Random Stress/Strain Tensors and Beyond

Peiliang Xu

Abstract:

Tensors in the Earth Sciences are practically random, since they are either directly measured or indirectly inverted from other types of geo-measurements. Although random tensors have its root in multivariate analysis and nuclear physics, they are now actively investigated more as an independent topic of research; the results of random eigenvalues and eigenvectors from these investigations are mainly of asymptotic nature. In the Earth Sciences, the only result on random tensors was the accuracy of the random spectra of a random stress/strain tensor with a first order approximation. Recently, we have been working on random stress/strain tensors, the results from which are clearly borne in mind for use in the Earth Sciences. The purpose of this paper is to preliminarily review the progress of our recent studies of random second-rank symmetric (SRS) tensors. More specifically, our reviews are mainly limited to: (i) the exact distribution of the random spectra, which is numerically manageable since the dimension of tensors of geo-interest is low; (ii) the biases of the random spectra, which are physically very important but not investigated; and (iii) the accuracy of higher order approximation, which is needed if the ratio of signal to noise in stress/strain measurements is not sufficiently large. Since the eigenvector parameters are as important as the eigenvalues in the Earth Sciences, we have been paying due attention to them. On the other hand, we often encounter constrained tensors (deviatoric stress/strain tensors, pure shear tensors and seismic moment tensors, for example) in the Earth Sciences. Thus we also include the spectral theory of constrained random SRS tensors.

1 Introduction

Stress/strain tensors physically describe the deformation state of a solid deformable body. In the Earth Sciences, they are important to gain knowledge on the deformation of the solid Earth or parts of it (global geology and tectonics, for instance), and could be used to analyze and understand Earth material fracture. Crustal stress/strain tensors can either be directly measured or indirectly inverted. Direct *in situ* stress measurement has been typically based on drilling boreholes. The accuracy of *in situ* stress measurements is generally not better than $10-20$ per cent in magnitude and $10-20^o$ in orientation (Amadei & Stephasson 1997). Currently the worldwide deepest borehole is about $11.6km$. Since drilling a deep borehole is very expensive and technically limited, we can hardly expect to conduct often direct measurement of crustal stresses beyond $10km$ in depth. Inversion methods are thus frequently used to derive crustal stress/strain tensors. Geodetic data have been used to compute crustal strain tensors in lands (see, *e.g.* Frank 1966; Grafarend 1986; Prescott 1981; Savage & Burford 1970). Geometrical data of faults and earthquake focal mechanisms can be inverted for crustal stress tensors in lands and under seas (see, *e.g.* Angelier et al. 1982; Angelier 1984, 1989; Gephart & Forsyth 1984; Horiuchi et al 1995; Lu et al. 1997; McKenzie 1969; Wyss et al. 1992). The significant advantages of inversion include integration of all types of geophysical, geological, seismological and geodetic data, and improvement of spatial (horizontal and in-depth) resolution of the crustal stress field.

Stress/strain tensors are practically random, since they are either directly measured or indirectly inverted from other geo-measurements. The study of second-rank random tensors started in problems of nuclear physics (see, *e.g.* Mehta 1990) and multivariate statistical analysis (see, *e.g.* Anderson 1958). Unlike deterministic second-rank tensors, second-rank random tensors have to be investigated from the statistical point of view. Given a probability density function (pdf) for a second-rank random tensor, it is profoundly difficult to obtain the exact distribution of the random eigenspectra. Therefore the mathematical interest of second-rank random tensors has been basically focused on approximate and/or limit distributions, for instance, of the products of random matrices and/or the random eigenspectra (see, *e.g.* Anderson 1958; Mehta 1990; Girko 1979, 1985, 1989, 1990a, b, 1992, 1993, 1997; Cohen, Kesten & Newman 1985; Furstenberg & Kesten 1960; Boutet de Monvel, Khorunzhy & Vasilchuk 1996; Oravecz & Petz 1997).

Random tensors have only recently been investigated from the statistical point of view in the Earth Sciences. Since the tensors in the Earth Sciences are physical quantities and their dimensions are generally low (≤ 3 for stress/strain tensors and ≤ 6 for elastic material tensors), mathematically approximate/limit distributions of the random eigenspectra are of limited practical value. In fact, the study of random stress/strain has been focused on the following four aspects: (1) the exact distribution of the random principal stress/strain components, since the dimension of stress/strain tensors is not greater than three and since the number of measurements is always finite; (2) the accuracy of the random eigenspectra. The accuracy is generally not investigated in the mathematical literature of second-rank random tensors. It is however a routine indicator that must be attached to any estimated/derived geo-quantity; (3) the biases of the random eigenspectra. Since the mapping between a stress/strain tensor and its eigenspectra is nonlinear, the random eigenspectra are biased. The biases of the eigenspectra, except for some inequality results on the biases of the random eigenvalues (see *e.g.* Cacoullos 1965), have not been well investigated in the mathematical literature on second-rank random tensors. They can have an important role to play in correctly interpreting the estimated stress/strain field geophysically, however; and (4) the eigendirections. The eigendirections have been almost always treated as nuisance parameters in nuclear physics and multivariate analysis. Geophysically, the eigendirections are very important and thus cannot be ignored.

The first work on random tensors in the Earth Science was to compute the first-order accuracy of the principal eigenvalues of a second-rank symmetric (SRS) random tensor (Angelier et al, 1982 as an appendix and probably independently, Soler & van Gelder 1991). Kagan & Knopoff (1985a, b) studied statistically the first two moments of stochastic three-dimensional (3D) seismic moment tensor invariants, which were then used to explain complex fault geometry (Kagan 1992a). By theoretical arguments and simulations, Kagan (1990, 1992b, 1994) used the Cauchy distribution to study earthquake focal mechanisms and incremental stress distributions. The statistics of random tensors have also contributed to the automatic detection of polarized seismic wave forms (see, *e.g.* Samson 1977; Cichowicz 1993; Dai & MacBeth 1997) and can be important for accurately estimating, interpolating and extrapolating crustal stress orientations (see, *e.g.* Hansen & Mount 1990). Given a probability density function (pdf) for an unconstrained SRS random tensor, Xu & Grafarend (1996a, b) systematically derived the joint and marginal pdfs of the random eigenvalues and eigenvector parameters. By a pdf of a random tensor, we mean in this paper a joint pdf of the random components of a random tensor in a given coordinate system. The pdfs of different random tensor component sets represented in different coordinate systems can be associated with each other using the theorem of transformation of random variables. Xu and Grafarend also computed the biases of the eigenspectral parameters, since the mapping from an SRS random tensor to its random eigenvalues and random eigenvectors is nonlinear. The relationship among the estimated, expected and true strain ellipsis has been shown and possible physical implications noted in Xu (1996). Taking the effect of nonlinearity into account, Xu & Grafarend (1996a, b) then extended the first order accuracy computation of Angelier et al. (1982) and Soler & van Gelder (1991) to the second order approximation. The effect of nonlinearity on the inference of the relative principal stress components has been illustrated in Xu & Shimada (1997). Recently, Xu (1999) further extended the statistical theory for unconstrained SRS random tensors to the case of constrained SRS random tensors.

