^{(0)}, \tag{6.14}$$

where the second equation is of the first order and the others are of the zeroth order in $u_{i,j}$.

In view of (6.11)–(6.14) and $d_t^2 r_i = d_t^2 u_i$, equations (6.1)–(6.5) reduce to their linearized forms:

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{6.15}$$

$$t_{ij,j} + \rho g_i = \rho\, d_t^2 u_i, \tag{6.16}$$

$$t_{ij} = m_{ijkl} e_{ij}, \tag{6.17}$$

which represent 15 scalar equations for 15 scalar unknowns (six components of t_{ij}, six components of e_{ij} and three components of u_i). In the case of isotropy, the last equation simplifies to

$$t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}. \tag{6.18}$$

We may eliminate e_{ij} and t_{ij} from (6.15)–(6.17). Assuming that m_{ijkl} is constant, we obtain

$$\tfrac{1}{2} m_{ijkl}(u_{k,jl} + u_{l,jk}) + \rho g_i = \rho\, d_t^2 u_i, \tag{6.19}$$

which, for isotropy, becomes

$$(\lambda + \mu) u_{j,ij} + \mu u_{i,jj} + \rho g_i = \rho\, d_t^2 u_i. \tag{6.20}$$

Either vector equation represents three scalar equations for the three components of u_i.

We continue using $r_i = X_i + u_i$ and assuming that the magnitudes of the components of $u_{i,j}$ are sufficiently small that only the lowest-order terms be retained. Then, (6.8) becomes

$$\sigma^{\mathrm{P}} = \sigma, \tag{6.21}$$

and, in view of (6.13), the interface conditions (6.9) and boundary conditions (6.10), respectively, reduce to their linearized forms

$$[u_i]^+_- = 0 \quad \text{and} \quad [t_{ij}]^+_- N_j^{\mathbf{X}+} = \sigma(d_t^2 u_i - g_i^{\pm}), \tag{6.22}$$

$$[u_i]^- = c_i \quad \text{or} \quad [t_{ij}]^- N_j^{\mathbf{X}+} = -\sigma(d_t^2 u_i - g_i^{\pm}). \tag{6.23}$$

On account of the linearity of the field equations, interface conditions and boundary conditions, the superposition principle applies. This means that, if $u_i^{(1)}$, $e_{ij}^{(1)}$, $t_{ij}^{(1)}$ are solutions to (6.15)–(6.17) for prescribed $g_i^{(1)}$ and $u_i^{(2)}$, $e_{ij}^{(2)}$, $t_{ij}^{(2)}$ are solutions for prescribed $g_i^{(2)}$, then $c_1 u_i^{(1)} + c_2 u_i^{(2)}$, $c_1 e_{ij}^{(1)} + c_2 e_{ij}^{(2)}$, $c_1 t_{ij}^{(1)} + c_2 t_{ij}^{(2)}$ are solutions for $c_1 g_i^{(1)} + c_2 g_i^{(2)}$, with c_1 and c_2 arbitrary constants. The solutions for prescribed interface and boundary conditions can be superposed in an analogous way.

6.2.3 Uniqueness theorem of linearized elastostatics

If the acceleration is sufficiently small, the term $\rho\, d_t^2 u_i$ can be neglected and (6.15)–(6.17) simplify to

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{6.24}$$

$$t_{ij,j} + \rho g_i = 0, \tag{6.25}$$

$$t_{ij} = m_{ijkl} e_{kl}, \tag{6.26}$$

which are the field equations of linearized elastostatics.

We now suppose that the fields are continuously differentiable everywhere on $\mathcal{X} \cup \partial\mathcal{X}^{(2)}$. We also assume perfect elasticity, which is equivalent to the existence of a non-negative strain energy per material unit volume of the form of (5.15):

$$w = \tfrac{1}{2} t_{ij} e_{ij} \geq 0, \tag{6.27}$$

where $w = 0$ applies to the undeformed state and $w > 0$ to the deformed state. The uniqueness theorem of elastostatics maintains that the solution to (6.24)–(6.26) is unique in $\mathcal{X}$ if g_i is prescribed in $\mathcal{X}$ and $u_i := [u_i]^-$ or $t_i := [t_{ij}]^- N_j^{\mathbf{X}+}$ is prescribed on $\partial\mathcal{X}^{(2)}$. To prove the theorem, we consider two solutions, $e_{ij}^{(1)}$, $t_{ij}^{(1)}$, $u_i^{(1)}$ and $e_{ij}^{(2)}$, $t_{ij}^{(2)}$, $u_i^{(2)}$, corresponding to the same volume forces in $\mathcal{X}$ and to the same boundary conditions on $\partial\mathcal{X}^{(2)}$. By the superposition principle, the solution $e_{ij} = e_{ij}^{(2)} - e_{ij}^{(1)}$, $t_{ij} = t_{ij}^{(2)} - t_{ij}^{(1)}$, $u_i = u_i^{(2)} - u_i^{(1)}$ then corresponds to $g_i = 0$ in $\mathcal{X}$ and to $t_i u_i = 0$ on $\partial\mathcal{X}^{(2)}$. Integration over $\partial\mathcal{X}^{(2)}$ gives

$$\int_{\partial\mathcal{X}^{(2)}} t_i u_i \, d^2X = 0, \tag{6.28}$$

which, using (3.17) and (4.3), is equivalent to

$$\int_{\mathcal{X}} (t_{ij} u_i)_{,j} \, d^3X = 0. \tag{6.29}$$

Since, on account of $g_i = 0$, the equation $t_{ij,j} = 0$ obtains, this simplifies to

$$\int_{\mathcal{X}} t_{ij} u_{i,j} \, d^3X = 0. \tag{6.30}$$

From (2.84)–(2.86), we have $u_{i,j} = e_{ij} + \alpha_{ij}$ and, in view of the symmetry properties of t_{ij}, e_{ij} and α_{ij}, we obtain

$$\int_{\mathcal{X}} t_{ij} e_{ij} \, d^3X = 0. \tag{6.31}$$

Considering (6.27), this is possible only if

$$t_{ij} e_{ij} = 0 \tag{6.32}$$

everywhere in $\mathcal{X}$. Since $w = 0$ applies to the undeformed state, $t_{ij} = e_{ij} = 0$, whence

$$e_{ij}^{(1)} = e_{ij}^{(2)}, \tag{6.33}$$

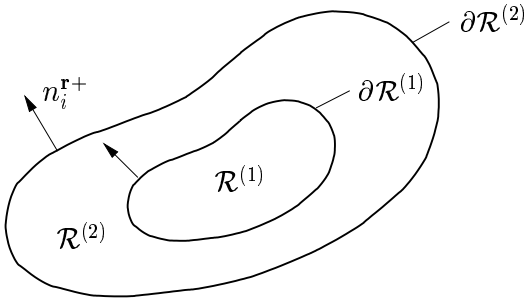


Figure 6.2: Domains employed in the Eulerian representation of viscodynamics. The symbols are explained in the text.

$$t_{ij}^{(1)} = t_{ij}^{(2)}. \tag{6.34}$$

In view of (6.24), we also have, except for a possible rigid-body displacement,

$$u_i^{(1)} = u_i^{(2)}. \tag{6.35}$$

Equations (6.33)–(6.35) confirm that, for perfect elasticity and with g_i prescribed in $\mathcal{X}$ and u_i or t_i prescribed on $\partial\mathcal{X}^{(2)}$, the solution to the linearized field equations of elastostatics is unique. The uniqueness theorem can be generalized in several ways and may be shown to be valid also in the case of a discontinuity on $\partial\mathcal{X}^{(1)}$.

