Geodesy beyond 2000: An attempt to unify geodesy by the geodesic flow in all branches

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Abstract

According to the idea of Prof. Grafarend, Department of Geodetic Science, Stuttgart University, we attempt to present a unified theory of geodesy by the geodesic flows. In the paper, the plumblines related to the orthometric heights, dynamics of satellite motions, signal transit path in the satellite geodesy, and the geodetic main problems in the geometric geodesy are uniformly interpreted as geodesic flows. Three fundamental elements in the unified theory of geodesy are discussed: First, the interpretation of geodesy problems as geodesic flows. Second, the definition the geodesic manifolds. Third, the determination of the embedding spaces.

Keywords: geodesic flow, metrics, variational principle, embedding spaces

1. Introduction

Geodesics, in particular minimal geodesics, are of focal geodetic interest. For example, the differential equations of geodesic flows are used for solving the geodetic main problems on the ellipsoid of revolution. Hotine (1966) and Marussi (1985, pp.169-172) used the Maupertuis-Euler-Lagrange-Jacobi variational principle of least action to describe the plumbline as geodesics. Some other related discussions in geodesy can be also found in e.g. Heitz, 1988, Moritz, 1994, Schwarz et al., 1993, You, 1995, Zund, 1994. They focused to describe some problems in geodesy as geodesic problems. The plumb line, trajectories of satellite orbital motions and the propagation paths of electromagnetic waves/light rays can be interpreted as geodesics according to their studies. Grafarend presented the unified concept of geodesy in 1973 and then try to establish the unified theory of geodesy (1973, 1989, 1994, 1995, 1997) such as in physics where the gravitation, magnetic force and electronic force etc. are described in a same mathematical form, a unified theory in physics. The central point of the unified theory is the geodesics. In the paper, we try to describe some problems in geodesy by using the same mathematical form, i.e. a unified theory.

2. Lagrange portray, Halmilton portray and Newton form of geodesics

Lagrange portray: If the Riemannian metric of general form

$$ds^2 = g_{\alpha\beta}\, dq^\alpha dq^\beta \,, \qquad \alpha,\beta \in \{1,2,\cdots,n\} \tag{1}$$

exists, the differential equations of the geodesic in the Riemannian manifold can be derived by the Euler-Lagrange variational principle for the fixed boundary points

$$\delta \int L \, ds = 0$$
$$L = \sqrt{g_{\alpha\beta} \frac{dq^\alpha}{ds} \frac{dq^\beta}{ds}} \tag{2}$$

The sufficient and necessary conditions

$$\frac{\partial L}{\partial q^\alpha} - \frac{d}{ds}\left(\frac{\partial L}{\partial \frac{dq^\alpha}{ds}} \right) = 0 \tag{3}$$

must be satisfied and then lead to the differential equations of the geodesic

$$\frac{d^2 q^\alpha}{ds^2} + \begin{Bmatrix} \alpha \\ \beta \quad \gamma \end{Bmatrix} \frac{dq^\beta}{ds} \frac{dq^\gamma}{ds} = 0, \qquad \alpha, \beta, \gamma \in \{1, 2, \cdots, n\} \tag{4}$$

Hamilton portray: One can describe the differential equations of geodesics as a system of differential equations of first order in phase space by using the Hamilton principle for the fixed points:

$$\delta \int (p_\alpha \frac{dq^\alpha}{ds} - \mathrm{H}) \, ds = 0$$
$$H = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta \tag{5}$$

where p is the general momentum. A minimal geodesic in phase space is now represented by

$$\frac{dq^\alpha}{ds} = \frac{\partial H}{\partial p_\alpha} = g^{\alpha\beta} p_\beta$$
$$\frac{dp_\alpha}{ds} = -\frac{\partial H}{\partial q^\alpha} = -\frac{1}{2} \frac{\partial g^{\beta\gamma}}{\partial q^\alpha} p_\beta p_\gamma \tag{6}$$

If the Riemannian manifold is characterized by conformal coordinates, the metric becomes

$$ds^2 = e^{2\sigma} \delta_{\alpha\beta} \, dq^\alpha dq^\beta \,, \qquad \alpha, \beta \in \{1, 2, \cdots, n\} \tag{7}$$

Then, a geodesic flow takes the form of the Newton law (Grafarend et al. 1995)

$$\frac{d^2 q^\alpha}{dt^2} - \frac{1}{2} \frac{\partial e^{2\sigma}}{\partial q^\alpha} = 0 \tag{8}$$

if the arc length s is replaced at the dynamic time t according to the Maupertuis gauge

$$ds = e^{2\sigma} \, dt \tag{9}$$

The factor of conformality $e^{2\sigma}$ is the source of the "conservative force field".