The purpose of this paper is basically to preliminarily review the progress on random stress/strain tensors. Although most tensors in geophysics are three-dimensional, the elastic material tensor is obviously six-dimensional. Thus we also consider n-D SRS random tensors whenever elegant results are obtainable. The paper is organized as follows. Section 2 first discusses the representation of 3D SRS stress/strain tensors and then n-D SRS tensors in the notation of rotation parameters. In Section 3, we present the differential forms and Jacobians of SRS tensors. Given a pdf for the original random tensor, we derive the joint and marginal pdfs of the random eigenvalues and eigendirection parameters in Section 4. Finally we discuss the biases and accuracy of the random eigenspectra in Section 5. Asympotic results of random singular values and random eigenvectors are not included, which can be found in a series of papers by Girko (see the references). Statistical correlation of invariants of stochastic 3D tensors is not included in this paper either, for which the reader is referred to Kagan & Knopoff (1985a, b) and Kagan (1992a).

2 Representation of symmetric second-rank tensors

2.1 Representation of 3D stress/strain tensors

The state of stress/strain at a point in the system of 3D rectangular Cartesian coordinates x, y, z can be described by a 3D SRS tensor as follows:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_z \end{bmatrix}, \tag{1}$$

where $\tau_{xy} = \tau_{xy}$, $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$. τ_x, τ_y and τ_z are the normal stresses/strains, and τ_{xy}, τ_{xz} and τ_{yz} the shear stresses/strains. Obviously a full stress/strain tensor has six independent components. Consider a second system of rectangular Cartesian coordinates x', y', z' with the same origin as the first Cartesian coordinate system x, y, z but different orientations. The relationship between these two Cartesian systems of coordinates is given by the following system of equations:

$$\left.\begin{array}{l} x' = u_{x'x}x + u_{x'y}y + u_{x'z}z \\ y' = u_{y'x}x + u_{y'y}y + u_{y'z}z \\ z' = u_{z'x}x + u_{z'y}y + u_{z'z}z \end{array}\right\},$$

or in matrix form:

$$\mathbf{x}' = \mathbf{U}^T\mathbf{x}, \tag{2}$$

where $u_{i'j}$ are the direction cosines of the i'-axis with respect to the j-axis, and

$$\mathbf{U} = \begin{bmatrix} u_{x'x} & u_{y'x} & u_{z'x} \\ u_{x'y} & u_{y'y} & u_{z'y} \\ u_{x'z} & u_{y'z} & u_{z'z} \end{bmatrix}, \tag{3}$$

$$\mathbf{x}' = (x',\ y',\ z')^T,$$

$$\mathbf{x} = (x,\ y,\ z)^T.$$

Using the law of transformation for second-rank Cartesian tensors, we have the stress/strain tensor at the point in the new Cartesian coordinate system as follows:

$$\boldsymbol{\tau}' = \mathbf{U}^T \boldsymbol{\tau} \mathbf{U}. \tag{4}$$

Among $\boldsymbol{\tau}'$ in (4), the one without off-diagonal elements is physically very important, whose diagonal elements are called the principal stress/strain components. Since $\mathbf{U}$ is orthogonal and if specially chosen, it is indeed mathematically possible to diagonalize the original stress/strain tensor $\boldsymbol{\tau}$. Denoting the diagonal $\boldsymbol{\tau}'$ by $\mathbf{Y}$ with three diagonal elements y_1, y_2 and y_3, we can rewrite (4) as follows:

$$\boldsymbol{\tau} = \mathbf{U}\mathbf{Y}\mathbf{U}^T, \tag{5}$$

where

$$\mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbf{I}. \tag{6}$$

The spectral decomposition (5) is not unique. In fact, the change of any column vector of $\mathbf{U}$ in sign will not change the decomposition (5). In order to make the spectral decomposition (5) unique, one has to pre-define the orientations of the three principal stress/strain components through the manipulation of the orthogonal matrix $\mathbf{U}$. Since the number of independent components of $\boldsymbol{\tau}$ is six, the total number of independent parameters that can be used to represent the orientations of the three principal stress/strain axes is equal to three. Practically, we may use one of the following methods to construct the matrix $\mathbf{U}$ of direction cosines in (3): (i) the three Euler angles; (ii) the four Rodrigues quaternion parameters; and (iii) three rotation angles. The three Euler angles have been defined as follows (Altmann 1986): first rotation $\mathbf{U}(\gamma\mathbf{z})$ by γ around the z-axis, second rotation $\mathbf{U}(\beta\mathbf{y})$ by β around the y-axis, and finally third rotation $\mathbf{U}(\alpha\mathbf{z})$ by α around the z-axis again. Thus the matrix $\mathbf{U}$ can be represented using the three Euler angles α, β and γ, as follows:

$$\begin{aligned}\mathbf{U} &= \mathbf{U}(\alpha\mathbf{z})\mathbf{U}(\beta\mathbf{y})\mathbf{U}(\gamma\mathbf{z}) \\ &= \begin{bmatrix} \cos\alpha\,\cos\beta\,\cos\gamma - \sin\alpha\,\sin\gamma & -\cos\alpha\,\cos\beta\,\sin\gamma - \sin\alpha\,\cos\gamma & \cos\alpha\,\sin\beta \\ \sin\alpha\,\cos\beta\,\cos\gamma + \cos\alpha\,\sin\gamma & -\sin\alpha\,\cos\beta\,\sin\gamma + \cos\alpha\,\cos\gamma & \sin\alpha\,\sin\beta \\ -\sin\beta\,\cos\gamma & \sin\beta\,\sin\gamma & \cos\beta \end{bmatrix},\end{aligned} \tag{7}$$

where

$$\mathbf{U}(\alpha\mathbf{z}) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{U}(\beta\mathbf{y}) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},$$

and

$$\mathbf{U}(\gamma\mathbf{z}) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the three Euler angles can represent any point on a unit sphere, the spectral decomposition (5) will not be unique. In order to uniquely define (5), we can trivially confine the definition domains of the three Euler angles to: $-\pi/2 \le \alpha \le \pi/2$, $0 \le \beta \le \pi$ and $-\pi/2 \le \gamma \le \pi/2$.