6.3 Viscodynamics

In this section, we summarize the field equations and interface conditions governing the flow of a Newtonian–viscous fluid. Since, in general, individual particles cannot be distinguished in fluids, we describe the flow using the Eulerian representation. We choose some spatial volume, $\mathcal{R}$, inside of the fluid and confine it by its spatial boundary, $\partial\mathcal{R}^{(2)}$. Furthermore, $\mathcal{R}$ is divided by the spatial interface $\partial\mathcal{R}^{(1)}$ into the internal spatial volume, $\mathcal{R}^{(1)}$, and the external spatial volume, $\mathcal{R}^{(2)}$. The outward spatial unit vectors normal to $\partial\mathcal{R}^{(1)}$ or $\partial\mathcal{R}^{(2)}$ are $n_i^{\mathbf{r}+}$ (Fig. 6.2). We allow prescribed discontinuities of the material parameters on $\partial\mathcal{R}^{(1)}$, but suppose continuity elsewhere. The field equations are thus defined for $\mathcal{R}^{(1)} \cup \mathcal{R}^{(2)}$, the interface conditions apply to $\partial\mathcal{R}^{(1)}$ and the boundary conditions are specified on $\partial\mathcal{R}^{(2)}$, with the individual domains in general time-dependent. In addition, appropriate initial conditions must be specified.

6.3.1 Non-linear theory

We restrict the analysis to isotropic Newtonian–viscous fluids. Then, from (2.113), (4.19), (4.40), (5.45) and (5.49), the field equations of viscodynamics in the Eulerian representation are

$$D_{ij} = \tfrac{1}{2}(V_{i,j} + V_{j,i}), \tag{6.36}$$

$$d_t\rho + \rho V_{i,i} = 0, \tag{6.37}$$

$$T_{ij,j} + \rho G_i = \rho\, d_t V_i, \tag{6.38}$$

$$\Pi = \Pi(\rho), \tag{6.39}$$

$$T_{ij} = -\Pi\delta_{ij} + \chi D_{kk}\delta_{ij} + 2\eta D_{ij}, \tag{6.40}$$

where $r_i \in \mathcal{R}^{(1)} \cup \mathcal{R}^{(2)}$. With G_i, Π, χ and η prescribed fields, these equations represent 17 scalar equations for 17 scalar unknowns (six components of T_{ij}, six components of D_{ij}, three components of V_i, Π and ρ). The number of equations and unknowns can be reduced by eliminating D_{ij} and T_{ij}. Assuming that χ and η are constant, (6.36)–(6.40) reduce to

$$d_t\rho + \rho V_{i,i} = 0, \tag{6.41}$$

$$\Pi = \Pi(\rho), \tag{6.42}$$

$$-\Pi_{,i} + (\chi + \eta)V_{j,ij} + \eta V_{i,jj} + \rho G_i = \rho\, d_t V_i, \tag{6.43}$$

which represent five scalar equations for five scalar unknowns (three components of V_i, Π and ρ). Equation (6.43) is called the Navier–Stokes equation.

If incompressibility is assumed, $d_t\rho = 0$ and $\Pi = P$ and (6.41)–(6.43) reduce to

$$V_{i,i} = 0, \tag{6.44}$$

$$-P_{,i} + \eta V_{i,jj} + \rho G_i = \rho\, d_t V_i. \tag{6.45}$$

If the fluid is also inviscid, $\eta = 0$ and we get

$$V_{i,i} = 0, \tag{6.46}$$

$$-P_{,i} + \rho G_i = \rho\, d_t V_i, \tag{6.47}$$

where the last equation is called the Euler equation. A simpification that, in a sense, is opposite to that of inviscidity results if $\rho\, d_t V_i$ can be neglected. Then, (6.44) and (6.45) become

$$V_{i,i} = 0, \tag{6.48}$$

$$-P_{,i} + \eta V_{i,jj} + \rho G_i = 0, \tag{6.49}$$

which are the quasi-static field equations.

From (4.112), (4.113), (4.119) and (4.120), the interface and boundary conditions applying to $r_i \in \partial\mathcal{R}^{(1)}$ and $r_i \in \partial\mathcal{R}^{(2)}$, respectively, are

$$[V_i]^+_- = 0 \quad \text{and} \quad [T_{ij}]^+_- n_j^{\mathbf{r}+} = \Sigma(d_t V_i - G_i^{\pm}), \tag{6.50}$$

$$[V_i]^- = C_i \quad \text{or} \quad [T_{ij}]^- n_j^{\mathbf{r}+} = -\Sigma(d_t V_i - G_i^{\pm}), \tag{6.51}$$

with Σ and C_i prescribed.

6.3.2 Linearized theory

The non-linearity of the field equations and interface conditions of viscodynamics arises from the terms $\rho V_{i,i}$ and $\rho\, d_t V_i = \rho(D_t V_i + V_j V_{i,j})$ in (6.41) and (6.42), but also appears in (6.50) or (6.51) if the spatial interface or spatial boundary is moving. Whereas the non-linearities associated with ρ vanish for incompressible fluids, the non-linearity associated with the advective part of the material time derivative of V_i remains in this case and even persists if the fluid is inviscid.

One method of linearizing the field equations is to assume that $d_t V_i$ can be neglected. This has been applied above and resulted in the quasi-static field equations (6.48) and (6.49). However, for compressible fluids, this method does not lead to complete linearization.

A different method of linearizating the field equations is the use of perturbation theory. The fundamental assumption of perturbation theory is that the *current* state of the fluid can be expanded into an *initial* state and an *incremental* state. Mathematically, this can be represented by a perturbation equation of the form

$$F_{ij\ldots} = F^{(0)}_{ij\ldots} + F^{(1)}_{ij\ldots}, \tag{6.52}$$

where $F^{(0)}_{ij\ldots}$ and $F^{(1)}_{ij\ldots}$ denote the initial and incremental parts, respectively, of $F_{ij\ldots}$. Substituting this into the non-linear field equations, interface conditions and boundary conditions results in a set of equations containing zeroth-, first- and second-order terms. In linear perturbation theory, it is assumed that the increments are sufficiently small that the second-order terms can be neglected. Assuming that the solution for the initial state is known, we then arrive at a set of linear equations for the first-order terms.