3. Some metrics in geodesy

In this section, we summarize some metrics that are often used in the geometric geodesy, physical geodesy and satellite geodesy.

(1) metric of an ellipsoid of revolution

a. in geographic coordinates
(Manifold: ellipsoid)

$$
\begin{aligned}
& ds^2 = M^2 d\phi^2 + N^2 \cos^2\phi \, d\lambda^2 \\
& \text{where} \\
& M = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}} \\
& N = \frac{a}{(1-e^2\sin^2\phi)^{1/2}} \\
& a : \text{semi-major axis} \\
& e : \text{eccentricity of ellipsoid} \\
& \phi : \text{geographic latitude} \\
& \lambda : \text{geographic longitude}
\end{aligned}
\tag{10}
$$

b. in conformal coordinates
(Manifold: ellipsoid)

$$
\begin{aligned}
& ds^2 = k^2(y)(dx^2 + dy^2) \\
& \text{where} \\
& x = a(\lambda - \lambda_0) \\
& y = a \ln\left(\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{e/2} \right) \\
& k^2(y) = 1 + c_1 y^2 + c_2 y^4 + \cdots \\
& c_1, c_2, \cdots : \text{constants (Yang, 1998)}
\end{aligned}
\tag{11}
$$

(2) metric of the manifold of plumb line

a. Maupertuis metric (Grafarend et al., 1994, 1995)
(Manifold: conformally flat Riemannian manifold)

$$
\begin{aligned}
& ds^2 = G^2(W)\left(dx^2 + dy^2 + dz^2\right) \\
& x, y, z : \text{3D cartesian coordinates} \\
& G^2(W) : \text{any function of terrestrial potential } W
\end{aligned}
\tag{12}
$$

b. Hotine metric (Hotine, 1966)
(manifold: conformally flat Riemannian manifold)

$$ds^2 = \| grad\ W \|^2\ f^2(W)\left(dx^2 + dy^2 + dz^2\right) \tag{13}$$

$f(W)$: any function of terrestrial potential W

c. Marussi metric (Marussi, 1985, pp.169-172, Moritz, 1994)
(Manifold: conformally flat Riemannian manifold)

$$ds^2 = \| grad\ W \|^2 \left(dx^2 + dy^2 + dz^2\right) \tag{14}$$

(3) metric of the manifold of satellite orbits

a. Maupertuis metric of configuration space (Goenner et al., 1994, Ong, 1975, Synge, 1926, You, 1995)
(Manifold: conformally flat Riemannian manifold)

$$ds^2 = 2(E - V)\left(dx^2 + dy^2 + dz^2\right) \tag{15}$$

x, y, z : cartesian coordinates
V : axialsymmetric potential
E : conservative energy

b. Maupertuis metric of impulse space (Moser, 1970, You, 1995)
Manifold: conformally flat Riemannian manifold)

$$ds^2 = \frac{4}{(\vec{p}^2 - 2F)^2}\left((dp^1)^2 + (dp^2)^2 + (dp^3)^2\right) \tag{16}$$

p^1, p^2, p^3 : impulses
F : Kepler energy

c. Space-time metric (e.g. Misner et al., 1973)
(Manifold: pseudo Riemannian manifold)

$$\begin{aligned} ds^2 = {} & (-1 - \frac{2V}{c^2} - \frac{2V^2}{c^4})c^2dt^2 + (1 - \frac{2V}{c^2})\left(dx^2 + dy^2 + dz^2\right) - \\ & - \frac{\omega_1}{c^2}(cdt \cdot dx) - \frac{\omega_2}{c^2}(cdt \cdot dy) - \frac{\omega_3}{c^2}(cdt \cdot dz) \end{aligned} \tag{17}$$

t, x, y, z : 4D harmonic coordinates
V : potential
c : velocity of light
$\vec{\omega}$: Thirring - Lense vertorial potential