The second approach to construct the matrix $\mathbf{U}$ is based on the rotation on a unit sphere. The law of motion on a unit sphere is actually the rotation around an axis by some angle (see, *e.g.* Altmann 1986), which can be symbolically written as $\mathbf{U}(\phi\mathbf{n})$, where $\phi \in [0,\, \pi]$ is the rotation angle and $\mathbf{n}$ is the unit vector of rotation. The matrix $\mathbf{U}(\phi\mathbf{n})$ can be elegantly represented using the Rodrigues quaternion parameters and is given as follows:

$$\mathbf{U} = \mathbf{U}(\phi\mathbf{n}) = \begin{bmatrix} \mu^2 + m_x^2 - m_y^2 - m_z^2 & 2(m_x m_y - \mu m_z) & 2(m_z m_x + \mu m_y) \\ 2(m_x m_y + \mu m_z) & \mu^2 - m_x^2 + m_y^2 - m_z^2 & 2(m_z m_y - \mu m_x) \\ 2(m_z m_x - \mu m_y) & 2(m_z m_y + \mu m_x) & \mu^2 - m_x^2 - m_y^2 + m_z^2 \end{bmatrix}, \quad (8)$$

(Altmann 1986), where $\mu = \cos(\phi/2)$ and $\mathbf{m} = \sin(\phi/2)\mathbf{n}$. Since a 3D orthogonal matrix has only three independent parameters, the four Rodrigues parameters μ and the unit direction vector $\mathbf{n}$ are not independent but have to satisfy $\mu^2 + ||\mathbf{m}||^2 = 1$. As in the case of the Euler angles, the matrix $\mathbf{U}$ of (8) by the four Rodrigues parameters cannot uniquely determine (5), since the vector of direction cosines $\mathbf{n}$ can point to any point on the unit sphere. A unique decomposition thus requires that $\mathbf{n}$ only point to the space of $z \geq 0$ and $x \geq 0$ (or $y \geq 0$).
The third method, which has been used in the study of random tensors by Xu (1996, 1999) and Xu & Grafarend (1996a, b), is to construct the matrix $\mathbf{U}$ by:

$$\mathbf{U} = \mathbf{U}_{32}\mathbf{U}_{31}\mathbf{U}_{21}, \quad (9)$$

where

$$\mathbf{U}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{32} & \sin\theta_{32} \\ 0 & -\sin\theta_{32} & \cos\theta_{32} \end{bmatrix},$$

$$\mathbf{U}_{31} = \begin{bmatrix} \cos\theta_{31} & 0 & \sin\theta_{31} \\ 0 & 1 & 0 \\ -\sin\theta_{31} & 0 & \cos\theta_{31} \end{bmatrix},$$

$$\mathbf{U}_{21} = \begin{bmatrix} \cos\theta_{21} & \sin\theta_{21} & 0 \\ -\sin\theta_{21} & \cos\theta_{21} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and the three angles θ_{21}, θ_{31} and θ_{32} are all defined between $-\pi/2$ and $\pi/2$. Thus the spectral decomposition (5) is unique.
Although the matrix $\mathbf{U}$ in (5) can be represented using different sets of rotations, even different ways of parameterization (see, *e.g.* Xu & Grafarend 1996a), all these different representations of $\mathbf{U}$ are mathematically equivalent. The relationship between the three Euler angles and the four Rodrigues quaternion parameters has been given in Altmann (1986). Given the joint pdf of one set of parameters for $\mathbf{U}$, one can also trivially write the joint pdf of the other set of parameters through the Jacobian between these two sets of parameters. However, the difficulty in the computation of Jacobians is different from one set of parameters to the other. On the other hand, it is not convenient to generalize the Euler representation (7) to the n-D case. The same is true for the Rodrigues quaternion representation. Even in the 3D case, it is more difficult to obtain the Jacobian by using the four Rodrigues quaternion parameters than by using our notation (9). Thus in the study of SRS random stress/strain tensors and beyond, we will confine ourselves to the representation (9) in this paper.

2.2 Representation of n-D SRS tensors

For an n-D SRS tensor $\mathbf{\Gamma}$, we can always spectrally decompose it as follows:

$$\mathbf{\Gamma} = \mathbf{U}\mathbf{Y}\mathbf{U}^T, \quad (10)$$

where $\mathbf{Y}$ is the diagonal matrix with the eigenvalues y_1, y_2, ..., y_n in decreasing order, *i.e.* $y_1 \geq y_2 \geq ... \geq y_n$, $\mathbf{U}$ is the orthonormal matrix of the eigenvectors, satisfying the following condition:

$$\mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbf{I}. \quad (11)$$

Because the n-D SRS tensor $\mathbf{\Gamma}$ has $N = n(n+1)/2$ independent components and because there are n independent eigenvalues, the normalized eigenvector matrix in (10) can only have $M = n(n-1)/2$ functionally independent components. If $\mathbf{U}$ is represented with M independent parameters, we can then solve for the n eigenvalues and M eigendirection parameters from the matrix equation (10). The solution $(\mathbf{Y}, \mathbf{U})$ is not unique, however. In order to obtain a unique solution $(\mathbf{U},\ \mathbf{Y})$ to (10), one can either impose positive phases to the first elements of all the eigenvectors (Girko 1985; Mehta 1990) or properly select the eigenvector parameters with proper definition domains (Xu & Grafarend 1996a, b).

Following Xu & Grafarend (1996b) and Xu (1999), we define the orthonormal $\mathbf{U}$ as a product of M rotation matrices $\mathbf{U}_{ij}$ $(i > j)$, where $\mathbf{U}_{ij}$ is given by

$$\mathbf{U}_{ij} = \begin{bmatrix} \mathbf{I}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cos\theta_{ij} & \mathbf{0} & \sin\theta_{ij} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\sin\theta_{ij} & \mathbf{0} & \cos\theta_{ij} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_3 \end{bmatrix}, \tag{12}$$

where $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$ are the identity matrices of different orders, $-\pi/2 \leq \theta_{ij} \leq \pi/2$, and $\mathbf{0}$ is either a zero matrix or a zero (row or column) vector. Thus the matrix $\mathbf{U}$ can be written as

$$\mathbf{U} = \mathbf{U}_{n(n-1)} ... \mathbf{U}_{32}\mathbf{U}_{n1} ... \mathbf{U}_{31}\mathbf{U}_{21}. \tag{13}$$

3 Differentials and Jacobians of SRS tensors

3.1 Differentials of SRS tensors

3.1.1 Differentials of SRS tensors without constraints

Differentiating (10) and (11) and then combining these two differentials, we obtain:

$$d\mathbf{\Gamma} = \mathbf{U}(d\mathbf{Y} + \mathbf{U}^T d\mathbf{U}\mathbf{Y} - \mathbf{Y}\mathbf{U}^T d\mathbf{U})\mathbf{U}^T, \tag{14}$$