To illustrate the method, we consider (6.36)–(6.40). On the assumption that the initial state is hydrostatic, $d_t F^{(0)}_{ij\ldots} = 0$, $D^{(0)}_{ij} = 0$ and $V^{(0)}_{i,j} = 0$. Then, the expanded forms of the equations are

$$D^{(1)}_{ij} = \tfrac{1}{2}(V^{(1)}_{i,j} + V^{(1)}_{j,i}), \tag{6.53}$$

$$D_t\rho^{(1)} + V^{(1)}_i \rho^{(1)}_{,i} + (\rho^{(0)} + \rho^{(1)}) V^{(1)}_{i,i} = 0, \tag{6.54}$$

$$-P^{(0)}_{,i} + T^{(1)}_{ij,j} + (\rho^{(0)} + \rho^{(1)})(G^{(0)}_i + G^{(1)}_i) = (\rho^{(0)} + \rho^{(1)})(D_t V^{(1)}_i + V^{(1)}_j V^{(1)}_{i,j}), \tag{6.55}$$

$$-P^{(0)}\delta_{ij} + T^{(1)}_{ij} = -(\Pi^{(0)} + \Pi^{(1)})\delta_{ij} + \chi D^{(1)}_{kk}\delta_{ij} + 2\eta D^{(1)}_{ij}, \tag{6.56}$$

$$\Pi^{(0)} + \Pi^{(1)} = \Pi(\rho^{(0)}) + K\rho^{(1)}, \tag{6.57}$$

where the fluid bulk modulus, $K = (d\Pi/d\rho)_{\rho^{(0)}}$, has been used in the last equation. Since these equations must be satisfied in the hydrostatic initial state, we may equate the zeroth-order terms, giving $P^{(0)}_{,i} = \rho^{(0)} G^{(0)}_i$, $\Pi^{(0)} = P^{(0)}$ and $\Pi^{(0)} = \Pi(\rho^{(0)})$. If we observe the zeroth-order equations and neglect the second-order terms, the linearized field equations of viscodynamics are

$$D^{(1)}_{ij} = \tfrac{1}{2}(V^{(1)}_{i,j} + V^{(1)}_{j,i}), \tag{6.58}$$

$$D_t\rho^{(1)} + \rho^{(0)}V_{i,i}^{(1)} = 0, \tag{6.59}$$

$$T_{ij,j}^{(1)} + \rho^{(0)}G_i^{(1)} + \rho^{(1)}G_i^{(0)} = \rho^{(0)}D_tV_i^{(1)}, \tag{6.60}$$

$$T_{ij}^{(1)} = -\Pi^{(1)}\delta_{ij} + \chi D_{kk}^{(1)}\delta_{ij} + 2\eta D_{ij}^{(1)}, \tag{6.61}$$

$$\Pi^{(1)} = K\rho^{(1)}. \tag{6.62}$$

Assuming that $G_i^{(0)}$, $G_i^{(1)}$, K, χ and η are prescribed fields, these equations represent 17 scalar equations for 17 scalar unknowns (six components of $T_{ij}^{(1)}$, six components of $D_{ij}^{(1)}$, three components of $V_i^{(1)}$, $\Pi^{(1)}$ and $\rho^{(1)}$). If K, χ and η are constant, we obtain, upon elimination of $D_{ij}^{(1)}$, $T_{ij}^{(1)}$ and $\Pi^{(1)}$, the relations

$$D_t\rho^{(1)} + \rho^{(0)}V_{i,i}^{(1)} = 0, \tag{6.63}$$

$$-K\rho_{,i}^{(1)} + (\chi+\eta)V_{j,ij}^{(1)} + \eta V_{i,jj}^{(1)} + \rho^{(0)}G_i^{(1)} + \rho^{(1)}G_i^{(0)} = \rho^{(0)}\,D_tV_i^{(1)}, \tag{6.64}$$

which represent four scalar equations for four scalar unknowns (three components of $V_i^{(1)}$ and $\rho^{(1)}$).

Assuming time-independent spatial domains, $n_i^{\mathbf{r}+} = n_i^{(0)\,\mathbf{r}+}$ and the interface conditions (6.50) and boundary conditions (6.51), respectively, can be expanded into

$$[V_i^{(1)}]_-^+ = 0 \quad \text{and} \quad [-P^{(0)}\delta_{ij} + T_{ij}^{(1)}]_-^+ n_j^{(0)\,\mathbf{r}+} = (\Sigma^{(0)} + \Sigma^{(1)})(d_tV_i^{(1)} - G_i^{(0)\pm} - G_i^{(1)\pm}), \tag{6.65}$$

$$[V_i^{(1)}]^- = C_i \quad \text{or} \quad [-P^{(0)}\delta_{ij} + T_{ij}^{(1)}]^- n_j^{(0)\,\mathbf{r}+} = -(\Sigma^{(0)} + \Sigma^{(1)})(d_tV_i^{(1)} - G_i^{(0)\pm} - G_i^{(1)\pm}). \tag{6.66}$$

With $[P^{(0)}]_-^+ = \Sigma^{(0)}\rho^{(0)}$ and $[P^{(0)}]^- = -\Sigma^{(0)}\rho^{(0)}$ in the hydrostatic initial state and the second-order terms neglected, the linearized interface and boundary conditions take the forms

$$[V_i^{(1)}]_-^+ = 0 \quad \text{and} \quad [T_{ij}^{(1)}]_-^+ n_j^{(0)\,\mathbf{r}+} = -\Sigma^{(0)}G_i^{(1)\pm} - \Sigma^{(1)}G_i^{(0)\pm}, \tag{6.67}$$

$$[V_i^{(1)}]^- = C_i \quad \text{or} \quad [T_{ij}^{(1)}]^- n_j^{(0)\,\mathbf{r}+} = \Sigma^{(0)}G_i^{(1)\pm} + \Sigma^{(1)}G_i^{(0)\pm}. \tag{6.68}$$

After linearization of the field equations, interface conditions and boundary conditions and with the spatial domains fixed, the superposition principle can be applied as discussed for linearized elastostatics (Sec. 6.2.2).

6.3.3 Uniqueness theorem of linearized quasi-static viscodynamics

If the acceleration is sufficiently small and the fluid incompressible, $\rho^{(0)}\,D_tV_i^{(1)} = 0$, $\rho^{(1)} = 0$ and $\Pi^{(1)} = P^{(1)}$. Dropping the superscripts, (6.58)–(6.62) then simplify to

$$D_{ij} = \tfrac{1}{2}(V_{i,j} + V_{j,i}), \tag{6.69}$$

$$V_{i,i} = 0, \tag{6.70}$$

$$T_{ij,j} + \rho G_i = 0, \tag{6.71}$$

$$T_{ij} = -P\delta_{ij} + 2\eta D_{ij}, \tag{6.72}$$

which are the linearized field equations of quasi-static viscodynamics for incompressibility.