(4) refraction metric

a. non-relativistic case (Born, 1980)
(Manifold: conformally flat Riemannian manifold)

$$ds^2 = n^2\left(dx^2 + dy^2 + dz^2\right)$$
$$x, y, z : \text{cartesian coordinates} \tag{18}$$
$$n : \text{index of refraction}$$

b. relativistic case (non-dispersive) (Ehlers, 1967)
(Manifold: pseudo Riemannian manifold)

$$ds^2 = \bar{g}_{\mu\nu} dq^{\mu} dq^{\nu}$$
where
$$q^0 = ct,\ q^1 = x,\ q^2 = y,\ q^3 = z \tag{19}$$
$$\bar{g}_{\mu\nu} = g_{\mu\nu} + (1 - \frac{1}{n^2})\frac{u^{\mu}u^{\nu}}{c^2}$$
$$g_{\mu\nu} : \text{space - time metric}$$

4. Embedding onto the flat spaces

Consider the metric

$$ds^2 = g_{\alpha\beta}\ dq^{\alpha} dq^{\beta}\ , \qquad \alpha, \beta \in \{1, 2, \cdots, n\} \tag{20}$$

of an n-dimensional Riemannian manifold. In order to get a better insight into the geometry of a geodesic and its corresponding manifold, it needs to embed the manifold into an m-dimensional flat space whose metric is

$$dS^2 = e_{ii}\ du^i du^i\ , \qquad i \in \{1, 2, \cdots, m\} \tag{21}$$

In general, we take the locally isometric embedding theory to find the embedding space, i.e. $dS^2 = ds^2$. Then, the sufficient and necessary conditions

$$e_{ii}\frac{\partial u^i}{\partial q^{\alpha}}\frac{\partial u^i}{\partial q^{\beta}} = g_{\alpha\beta} \tag{22}$$

must be satisfied. After solving the above partial differential equations, the embedding space can be performed. But it is usually difficult to solve analytically this system of partial differential equations. Some authors use the transformation method as an alternative approach to find the embedding space, see e.g. Brinkmann, 1923, Goenner et al., 1994 and Rosen, 1965. One can easily obtain the embedding space of a conformally flat manifold by using the transformation of Brinkmann

According of the transformation methods, the embedding spaces of Marussi metric of plumblines in spherical symmetric fields are gives in the following:

(a) exterior space

$$\begin{aligned} ds^2 &= \frac{\mu}{r^4}^2 \left(dx^2 + dy^2 + dz^2\right) \\ &= \frac{\mu}{r^4}^2 (dr^2 + r^2 d\phi^2 + r^2 \cos\phi^2 d\lambda^2) \\ &= (du^1)^2 + (du^2)^2 + (du^3)^2 \end{aligned} \tag{23}$$

Embedding space $E^3(3,0)$:

$$\begin{aligned} u^1 &= \frac{\mu}{r} \cos\phi \ \cos\lambda \\ u^2 &= \frac{\mu}{r} \cos\phi \ \sin\lambda \\ u^3 &= \frac{\mu}{r} \sin\phi \end{aligned} \tag{24}$$

(b) interior space

$$\begin{aligned} ds^2 &= \frac{\mu}{R^4}^2 r^2 \left(dx^2 + dy^2 + dz^2\right) \\ &= \frac{\mu}{R^4}^2 r^2 (dr^2 + r^2 d\phi^2 + r^2 \cos\phi^2 d\lambda^2) \\ &= (du^1)^2 + (du^2)^2 + (du^3)^2 - (du^4)^2 \end{aligned} \tag{25}$$

R : constant radius

Embedding space $E^4(3,1)$:

$$\begin{aligned} u^1 &= \frac{\mu}{R^3} r^2 \cos\phi \ \cos\lambda \\ u^2 &= \frac{\mu}{R^3} r^2 \cos\phi \ \sin\lambda \\ u^3 &= \frac{\mu}{R^3} r^2 \sin\phi \\ u^4 &= \frac{\sqrt{3}}{2} \frac{\mu}{R^3} r^2 \end{aligned} \tag{26}$$

The embedding space of the Maupertuis manifold of satellite orbits can be found in Goenner (1994) and You (1998).

5. References

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