(see *e.g.* Xu & Grafarend 1996b).
Substituting (13) into (14) yields:

$$d\mathbf{\Gamma} = \mathbf{U}(d\mathbf{Y} + \sum_{k=1}^{M} \mathbf{H}_k d\theta_k)\mathbf{U}^T, \tag{15}$$

where

$$\begin{aligned} d\theta_k &= d\theta_{ij}, \\ \mathbf{H}_k &= \mathbf{U}^T_{II(k)}\mathbf{O}_{ij(k)}\mathbf{U}_{II(k)}\mathbf{Y} - \mathbf{Y}\mathbf{U}^T_{II(k)}\mathbf{O}_{ij(k)}\mathbf{U}_{II(k)}, \\ \mathbf{O}_{ij(k)} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -1 & \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \\ \mathbf{U}_{I(k)} &= \mathbf{U}_{n(n-1)}\mathbf{U}_{n(n-2)} ... \mathbf{U}_{i(j+1)}, \\ \mathbf{U}_{II(k)} &= \mathbf{U}_{i(j-1)} ... \mathbf{U}_{32}\mathbf{U}_{n1} ... \mathbf{U}_{31}\mathbf{U}_{21}. \end{aligned}$$

Since the tensor $\mathbf{\Gamma}$ is symmetric, we vectorize both sides of (15) by eliminating the elements in the upper triangle of each symmetric tensor and obtain:

$$v(d\mathbf{\Gamma}) = \mathbf{D}_n^+(\mathbf{U} \otimes \mathbf{U})\mathbf{D}_n \ v\{d\mathbf{Y} + \sum_{k=1}^{M} \mathbf{H}_k d\theta_k\}, \tag{16}$$

where $\mathbf{D}_n$ is the *duplication matrix* and $\otimes$ stands for Kronecker product of matrices (Magnus & Neudecker 1988).
For a full 3D stress/strain tensor, (16) becomes

$$v(d\boldsymbol{\tau}) = \mathbf{D}_3^+(\mathbf{U}\otimes\mathbf{U})\mathbf{D}_3\; v\{d\mathbf{Y} + \mathbf{H}_{21}d\theta_{21} + \mathbf{H}_{31}d\theta_{31} + \mathbf{H}_{32}d\theta_{32}\}, \tag{17}$$

where $\mathbf{U}$ is given by (9), the three non-zero elements of $d\mathbf{Y}$ are dy_1, dy_2 and dy_3, and the three matrices $\mathbf{H}_{21}$, $\mathbf{H}_{31}$ and $\mathbf{H}_{32}$ are respectively given by:

$$\mathbf{H}_{21} = \begin{bmatrix} 0 & y_2 - y_1 & 0 \\ & 0 & 0 \\ symm & & 0 \end{bmatrix},$$

$$\mathbf{H}_{31} = \begin{bmatrix} 0 & 0 & (y_3 - y_1)\cos\theta_{21} \\ & 0 & (y_3 - y_2)\sin\theta_{21} \\ symm & & 0 \end{bmatrix},$$

$$\mathbf{H}_{32} = \begin{bmatrix} 0 & (y_2 - y_1)\sin\theta_{31} & -(y_3 - y_1)\sin\theta_{21}\cos\theta_{31} \\ & 0 & (y_3 - y_2)\cos\theta_{21}\cos\theta_{31} \\ symm & & 0 \end{bmatrix}.$$

3.1.2 Differentials of constrained SRS tensors

Many important SRS tensors in the Earth Sciences are conditionally constrained. A deviatoric 3D stress/strain tensor has only five independent components and is subject to the traceless constraint. In physical geodesy, The Laplace equation of the geopotential field also demands that all gravity tensors be traceless. A double-couple (DC) point source in seismology can be represented by a 3D SRS seismic moment tensor whose components are subject to zero isotropic and zero intermediate principal component constraints (Aki & Richards 1980; Kostrov & Das 1988; Lay & Wallace 1995). Mathematically, the pure shear tensor on a certain oriented plane is of the same form as a DC seismic moment tensor in seismology.
Assume that the components of a 3D SRS tensor $\boldsymbol{\tau}$ are constrained by:

$$\mathbf{h}(\boldsymbol{\tau}) = 0, \tag{18}$$

where $\mathbf{h}(\boldsymbol{\tau})$ is an m_c-dimensional function vector of the components of the tensor $\boldsymbol{\tau}$. The spectral decomposition of a constrained 3D SRS tensor is now equivalent to simultaneously solving (5) and (18) for the eigenvalues and the eigendirection parameters. Since (18) imposes m_c conditions on the components of the 3D SRS tensor $\boldsymbol{\tau}$, the total number of independent equations for the spectral decomposition is equal to $(6 - m_c)$. Thus there can only be $(6 - m_c)$ independent eigenvalues and eigendirection parameters. In the similar manner, in order to derive the differential relationship between a set of $(6 - m_c)$ independent tensor components and $(6 - m_c)$ eigenvalues and eigendirection parameters, we have to first differentiate (18) and then combine it with (17). For more details, we refer the reader to Xu (1999). To summarize, we give the differential relations between constrained 3D SRS tensor components and their eigenvalues and eigendirection parameters as follows:

- **for 3D deviatoric stress/strain tensors:**
  The constraint condition of a 3D deviatoric stress/strain tensor can be expressed either by

  $$\tau_x + \tau_y + \gamma_z = 0,$$

  in terms of the original tensor components, or by

  $$y_1 + y_2 + y_3 = 0,$$

in terms of the eigenvalues ($y_1 \geq y_2 \geq y_3$).

For 3D deviatoric (traceless) stress/strain tensors with (9), (17) becomes:

$$v_{c_1}(d\boldsymbol{\tau}) = \mathbf{D}_{3c_1}^{+}(\mathbf{U} \otimes \mathbf{U})\mathbf{D}_{3c_1} v_{c_1}\{d\mathbf{Y} + \mathbf{H}_{21}d\theta_{21} + \mathbf{H}_{31}d\theta_{31} + \mathbf{H}_{32}d\theta_{32}\}, \tag{19}$$

where $v_{c_1}(d\boldsymbol{\tau})$ stands for the vectorization operation under the traceless constraint, which is actually equal to $v(d\boldsymbol{\tau})$ without the last element if τ_z and y_3 are taken as nuisance parameters, the matrices $\mathbf{U}$, $\mathbf{H}_{21}$, $\mathbf{H}_{31}$ and $\mathbf{H}_{32}$ are the same as in (17), $\mathbf{D}_{3c_1}$ is the *duplication matrix* in the presence of the traceless constraint and is given by:

$$\mathbf{D}_{3c_1} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 \end{array}\right].$$

- **for the pure shear tensor on an oriented plane:**
  A pure shear tensor on an oriented plane or a 3D DC SRS seismic moment tensor can be represented by four independent tensor components (see, *e.g.* Brekhovskikh & Goncharov 1994; Lay & Wallace 1995). The two constraints for this type of tensors can be written as follows:

$$y_1 + y_3 = 0;$$
$$y_2 = 0,$$

in terms of the eigenvalues, or equivalently,

$$\tau_x + \tau_y + \tau_z = 0;$$
$$det\{\boldsymbol{\tau}\} = 0,$$

in terms of the original tensor components.