We now assume that the fields are continuously differentiable everywhere on $\mathcal{R} \cup \partial\mathcal{R}^{(2)}$. We also recall (5.62), *i.e.* the definition of the stress power per spatial unit volume, which, for an incompressible fluid, is purely dissipative and, therefore, non-negative (Sec. 5.3.6):

$$Z := T_{ij} D_{ij} \geq 0. \tag{6.73}$$

The uniqueness theorem of quasi-static viscodynamics maintains that the solution to (6.69)–(6.72) is unique in $\mathcal{R}$ if G_i is prescribed in $\mathcal{R}$ and $V_i := [V_i^{(1)}]^-$ or $T_i := [T_{ij}^{(1)}]^- n_j^{(0)+}$ is prescribed on $\partial\mathcal{R}^{(2)}$. To prove the theorem, we consider two solutions, $D_{ij}^{(1)}$, $T_{ij}^{(1)}$, $V_i^{(1)}$ and $D_{ij}^{(2)}$, $T_{ij}^{(2)}$, $V_i^{(2)}$, corresponding to the same volume forces in $\mathcal{R}$ and to the same boundary conditions on $\partial\mathcal{R}^{(2)}$. By the superposition principle, the solution $D_{ij} = D_{ij}^{(2)} - D_{ij}^{(1)}$, $T_{ij} = T_{ij}^{(2)} - T_{ij}^{(1)}$, $V_i = V_i^{(2)} - V_i^{(1)}$ then corresponds to $G_i = 0$ in $\mathcal{R}$ and to $T_i V_i = 0$ on $\partial\mathcal{R}^{(2)}$. Upon integration over $\partial\mathcal{R}^{(2)}$, we get

$$\int_{\partial\mathcal{R}^{(2)}} T_i V_i \, d^2 r = 0, \tag{6.74}$$

which, using (2.30), (3.17) and (4.1), becomes

$$\int_{\mathcal{R}} (T_{ij} V_i)_{,j} \, d^3 r = 0. \tag{6.75}$$

In view of (6.71), this reduces to

$$\int_{\mathcal{R}} T_{ij} V_{i,j} \, d^3 r = 0. \tag{6.76}$$

From (2.112)–(2.114), we have $V_{i,j} = D_{ij} + \Omega_{ij}$. Exploiting the symmetry properties of T_{ij}, D_{ij} and Ω_{ij}, we get

$$\int_{\mathcal{R}} T_{ij} D_{ij} \, d^3 r = 0. \tag{6.77}$$

On account of (6.73) this is possible only if

$$T_{ij} D_{ij} = 0 \tag{6.78}$$

everywhere in $\mathcal{R}$. Since $Z = 0$ is valid only in the hydrostatic state, even $T_{ij} = D_{ij} = 0$ applies and, therefore,

$$D_{ij}^{(1)} = D_{ij}^{(2)}, \tag{6.79}$$

$$T_{ij}^{(1)} = T_{ij}^{(2)}. \tag{6.80}$$

In view of (6.69), we also have, except for a uniform motion,

$$V_i^{(1)} = V_i^{(2)}. \tag{6.81}$$

The last three equations confirm that, for incompressible Newtonian viscosity, G_i prescribed in $\mathcal{R}$ and V_i or T_i prescribed on $\partial\mathcal{R}^{(2)}$, the solution to the linearized field equations of quasi-static viscodynamics is unique. The uniqueness theorem, here proved only for incompressibility, can be generalized in several ways and, in particular, may be shown to be valid also in the case of a discontinuity on $\partial\mathcal{R}^{(1)}$.

6.4 Viscoelastodynamics

Since the constitutive behaviour of viscoelastic bodies is intermediate between that of solids and fluids, both the Lagrangian and the Eulerian representations may profitably be employed. In this section, we suppose the existence of an undeformed state and adopt the Lagrangian representation, with this state serving as the initial state. Hence, the domains of definition used in the Lagrangian representation of the field equations, interface conditions and boundary conditions of elastodynamics (Fig. 6.1) continue to apply. As in Sec. 5.4.1, the initial conditions are $e_{ij}^{\mathrm{G}}(t-\tau) = 0$ for $\tau \geq t$.

6.4.1 Non-linear theory

From (2.4), (2.105), (3.23), (3.35), (4.28), (4.43) and (5.70), respectively, the field equations of viscoelastodynamics in the Lagrangian representation are

$$j^{\mathbf{X}} := \det r_{i,j}, \tag{6.82}$$

$$e_{ij}^{\mathrm{G}} := \tfrac{1}{2}(r_{k,i}r_{k,j} - \delta_{ij}), \tag{6.83}$$

$$t_{ij}^{\mathrm{P}} := j^{\mathbf{X}} r_{j,k}^{-1} t_{ik}, \tag{6.84}$$

$$t_{ij}^{\mathrm{K}} := j^{\mathbf{X}} r_{i,k}^{-1} r_{j,l}^{-1} t_{kl}, \tag{6.85}$$

$$j^{\mathbf{X}}\rho = \rho^{(0)}, \tag{6.86}$$

$$t_{ij,j}^{\mathrm{P}} + \rho^{(0)} g_i = \rho^{(0)}\, d_t^2 r_i, \tag{6.87}$$

$$t_{ij}^{\mathrm{K}} = \int_0^t m_{ijkl}(t-t')\, d_{t'} e_{kl}^{\mathrm{G}}(t')\, dt', \tag{6.88}$$

where $X_i \in \mathcal{X}^{(1)} \cup \mathcal{X}^{(2)}$. With g_i, $m_{ijkl}(t-t')$ and $\rho^{(0)}$ prescribed fields, these are 26 scalar equations for 26 scalar unknowns (nine components of t_{ij}^{P}, six components of t_{ij}^{K}, six components of e_{ij}^{G}, three components of r_i, $j^{\mathbf{X}}$ and ρ).

From (4.3), we have

$$\sigma_i^{\mathrm{P}} = j^{\mathbf{X}} r_{i,j}^{-1} \sigma_j, \tag{6.89}$$

and, from (4.125), (4.126), (4.130) and (4.131), the interface and boundary conditions applying to $X_i \in \partial\mathcal{X}^{(1)}$ and $X_i \in \partial\mathcal{X}^{(2)}$, respectively, have the forms

$$[r_i]^+_- = 0 \quad \text{and} \quad [t^{\mathrm{P}}_{ij}]^+_- \, N_j^{\mathbf{X}+} = \sigma^{\mathrm{P}}(d_t^2 r_i - g_i^{\pm}), \tag{6.90}$$

$$[r_i]^- = c_i \quad \text{or} \quad [t^{\mathrm{P}}_{ij}]^- \, N_j^{\mathbf{X}+} = -\sigma^{\mathrm{P}}(d_t^2 r_i - g_i^{\pm}), \tag{6.91}$$

with $\sigma^{\mathrm{P}} := \sigma_i^{\mathrm{P}} N_i^{\mathbf{X}+}$ and c_i prescribed.