The two constraints for the pure shear tensor on an oriented plane will make only four independent equations from (5). Here we choose τ_x, τ_{xy}, τ_{xz} and τ_{yz}, and collect them into a vector,

$$v_{c_2}(d\boldsymbol{\tau}) = (\tau_x, \tau_{xy}, \tau_{xz}, \tau_{yz})^T.$$

Note, however, that the other two components τ_y and τ_z cannot be arbitrary but have to be solved using the two constraints specified in the above. Left-multiplying both sides of (17) by the matrix $\mathbf{D}_{c_2}$:

$$\mathbf{D}_{c_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

we obtain the differential form for the pure shear tensor,

$$v_{c_2}(d\boldsymbol{\tau}) = \mathbf{D}_{c_2}\mathbf{D}_3^{+}(\mathbf{U} \otimes \mathbf{U})\mathbf{D}_3\ v\{d\mathbf{Y} + \mathbf{H}_{21}d\theta_{21} + \mathbf{H}_{31}d\theta_{31} + \mathbf{H}_{32}d\theta_{32}\}. \tag{20}$$

- **for 3D SRS tensors with only one non-zero eigenvalue:**
  A 3D SRS tensor with only one non-zero eigenvalue has three distinct components and physically represents a completely polarized wave field (Samson 1977). Mathematically, it can be rewritten as follows:

$$\boldsymbol{\tau} = \mathbf{U} \begin{bmatrix} y & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^T = y\mathbf{u}_1\mathbf{u}_1^T, \tag{21}$$

  where y is the only non-zero eigenvalue. $\mathbf{U}$ is defined as in (9), and $\mathbf{u}_1$ is the first column vector of $\mathbf{U}$. Obviously the SRS tensor $\boldsymbol{\tau}$ in (21) has only three independent components. Except for the only non-zero eigenvalue y, and since $||\mathbf{u}_1|| = 1$, there can only be two independent components for $\mathbf{U}$. Without loss of generality, we can assume that y is non-negative and represent $\mathbf{U}$ in (9) simply as follows:

$$\mathbf{U} = \mathbf{U}_{31}\mathbf{U}_{21},$$

  or elementwise,

$$u_{11} = \cos\theta_{31}\cos\theta_{21},$$
$$u_{21} = -\sin\theta_{21}; \quad u_{31} = -\sin\theta_{31}\cos\theta_{21}.$$

  Differentiating both sides of (21), we have

$$d\boldsymbol{\tau} = \mathbf{u}_1\mathbf{u}_1^T dy + y\, d\mathbf{u}_1\mathbf{u}_1^T + y\, \mathbf{u}_1 d\mathbf{u}_1^T. \tag{22}$$

  Selecting the second column of $\boldsymbol{\tau}$ for (22), we have

$$\frac{\partial(\tau_{xy}, \tau_y, \tau_{yz})^T}{\partial(y, \theta_{21}, \theta_{31})} = \mathbf{M}_\theta \mathbf{Y}_y, \tag{23}$$

  where

$$\mathbf{Y}_y = \begin{bmatrix} \sin\theta_{21} & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y\sin\theta_{21}\cos\theta_{21} \end{bmatrix},$$

$$\mathbf{M}_\theta = \begin{bmatrix} -\cos\theta_{21}\cos\theta_{31} & -\cos 2\theta_{21}\cos\theta_{31} & \sin\theta_{31} \\ \sin\theta_{21} & \sin 2\theta_{21} & 0 \\ \cos\theta_{21}\sin\theta_{31} & \cos 2\theta_{21}\sin\theta_{31} & \cos\theta_{31} \end{bmatrix}.$$

  For the n-D case and more details of the derivation, the reader is referred to Xu (1999).

3.2 Jacobians of SRS tensors

For the tensor equation (10) with m_c constraints, if we pre-select a set of $(n(n+1)/2 - m_c)$ independent tensor components and the same number of independent eigenvalues y_i (in non-increasing order) and eigendirection parameters α_i, then the Jacobian of the components of the constrained SRS tensor $\boldsymbol{\Gamma}$ with respect to $\mathbf{Y}$ and $\mathbf{U}$ is defined as follows:

$$J(\mathbf{y}, \boldsymbol{\alpha}) = \left| det \left\{ \frac{\partial(\gamma_1, \gamma_2, ..., \gamma_{(n(n+1)/2-m_c)})^T}{\partial(y_1, y_2, ..., y_i, \alpha_1, \alpha_2, ..., \alpha_M)} \right\} \right|, \tag{24}$$

where $det\{.\}$ stands for the determinant of a square matrix, γ_j are the functionally independent components of the constrained SRS tensor $\boldsymbol{\Gamma}$ and $M = (n-1)n/2 - m_c - i$. In the case of $m_c = 0$, the SRS tensor $\boldsymbol{\Gamma}$ becomes unconstrained. The Jacobian of an unconstrained SRS tensor with respect to n eigenvalues and an arbitrary set of eigendirection parameters was given in implicit function form of the eigenparameters in Anderson (1958), Mehta (1990) and Girko (1985), which can be summarized as Theorem 1:

Theorem 1 *Let $\mathbf{\Gamma}$ be a real n-D SRS tensor, whose eigenvalues and eigendirection parameters are respectively denoted by $\mathbf{y} = (y_1, y_2, ..., y_n)$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_M)$. Then the Jacobian of $\mathbf{\Gamma}$ with respect to $\mathbf{y}$ and $\boldsymbol{\alpha}$ is given by*

$$J(\mathbf{y}, \boldsymbol{\alpha}) = \left| det \left\{ \frac{\partial v(\mathbf{\Gamma})}{\partial(\mathbf{y}, \boldsymbol{\alpha})} \right\} \right| = \prod_{i<j} (y_i - y_j) s(\boldsymbol{\alpha}), \tag{25}$$

where $s(\boldsymbol{\alpha})$ is an implicit function of $\boldsymbol{\alpha}$ but independent of $\mathbf{y}$.