6.4.2 Linearized theory

We consider $r_i = X_i + u_i$, $d_t^2 r_i = d_t^2 u_i$ and employ the same approximations used in Sec. 6.2.2. Then, (6.11)–(6.14) apply and (6.82)–(6.88) take the linearized forms

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{6.92}$$

$$t_{ij,j} + \rho g_i = \rho \, d_t^2 u_i, \tag{6.93}$$

$$t_{ij} = \int_0^t m_{ijkl}(t - t') \, d_{t'} e_{kl}(t') \, dt', \tag{6.94}$$

which represent 15 scalar equations for 15 scalar unknowns (six components of t_{ij}, six components of e_{ij} and three components of u_i). Assuming isotropy, the last equation simplifies to

$$t_{ij} = \int_0^t m_1(t - t') \, d_{t'} e_{kk}(t') \, \delta_{ij} \, dt' + 2 \int_0^t m_2(t - t') \, d_{t'} e_{ij}(t') \, dt'. \tag{6.95}$$

The number of unknowns can be reduced by eliminating e_{ij} and t_{ij} from (6.92)–(6.94). Assuming that $m_{ijkl}(t - t')$ is spatially constant, we obtain

$$\tfrac{1}{2} \int_0^t m_{ijkl}(t - t') \, d_{t'} [u_{k,jl}(t') + u_{l,jk}(t')] \, dt' + \rho g_i = \rho \, d_t^2 u_i, \tag{6.96}$$

which, for isotropy, becomes

$$\int_0^t [m_1(t - t') + m_2(t - t')] \, d_{t'} u_{j,ij}(t') \, dt' + \int_0^t m_2(t - t') \, d_{t'} u_{i,jj}(t') \, dt' + \rho g_i = \rho \, d_t^2 u_i. \tag{6.97}$$

Either equation represents three scalar equations for the three components of u_i.

With $r_i = X_i + u_i$ and the magnitudes of the components of $u_{i,j}$ sufficiently small that only the lowest-order terms be retained, $t^{\mathrm{P}}_{ij} = t_{ij}$ and $\sigma^{\mathrm{P}} = \sigma$ continue to apply and the linearized forms of the interface and boundary conditions (6.90) and (6.91), respectively, reduce to

$$[u_i]^+_- = 0 \quad \text{and} \quad [t_{ij}]^+_- \, N_j^{\mathbf{X}+} = \sigma(d_t^2 u_i - g_i^{\pm}), \tag{6.98}$$

$$[u_i]^- = c_i \quad \text{or} \quad [t_{ij}]^- \, N_j^{\mathbf{X}+} = -\sigma(d_t^2 u_i - g_i^{\pm}). \tag{6.99}$$

We note that, except for the constitutive equation, the field equations, interface conditions and boundary conditions of viscoelastodynamics are identical to those of elastodynamics.

Another way of displaying the formal relationship between the two systems of equations is to consider the Laplace transforms with respect to the time, t, of the field equations, interface conditions and boundary conditions of viscoelastodynamics. Here, we only discuss this relationship for the linearized equations. Assuming that the perturbations are quasi-static, $d_t^2 u_i = 0$, and that the volume-mass density is time-independent, $\rho = \rho^{(0)}$, and denoting the Laplace transform of $f_{ij\ldots}$ with respect to t by $\widetilde{f}_{ij\ldots}$, the Laplace transforms of (6.92)–(6.94) take the forms (App. A.1)

$$\widetilde{e}_{ij} = \tfrac{1}{2}(\widetilde{u}_{i,j} + \widetilde{u}_{j,i}), \tag{6.100}$$

$$\widetilde{t}_{ij,j} + \rho \widetilde{g}_i = 0, \tag{6.101}$$

$$\widetilde{t}_{ij} = s\widetilde{m}_{ijkl}\widetilde{e}_{kl}. \tag{6.102}$$

The Laplace transforms of (6.98) and (6.99) are

$$[\widetilde{u}_i]^+_- = 0 \quad \text{and} \quad [\widetilde{t}_{ij}]^+_- N_j^{\mathbf{X}+} = -\widetilde{\sigma}\widetilde{g}_i^{\pm}, \tag{6.103}$$

$$[\widetilde{u}_i]^- = \widetilde{c}_i \quad \text{or} \quad [\widetilde{t}_{ij}]^- N_j^{\mathbf{X}+} = \widetilde{\sigma}\widetilde{g}_i^{\pm}. \tag{6.104}$$

In these equations, the inverse Laplace time, s, of Laplace-transformed fields and the relaxation tensor has been suppressed as the argument. We note that the expressions are formally identical to the field equations, interface conditions and boundary conditions of elastostatics, provided that the Laplace-transformed viscoelastic fields are associated with the corresponding elastic field quantities and provided that $s\widetilde{m}_{ijkl}$ is associated with the elasticity tensor, m_{ijkl}. For a particular problem in quasi-static viscoelastodynamics, the Laplace-transformed solution is therefore obtained from the solution of the corresponding elastostatic problem if m_{ijkl} is replaced by $s\widetilde{m}_{ijkl}$. To find the solution to the viscoelastic problem in the time domain, the inverse Laplace transform (App. A.2) must be taken. The formal agreement between the Laplace-transformed viscoelastic solution and the corresponding elastic solution is called elastic–viscoelastic correspondence principle. Note that the assumption of quasi-staticity is essential to this correspondence.

6.4.3 Uniqueness theorem of linearized, quasi-static viscoelastodynamics

Assuming isotropy and with the acceleration sufficiently small, (6.92)–(6.94) reduce to

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{6.105}$$

$$t_{ij,j} + \rho g_i = 0, \tag{6.106}$$

$$t_{ij} = \int_0^t m_1(t-t')\, d_{t'} e_{kk}(t')\, \delta_{ij}\, dt' + 2\int_0^t m_2(t-t')\, d_{t'} e_{ij}(t')\, dt', \tag{6.107}$$

which are the field equations of linearized, isotropic, quasi-static viscoelastodynamics.