For the proof of Theorem 1, the reader is referred to Mehta (1990) or Girko (1990b, Chap.3). Since the eigendirection parameters are generally of no interest in multivariate analysis and nuclear physics, the function $s(\boldsymbol{\alpha})$ is not derived explicitly. In mechanics and the Earth Sciences, the eigendirection parameters are as important as (if not more imporant than) the eigenvalues. Thus we have been trying to work out $s(\boldsymbol{\alpha})$, the results of which can be summarized by the following theorem:

Theorem 2 *Let $\mathbf{\Gamma}$ be a real n-D SRS tensor, whose eigenvalues and eigendirection parameters are respectively denoted by $\mathbf{y} = (y_1, y_2, ..., y_n)$ and $\boldsymbol{\theta} = (\theta_{21}, \theta_{31}, ..., \theta_{n1}, \theta_{32}, ..., \theta_{n(n-1)})$, as specified in (13). Then the Jacobian of $\mathbf{\Gamma}$ with respect to $\mathbf{y}$ and $\boldsymbol{\theta}$ is given by*

$$J(\mathbf{y}, \boldsymbol{\theta}) = \left| det \left\{ \frac{\partial v(\mathbf{\Gamma})}{\partial(\mathbf{y}, \boldsymbol{\theta})} \right\} \right| = \prod_{i<j} (y_i - y_j) \prod_{i=1}^{n-2} \prod_{j=i+2}^{n} (\cos\theta_{ji})^{j-i-1}. \tag{26}$$

For the proof of Theorem 2, the reader is referred to Xu (1999). In the 3D case of a full stress/strain tensor, the Jacobian (26) becomes:

$$J(y_1, y_2, y_3, \theta_{21}, \theta_{31}, \theta_{32}) = (y_1 - y_2)(y_2 - y_3)(y_1 - y_3) \ \cos\theta_{31}. \tag{27}$$

Using the differential results of (19), (20) and (23), we can readily summarize the Jacobian results of deviatoric stress/strain and pure shear tensors and the SRS tensor with only one non-zero eigenvalue in Table 1, where the four elements of $\mathbf{M}_{c2S}$ are given by

$$m_{c2S}^{11} = \cos 2\theta_{31}/3 + (3\cos^2\theta_{21} - 1)(2 - \cos^2\theta_{31})/3,$$

$$m_{c2S}^{12} = \sin 2\theta_{21} \sin\theta_{31}(2 - \cos^2\theta_{31}),$$

$$m_{c2S}^{21} = \sin^2\theta_{21} \sin 2\theta_{32},$$

and

$$m_{c2S}^{22} = \cos 2\theta_{32} - \sin 2\theta_{21} \sin\theta_{31} \sin 2\theta_{32}.$$

4 Probability distributions of random spectra

Let $\mathbf{T}$ be an n-D SRS random tensor whose components are subject to m_c equality constraints. Then in the similar manner to (10), we can decompose $\mathbf{T}$ as:

$$\mathbf{T} = \mathbf{G}\mathbf{\Lambda}\mathbf{G}^T, \tag{28}$$

subject to the m_c constraints, where $\mathbf{\Lambda}$ has i functionally independent random eigenvalues satisfying $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_i$, $\mathbf{G}$ is a random orthonormal matrix represented by $(n(n+1)/2 - m_c - i)$ functionally independent random eigendirection parameters. To represent $\mathbf{G}$, we use our rotation notations but not an arbitrary set of parameters as in Mehta (1990) or Girko (1985), for instance. Thus $\mathbf{G}$ is the same as $\mathbf{U}$ except that θ_{ij} in (10) are replaced with

Table 1: Jacobians of unconstrained and constrained 3D SRS tensors: Unconstrained – unconstrained 3D stress/strain tensors; Isotropic – traceless 3D stress/strain tensors; Pure Shear – pure shear tensors; Single Eigenvalue – 3D SRS tensors with one non-zero eigenvalue; and $g(\theta_{31}) = \cos^3\theta_{31}(1 + 2\sin^2\theta_{31})$. (modified from Xu (1999))

Models	y_1, y_2, y_3, θ_{21}, θ_{31}, θ_{32}
Unconstrained	$(y_1 - y_2)(y_1 - y_3)(y_2 - y_3)\cos\theta_{31}$
Isotropic	$(y_1 - y_2)(2y_1 + y_2)(y_1 + 2y_2)g(\theta_{31})$
Pure Shear	$2y_1^3 \cos\theta_{31}\lvert det\{\mathbf{M}_{c2S}\}\rvert$
Single Eigenvalue	$y^2\lvert\sin\theta_{21}\rvert^3\cos\theta_{21}$

ϕ_{ij}. Denote the probability density function (pdf) of the random tensor $\mathbf{T}$ by $f_T(\mathbf{\Gamma})$. Very often, if the components of the random SRS tensor can be approximately derived linearly from geodetic measurements, $\mathbf{T}$ has a Gaussian distribution; in multivariate analysis, $\mathbf{T}$ has a Wishart distribution if the observations are normally distributed. In this section, we will first discuss the joint pdf of the random eigenvalues and random rotations of $\mathbf{T}$ without constraints, and then focus on the random eigenvalues and random rotations of 3D random stress/strain tensors.

4.1 Probability distributions of n-D random spectra

For a full n-D SRS random tensor $\mathbf{T}$, we denote the n random eigenvalues and $n(n-1)/2$ random eigenvector parameters by $\boldsymbol{\lambda}$ and $\mathbf{\Phi}$, respectively. Then with the Jacobian (26), we obtain the joint pdf of the random eigenvalues $\boldsymbol{\lambda}$ and the random rotations $\mathbf{\Phi}$:

$$
\begin{aligned}
f_{\lambda\Phi}(\mathbf{y}, \boldsymbol{\theta}) &= f_T(\mathbf{UYU}^T)\left| det\left\{\frac{\partial v(\mathbf{\Gamma})}{\partial(\mathbf{y}, \boldsymbol{\theta})}\right\}\right| \\
&= \prod_{i<j}(y_i - y_j)\ f_T(\mathbf{UYU}^T)\prod_{i=1}^{n-2}\prod_{j=i+2}^{n}(\cos\theta_{ji})^{j-i-1}, \qquad (29)
\end{aligned}
$$

where $v(\mathbf{\Gamma})$ is the v-operation of $\mathbf{\Gamma}$, $\mathbf{y}$ is the vector of the eigenvalues, and $\boldsymbol{\theta}$ consists of all the rotations used to represent (13).
Hence the marginal (joint) pdf of the distinct random eigenvalues can be obtained by integrating (29) over the definition domain of $\boldsymbol{\theta}$:

$$
f_\lambda(\mathbf{y}) = \int_{\Omega_\theta}\prod_{i<j}(y_i - y_j)\ f_T(\mathbf{UYU}^T)\prod_{i=1}^{n-2}\prod_{j=i+2}^{n}(\cos\theta_{ji})^{j-i-1}\,d\Omega_\theta, \qquad (30)
$$

where the domain Ω_θ of $\boldsymbol{\theta}$ is defined by $-\pi/2 \leq \theta_{ij} \leq \pi/2$ for all $i < j \leq n$. In the similar manner, we can obtain the marginal pdf of the random rotations:

$$
f_\Phi(\boldsymbol{\theta}) = \int_{\Omega_y}\prod_{i<j}(y_i - y_j)\ f_T(\mathbf{UYU}^T)\prod_{i=1}^{n-2}\prod_{j=i+2}^{n}(\cos\theta_{ji})^{j-i-1}\,d\Omega_y, \qquad (31)
$$

where the domain Ω_y is defined by $-\infty < y_n \leq ... \leq y_2 \leq y_1 < \infty$. If the n-D SRS random tensor $\mathbf{T}$ is positive definite (the estimated variance-covariance matrix in multivariate analysis, for example), $y_n > 0$.

The marginal pdfs (30) and (31) of the random eigenvalues and random rotations can hardly have a simple analytical expression generally, unless $f_T(\boldsymbol{Gamma})$ takes a certain special form. In this case, the probability of the random eigenvalues or random rotations can be too approximate to use if it is computed by using the formulas in Stroud (1971) if the dimension n is large. However, for a certain class of pdfs of n-D SRS random tensors $\mathbf{T}$, we can have very elegant formulas for the marginal pdfs of the random eigenvalues and random rotations, which is summarized in the following theorem.

Theorem 3 *Let* $\mathbf{T}$ *be a real n-D SRS random tensor, whose random eigenvalues and random rotation parameters are respectively denoted by* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n)$ *with its elements satisfying* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ *and* $\boldsymbol{\Phi} = (\phi_{21}, \phi_{31}, ..., \phi_{n(n-1)})$. *If the pdf* $f_T(\boldsymbol{\Gamma})$ *of* $\mathbf{T}$ *is invariant under the rotation group,* i.e. $f_T(\boldsymbol{\Gamma}) = f_T(\mathbf{y})$, *then the random eigenvalues* $\boldsymbol{\lambda}$ *and the random eigenrotations* $\boldsymbol{\Phi}$ *are stochastically independent and respectively have the following marginal distributions:*

$$f_\lambda(\mathbf{y}) \sim f_T(\mathbf{y}) \prod_{i<j} (y_i - y_j) \tag{32}$$

for the n random eigenvalues, and

$$f_\Phi(\boldsymbol{\theta}) \sim \prod_{i=1}^{n-2} \prod_{j=i+2}^{n} (\cos\theta_{ji})^{j-i-1} \tag{33}$$

for the $n(n-1)/2$ *random rotations.*

For the proof and usage of (32), the reader is referred to Mehta (1990), Girko (1985) or Xu & Grafarend (1996b); (33) is a consequence of Theorem 2 (see also Xu 1999). In particular, it is very important to note that (33) of Theorem 3 indicates that all the random eigenrotations are also stochastically independent.

4.2 Probability distributions of 3D random spectra

For 3D random tensors $\mathbf{T}$ of geo-interest, the three random eigenvalues and three random rotations are denoted by $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ satisfying $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and $\boldsymbol{\Phi} = (\phi_{21}, \phi_{31}, \phi_{32})$ as in the above n-D case, respectively. Assume that the 3D SRS random tensor $\mathbf{T}$ has a pdf $f_T(\boldsymbol{\tau})$. Then in the similar manner to (29) to (31), for a full 3D random stress/strain tensor $\mathbf{T}$, we can readily obtain the joint and marginal pdfs of the three random principal stress/strain components and three random rotations as follows:

$$f_{\lambda\Phi}(\mathbf{y}, \boldsymbol{\theta}) = \prod_{1\leq i<j\leq 3} (y_i - y_j)\ f_T(\mathbf{U}\mathbf{Y}\mathbf{U}^T)\cos\theta_{31}, \tag{34}$$

$$f_\lambda(y_1, y_2, y_3) = \int_{\Omega_\theta} \prod_{1\leq i<j\leq 3} (y_i - y_j)\ f_T(\mathbf{U}\mathbf{Y}\mathbf{U}^T)\cos\theta_{31}\, d\Omega_\theta, \tag{35}$$

and

$$f_\Phi(\theta_{21}, \theta_{31}, \theta_{32}) = \int_{\Omega_y} \prod_{1\leq i<j\leq 3} (y_i - y_j)\ f_T(\mathbf{U}\mathbf{Y}\mathbf{U}^T)\cos\theta_{31}\, d\Omega_y, \tag{36}$$

where $\mathbf{U}$ and $\mathbf{Y}$ have been defined in (5) and (9), the domains Ω_y and Ω_θ are respectively defined by $-\infty < y_3 \leq y_2 \leq y_1 < \infty$ and $-\pi/2 \leq \theta_{21}, \theta_{31}, \theta_{32} \leq \pi/2$.
If the pdf $f_T(\boldsymbol{\tau})$ takes the form:

$$\begin{aligned} f_T(\boldsymbol{\tau}) &= exp\{-a\, tr(\boldsymbol{\tau}^2) + b\, tr(\boldsymbol{\tau}) + c\} \\ &= exp\{-a \sum_{i=1}^{3} y_i^2 + b \sum_{i=1}^{3} y_i + c\}, \end{aligned}$$

where $a > 0$, b and c must satisfy the condition of unity probability, then the marginal pdfs of the three random principal stress/strain components and three random rotations become:

$$f_\lambda(y_1, y_2, y_3) = 2\pi^2 \prod_{1\le i<j\le 3} (y_i - y_j) exp\{-a\sum_{i=1}^{3} y_i^2 + b\sum_{i=1}^{3} y_i + c\},$$

and

$$f_\Phi(\theta_{21}, \theta_{31}, \theta_{32}) = \frac{1}{2\pi^2} \cos\theta_{31},$$

respectively (Xu & Grafarend 1996b).

Now we shall further examine the distributions of the random spectra of the 3D deviatoric stress and pure shear random tensors, with numerical demonstrations.