We now suppose that the fields are continuously differentiable everywhere on $\mathcal{X} \cup \partial\mathcal{X}^{(2)}$. The uniqueness theorem of the restricted form of viscoelastodynamics considered maintains that the solution to (6.105)–(6.107) is unique in $\mathcal{X}$ if g_i is prescribed in $\mathcal{X}$ and $u_i := [u_i]^-$ or $t_i := [t_{ij}]^- N_j^{\mathbf{X}+}$ is prescribed on $\partial\mathcal{X}^{(2)}$. To prove the theorem, we follow Sec. 6.2.3 and consider two solutions, $e_{ij}^{(1)}$, $t_{ij}^{(1)}$, $u_i^{(1)}$ and $e_{ij}^{(2)}$, $t_{ij}^{(2)}$, $u_i^{(2)}$, corresponding to the same volume forces in $\mathcal{X}$ and to the same boundary conditions on $\partial\mathcal{X}^{(2)}$. By the superposition principle, the solution $e_{ij} = e_{ij}^{(2)} - e_{ij}^{(1)}$, $t_{ij} = t_{ij}^{(2)} - t_{ij}^{(1)}$, $u_i = u_i^{(2)} - u_i^{(1)}$ then corresponds to $g_i = 0$ in $\mathcal{X}$ and to $t_i u_i = 0$ on $\partial\mathcal{X}^{(2)}$. Integration over $\partial\mathcal{X}^{(2)}$ gives

$$\int_{\partial\mathcal{X}^{(2)}} t_i u_i \, d^2X = 0, \tag{6.108}$$

which, repeating the steps shown in Sec. 6.2.3, leads to

$$\int_{\mathcal{X}} t_{ij} e_{ij} \, d^3X = 0 \tag{6.109}$$

everywhere in $\mathcal{X}$. Substituting (6.107), this can be rewritten as

$$\int_{\mathcal{X}} \left[\int_0^t m_1(t-t') \, d_{t'} e_{kk}(t') \, \delta_{ij} \, dt' + 2 \int_0^t m_2(t-t') \, d_{t'} e_{ij}(t') \, dt' \right] e_{ij}(t) \, d^3X = 0, \tag{6.110}$$

where, for clarity, the arguments t and t' are reintroduced. Consideration of (5.82) and the initial condition $e_{ij}(0) = 0$ followed by integration by parts leads to

$$\begin{aligned} \int_{\mathcal{X}} \Big[& m_3(0) \, e_{ii}(t) \, e_{jj}(t) + 2 m_2(0) \, e_{ij}(t) \, e_{ij}(t) \; + \\ & \int_0^t d_{t'} m_3(t-t') \, e_{ii}(t) \, e_{jj}(t') \, dt' + 2 d_{t'} m_2(t-t') \, e_{ij}(t) \, e_{ij}(t') \, dt' \Big] d^3X \; = \; 0. \end{aligned} \tag{6.111}$$

This equation simplifies if we introduce the following notational changes:

$$f_n(t) := \begin{cases} [m_3(0)]^{\frac{1}{2}} e_{ii}(t), & n = 1 \\ [2m_2(0)]^{\frac{1}{2}} e_{ij}(t), & n = 2, ..., 10, \quad i,j = 1,2,3 \end{cases}, \tag{6.112}$$

$$g_n(t-t') := \begin{cases} -\frac{d_{t'} m_3(t-t')}{m_3(0)}, & n = 1 \\ -\frac{d_{t'} m_2(t-t')}{m_2(0)}, & n = 2, ..., 10, \quad i,j = 1,2,3 \end{cases}, \tag{6.113}$$

where $f_n(t)$ and $g_n(t-t')$ are auxiliary functions. With these definitions and after some manipulations, (6.111) becomes

$$\int_{\mathcal{X}} \sum_{n=1}^{10} \left[f_n(t) \, f_n(t) + \int_0^t g_n(t-t') \, f_n(t) \, f_n(t') \, dt' \right] d^3X = 0. \tag{6.114}$$

It can be shown that, with $f_n(t)$ continuous in $\mathcal{X}$ and $g_n(t-t') > 0$, it follows from (6.114) that $f_n(t) = 0$ for $n = 1, ..., 10$. Considering $m_2(0) > 0$, $m_3(0) > 0$ according to (5.109) and using (6.112), this implies that $e_{ij} = 0$ and, with (6.109), also that $t_{ij} = 0$. We therefore obtain

$$e_{ij}^{(1)} = e_{ij}^{(2)}, \tag{6.115}$$

$$t_{ij}^{(1)} = t_{ij}^{(2)} \tag{6.116}$$

and, with (6.105) and except for a possible rigid-body displacement, also

$$u_i^{(1)} = u_i^{(2)}. \tag{6.117}$$

A Laplace transform

A.1 Forward transform

The forward Laplace transform, $\mathcal{L}[f(t)]$, of a function, $f(t)$, is defined by

$$\mathcal{L}[f(t)] := \int_0^\infty f(t)\, e^{-st} dt, \quad s \in \mathcal{C}, \tag{A.1}$$

where $\mathcal{L}$ is the Laplace-transform functional, s the inverse Laplace time and $\mathcal{C}$ the complex domain. We here assume that $f(t)$ is continuous for all $t \in [\,0, \infty)$ and of exponential order as $t \to \infty$, which are sufficient conditions for the convergence of the Laplace integral in (A.1) for $\mathrm{Re}\, s$ larger than some value, s_R. Defining $\widetilde{f}(s) := \mathcal{L}[f(t)]$ and assuming the same properties for $g(t)$, elementary consequences of (A.1) are

$$\mathcal{L}[a\, f(t) + b\, g(t)] = a\, \widetilde{f}(s) + b\, \widetilde{g}(s), \quad a, b = \text{constant}, \tag{A.2}$$

$$\mathcal{L}[d_t f(t)] = s\, \widetilde{f}(s) - f(0), \tag{A.3}$$

$$\mathcal{L}\Big[\int_0^t f(t')\, dt'\Big] = \frac{\widetilde{f}(s)}{s}, \tag{A.4}$$

$$\mathcal{L}\Big[\int_0^t f(t - t')\, g(t')\, dt'\Big] = \widetilde{f}(s)\, \widetilde{g}(s), \tag{A.5}$$

$$\mathcal{L}[1] = \frac{1}{s}\,, \tag{A.6}$$

$$\mathcal{L}[e^{-s_0 t}] = \frac{1}{s + s_0}\,, \quad s_0 = \text{constant}. \tag{A.7}$$

A.2 Inverse transform

If $\mathcal{L}[f(t)]$ is the forward Laplace transform of $f(t)$, then $f(t)$ is called the inverse Laplace transform of $\mathcal{L}[f(t)]$. This is expressed by $\mathcal{L}^{-1}\{\mathcal{L}[f(t)]\} \equiv f(t)$, with $\mathcal{L}^{-1}$ the inverse Laplace-transform functional. Since $\widetilde{f}(s) := \mathcal{L}[f(t)]$, it follows that

$$\mathcal{L}^{-1}[\widetilde{f}(s)] = f(t), \quad t \in [\,0, \infty), \tag{A.8}$$

which admits the immediate inversion of the forward transforms listed above.

B List of important symbols

The entries in the third column denote the section numbers of the first reference to the corresponding symbols. Except for the arbitrary Cartesian tensor field, for which all quantities are given both in the Eulerian representation and in the Lagrangian representation, all other fields are given only in the kinematic representation of their first reference.

B.1 Latin symbols

Symbol	*Name*	*Reference*
$\mathcal{A}$	current area of thin disk	4.11.1
$a^{(k)}$	kth invariant of e_{ij}	2.7.1
B	entropy-production rate per unit mass	4.8.1
$B^{(k)}$	kth invariant of T_{ij}	3.6.1
C	heat-production rate per unit mass	4.7.1
$\mathcal{C}$	complex domain	A.1
c_1	first creep function	5.4.2
c_2	second creep function (shear-creep function)	5.4.2
c_3	bulk-creep function	5.4.3
$c_{ijkl}(\tau)$	creep tensor	5.4.1
D	dissipation function	4.10
D_i	strain-rate vector	2.6
D_{ij}	strain-rate tensor	2.6
$\mathcal{E}$	total energy	4.7.1
$\mathcal{E}^{\mathrm{K}}$	integral kinetic energy	4.7.1
$\mathcal{E}^{\mathrm{U}}$	integral internal energy	4.7.1
$\delta_t\mathcal{E}^{\mathrm{Q}}$	integral heat-input rate	4.7.1
$\delta_t\mathcal{E}^{\mathrm{W}}$	integral work-input rate	4.7.1
E^{C}_{ij}	Cauchy strain	2.5.1
e	2.71828...	—
e_i	infinitesimal strain vector	2.4
$e^{(k)}$	principal value of e_{ij}	2.7.1
e_{ij}	infinitesimal strain tensor	2.4

Symbol	*Name*	*Reference*
$e_{(ij)}$	spherical part of e_{ij}	2.7.2
$e_{[ij]}$	deviatoric of part of e_{ij}	2.7.2
e_{ii}	dilatation	2.7.2
e_{ij}^{G}	Green strain	2.5.2
F_i	force per spatial unit volume	3.2.1
dF_i^{S}	(actual) differential surface force for fixed t	3.3
$d\hat{F}_i^{\mathrm{S}}$	fictious differential surface force for fixed t	3.5.2
$\mathcal{F}_i$	integral force	3.2.1
$\mathcal{F}_i^{\mathrm{S}}$	integral surface force	3.2.1
$\mathcal{F}_i^{\mathrm{V}}$	integral volume force	3.2.1
$F_{ij\ldots}$	Eulerian representation of Cartesian tensor field on $\mathcal{R} \cup \partial\mathcal{R}$	2.2.1
$F_{ij\ldots}^{-}$	Eulerian representation of Cartesian tensor field in $\mathcal{R}^{(1)}$	4.11.1
$F_{ij\ldots}^{+}$	Eulerian representation of Cartesian tensor field in $\mathcal{R}^{(2)}$	4.11.1
$[F_{ij\ldots}]^{-}$	internal limit of $F_{ij\ldots}$	4.11.1
$[F_{ij\ldots}]^{+}$	external limit of $F_{ij\ldots}$	4.11.1
$[F_{ij\ldots}]_{-}^{+}$	jump of $F_{ij\ldots}$	4.11.1
$F_{ij\ldots}^{\pm}$	arithmetic mean of $[F_{ij\ldots}]^{-}$ and $[F_{ij\ldots}]^{+}$	4.11.1
$F_{ij\ldots}^{(0)}$	initial part of $F_{ij\ldots}$	4.4.2
$F_{ij\ldots}^{(1)}$	incremental part of $F_{ij\ldots}$	6.3.2
$\mathcal{F}_{ij\ldots}$	integral of $F_{ij\ldots}$ over $\mathcal{R}$ or integral of $f_{ij\ldots}$ over $\mathcal{X}$	2.2.4
$dF_{ij\ldots}$	differential of $F_{ij\ldots}$ for fixed t	2.2.4
$F_{ij\ldots,k}$	gradient of $F_{ij\ldots}$ with respect to r_k	2.2.2
f_n	auxiliary function	6.4.3
$f_{ij\ldots}$	Lagrangian representation of Cartesian tensor field on $\mathcal{X} \cup \partial\mathcal{X}$	2.2.1
$f_{ij\ldots}^{-}$	Lagrangian representation of Cartesian tensor field in $\mathcal{X}^{(1)}$	4.11.2
$f_{ij\ldots}^{+}$	Lagrangian representation of Cartesian tensor field in $\mathcal{X}^{(2)}$	4.11.2
$\widetilde{f}_{ij\ldots}$	Laplace transform of $f_{ij\ldots}$	6.4.2, A.1
$[f_{ij\ldots}]^{-}$	internal limit of $f_{ij\ldots}$	4.11.2
$[f_{ij\ldots}]^{+}$	external limit of $f_{ij\ldots}$	4.11.2
$[f_{ij\ldots}]_{-}^{+}$	jump of $f_{ij\ldots}$	4.11.2
$f_{ij\ldots}^{\pm}$	arithmetic mean of $[f_{ij\ldots}]^{-}$ and $[f_{ij\ldots}]^{+}$	4.11.2
$f_{ij\ldots}^{(0)}$	initial part of $f_{ij\ldots}$	4.4.2
$df_{ij\ldots}$	differential of $f_{ij\ldots}$ for fixed t	2.2.4
$f_{ij\ldots,k}$	gradient of $f_{ij\ldots}$ with respect to X_k	2.2.2
$f_{ij\ldots}^{\mathrm{P}}$	Cartesian tensor field per material unit area (Piola field)	4.2.2

Symbol	*Name*	*Reference*
$f^*_{ij\ldots}$	spatial mean of $f_{ij\ldots}$	3.4.1
$\mathcal{G}$	integral entropy-input rate	4.8.1
$G_{ij\ldots}$	$F_{ij\ldots}$ per unit mass	4.4.1
g_n	auxiliary function	6.4.3
g_i	force per unit mass (gravity)	3.2.1
H_+	right-handed Heaviside step function	5.4.5
H_i	entropy flux per spatial unit area (entropy-flux density)	4.8.1
H^{C}_{ij}	Cauchy deformation	2.5.1
h	thickness of thin disk	4.11.1
h'	coordinate normal to $\partial\mathcal{R}^{(1)}$	4.11.1
h^{P}_i	entropy flux per material unit area (Piola enropy-flux density)	4.8.2
h^{G}_{ij}	Green deformation	2.5.2
$J^{\mathbf{r}}$	spatial Jacobian determinant	2.2.1
$j^{\mathbf{X}}$	material Jacobian determinant	2.2.1
K	fluid bulk modulus	6.3.2
k	elastic bulk modulus	5.2.5
$\mathcal{L}$	Laplace-transform functional	A.1
$\mathcal{L}^{-1}$	inverse Laplace-transform functional	A.2
$\mathcal{M}$	integral mass	4.4.1
m_1	first relaxation function	5.4.2
m_2	second relaxation function (shear-relaxation function)	5.4.2
m_3	bulk-relaxation function	5.4.3
m_{ijkl}	elasticity tensor	5.2
$m_{ijkl}(\tau)$	relaxation tensor	5.4.1
$N^{\mathbf{X}}_i$	material unit vector collinear with d^2X_i	2.2.4
$N^{\mathbf{X}+}_i$	outward material unit vector collinear with d^2X_i	4.11.2
N_{ijkl}	viscosity tensor	5.3.2
$n^{\mathbf{r}}_i$	spatial unit vector collinear with d^2r_i	2.2.4
$n^{\mathbf{r}+}_i$	outward spatial unit vector collinear with d^2r_i	3.3
$n^{\mathbf{r}(k)}_i$	principal direction of T_{ij}	3.6.1
o	origin of r_i coordinate system	3.4.1
$\mathcal{P}_i$	integral linear momentum	4.5.1
$P^{\mathbf{X}}_i$	material unit vector collinear with dX_i	2.2.4
$P^{\mathbf{X}(k)}_i$	principal direction of e_{ij}	2.7.1
p	mechanical pressure	3.6.2

Symbol	*Name*	*Reference*
$p_i^{\mathbf{r}}$	spatial unit vector collinear with dr_i	2.2.4
Q_i	heat flux per spatial unit area (heat-flux density)	4.7.1
$Q_i^{\mathbf{X}}$	material unit vector collinear with dX_i	2.4
q_i^{P}	heat flux per material unit area (Piola heat-flux density)	4.7.2
$q_i^{\mathbf{r}}$	spatial unit vector collinear with dr_i	2.6
$\mathcal{R}$	spatial 3-D domain (spatial volume)	2.2.1
$\partial\mathcal{R}$	spatial spatial 2-D domain (spatial boundary) confining $\mathcal{R}$	3.2.1
$\mathcal{R}^{(1)}$	internal spatial 3-D domain (internal spatial volume)	6.3
$\partial\mathcal{R}^{(1)}$	spatial interface between $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$	6.3
$\mathcal{R}^{(2)}$	external spatial 3-D domain (external spatial volume)	6.3
$\partial\mathcal{R}^{(2)}$	spatial boundary confining $\mathcal{R}^{(2)}$	6.3
$\mathbf{r}$	spatial position	2.2.1
r_i	Lagrangian representation of $\mathbf{r}$ (current position)	2.2.1
$r_{i,j}^{-1}$	inverse of $r_{i,j}$	2.2.3
dr	magnitude of dr_i	2.2.4
dr_i	spatial 1-D differential (spatial differential length)	2.2.4
d^2r	magnitude of d^2r_i	2.2.4
d^2r_i	spatial 2-D differential (spatial differential area)	2.2.4
d^3r	spatial 3-D differential (spatial differential volume)	2.2.4
S	entropy per unit mass	4.8.1
$\mathcal{S}$	integral entropy-increase rate	4.8.1
s	inverse Laplace time	6.4.2, A.1
T_i	force per spatial unit area (Cauchy traction)	3.2.1
$T_i^{(j)}$	Cauchy traction across plane normal to r_j coordinate	3.3
$T_i^{(\mathbf{n})}$	Cauchy traction across plane normal to n_i	3.3
$T_i^{(k)}$	principal value of T_{ij}	3.6.1
T_{ij}	Cauchy stress	3.4.1
$T_{(ij)}$	spherical part of T_{ij}	3.6.2
$T_{[ij]}$	deviatoric part of T_{ij}	3.6.2
T_{ij}^{C}	conservative Cauchy stress	4.10
T_{ij}^{D}	dissipative Cauchy stress	4.10
T_{ij}^{K}	Kirchhoff stress	3.5.3
t	current time epoch	2.2.1
t'	excitation time epoch	5.4.1
D/Dt	spatial time-derivative operator	2.3.1

Symbol	*Name*	*Reference*
D_t	spatial time-derivative operator	4.3
d/dt	material time-derivative operator	2.3.1
d_t	material time-derivative operator	4.3
δ_t	material time-derivative operator applied to work or heat	4.7
t_i^{P}	force per material unit area (Piola traction)	3.2.2
t_{ij}^{F}	Finger stress	3.5.2
t_{ij}^{P}	Piola stress	3.5.1
U	internal energy per unit mass	4.7.1
u_i	displacement	2.2.1
$\mathcal{V}$	current volume of thin disk	4.11.1
v_i	velocity	2.3.1
w	strain energy per material unit volume	5.2.3
$\mathcal{X}$	material 3-D domain (material volume)	2.2.1
$\partial\mathcal{X}$	material 2-D domain (material boundary) confining $\mathcal{X}$	2.2.1
$\mathcal{X}^{(1)}$	internal material 3-D domain (internal material volume)	6.2
$\partial\mathcal{X}^{(1)}$	material interface between $\mathcal{X}^{(1)}$ and $\mathcal{X}^{(2)}$	6.2
$\mathcal{X}^{(2)}$	external material 3-D domain (external material volume)	6.2
$\partial\mathcal{X}^{(2)}$	material boundary of $\mathcal{X}^{(2)}$	6.2
$\mathbf{X}$	material position	2.2.1
X_i	Eulerian representation of $\mathbf{X}$ (initial position)	2.2.1
$X_{i,j}^{-1}$	inverse of $X_{i,j}$	2.2.3
dX	magnitude of dX_i	2.2.4
dX_i	material 1-D differential (material differential length)	2.2.4
d^2X	magnitude of d^2X_i	2.2.4
d^2X_i	material 2-D differential (material differential area)	2.2.4
d^3X	material 3-D differential (material differential volume)	2.2.4
Z	stress power per spatial unit volume	5.3.6

B.2 Greek symbols

Symbol	*Name*	*Reference*
α_i	infinitesimal rotation vector	2.4
α_{ij}	infinitesimal rotation tensor	2.4
δ	Dirac delta function	4.11.1

Symbol	*Name*	*Reference*
δ_{ij}	Kronecker symbol	2.2.3
∂	partial-derivative operator	2.2.1
η	second viscosity parameter (shear viscosity)	5.3.2
Θ	thermodynamic temperature	4.8.1
κ	bulk viscosity	5.3.3
λ	first Lamé parameter	5.2.4
μ	second Lamé parameter (elastic shear modulus)	5.2.4
ϵ_{ijk}	Levi–Civita symbol	2.2.3
ρ	mass per spatial unit volume (volume-mass density)	3.2.1
τ	lapse time interval	5.4.1
π	3.14159...	—
Π	thermodynamic pressure	5.3.1
Σ	mass per spatial unit area (interface-mass density)	4.11.1
σ^{P}	mass per material unit area (Piola interface-mass density)	4.11.2
χ	first viscosity parameter	5.3.2
Ψ	Helmholtz free energy per unit mass	4.9.1
Ω	angular speed	2.6
Ω_i	vorticity vector	2.6
Ω_{ij}	vorticity tensor	2.6

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