- **3D deviatoric random stress/strain tensors.**
  Given a pdf $f_T(\boldsymbol{\tau})$ for the deviatoric random stress $\mathbf{T}$, and with the corresponding Jacobian in Table 1, we readily obtain the joint pdf of the two random principal stress components and the three random rotations:

$$f_{\lambda\Phi}(\mathbf{y}, \boldsymbol{\theta}) = f_T(\mathbf{UYU}^T)(y_1 - y_2)(2y_1 + y_2)(y_1 + 2y_2)\cos^3\theta_{31}(1 + 2\sin^2\theta_{31}), \tag{37}$$

  the marginal pdf of the two random principal stress components:

$$f_\lambda(y_1, y_2) = \int_{\Omega_\theta} f_T(\mathbf{UYU}^T)(y_1 - y_2)(2y_1 + y_2)(y_1 + 2y_2)\cos^3\theta_{31}(1 + 2\sin^2\theta_{31})\, d\Omega_\theta, \tag{38}$$

  and the marginal pdf of the three random rotations:

$$f_\Phi(\theta_{21}, \theta_{31}, \theta_{32}) = \int_{\Omega_y} f_T(\mathbf{UYU}^T)(y_1 - y_2)(2y_1 + y_2)(y_1 + 2y_2)\cos^3\theta_{31}(1 + 2\sin^2\theta_{31})\, d\Omega_y, \tag{39}$$

  respectively, where $-\infty < y_2 \le y_1 < \infty$.

  For numerical demonstrations of (38) and (39), we construct a 3D deviatoric random stress tensor with the three principal stresses 15.1, -1.6 and -13.5 MPa, based on the real stress data taken from Amadei & Stephasson (1997). The three orientations are generated artificially for making a full deviatoric stress tensor. The five independent stress components are assumed to have a relative error of 15 per cent and to be statistically independent. Then we assume the Gaussian and Laplacian probability models for the generated random stress components. The marginal pdfs of the two random principal stresses are shown in Fig.1 and Fig.2, and the marginal pdfs of the three random rotations in Fig.3. These figures have shown that the pdfs of the random eigenvalues are significantly different from normal and the pdfs of the random rotations significantly different from either normal or standard (Fisher's) pdf model for directional data. For more details on the example, the reader is referred to Xu (1999).

- **3D pure shear random tensors.**
  In a similar manner to (37), (38) and (39), we have the joint pdf of the only random principal stress component and the three random rotations:

$$f_{\lambda\Phi}(y, \boldsymbol{\theta}) = f_T(\mathbf{UYU}^T) 2y_1^3 \cos\theta_{31} |det\{\mathbf{M}_{c_{2S}}\}|, \tag{40}$$

  the marginal pdf of the two random principal stress components:

$$f_\lambda(y) = \int_{\Omega_\theta} f_T(\mathbf{UYU}^T) 2y_1^3 \cos\theta_{31} |det\{\mathbf{M}_{c_{2S}}\}|\, d\Omega_\theta, \tag{41}$$

and the marginal pdf of the three random rotations:

$$f_{\Phi}(\theta_{21}, \theta_{31}, \theta_{32}) = \int_{\Omega_y} f_T(\mathbf{U}\mathbf{Y}\mathbf{U}^T) 2y_1^3 \cos\theta_{31} |det\{\mathbf{M}_{c2S}\}| \, d\Omega_y, \tag{42}$$

respectively, where $-\infty < y < \infty$.

For illustrative purposes, we have constructed a pure shear random tensor (exactly the same as the DC seismic moment example in Xu (1999) except the difference in unit). The pdfs of the random principal stress with the Gaussian and Laplacian models for the original random tensor components are shown in Fig.4. It is very clear again that the pdfs of the random eigenvalue is significantly different from normal. Fig.5 has plotted the pdfs of the three random rotations, which also have shown that Fisher's model is not representative of the rotations of the 3D pure shear random tensor.

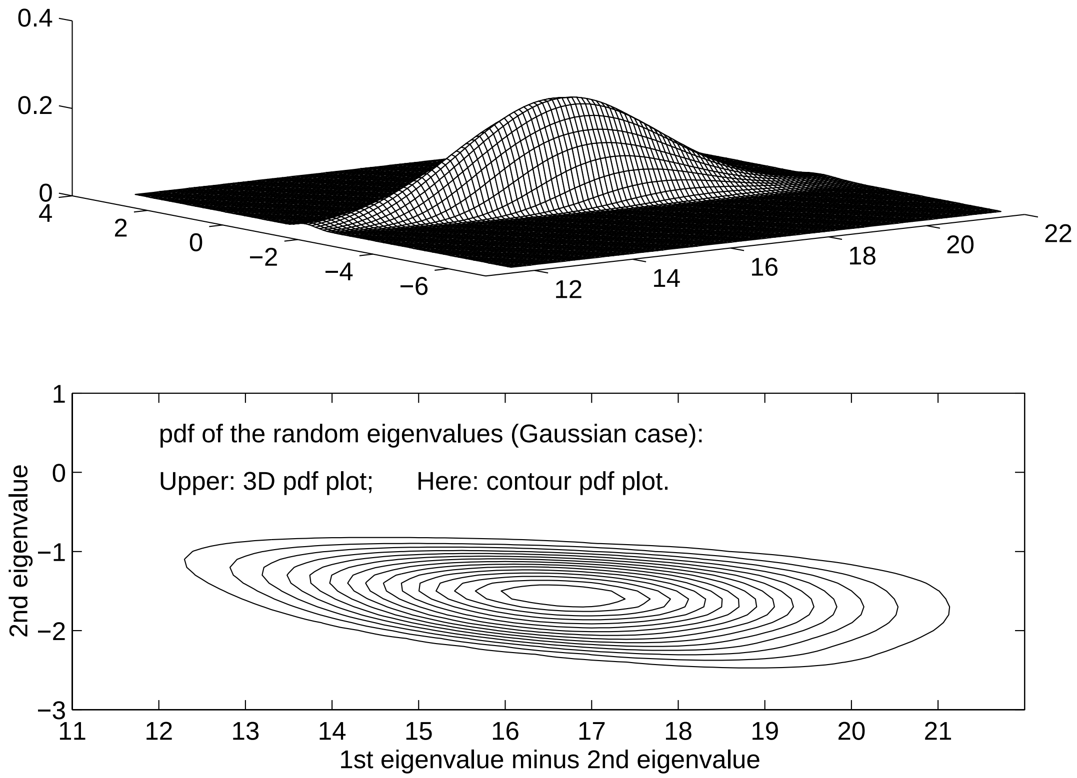

Figure 1: The probability density function of the two random principal stresses of the 3D deviatoric random stress tensor (after Xu 1999).

5 Accuracy and biases of the random eigenspectra

Although 3D SRS stress/strain tensors are practically random, the statistical issue of the random principal stress/strain components and the random orientations of the principal axes has been paid much attention only recently. The variance-covariance matrix of the three random eigenvalues $\boldsymbol{\lambda}$ and three random rotations $\boldsymbol{\Phi}$ with the first order approximation was first hinted at by Angelier et al. (1982) and then further systematically (and independently) worked out by Soler & van Gelder (1991). Since the ratio of stress/strain signal to noise is generally small (see, *e.g.* Amadei & Stephasson 1997), the first-order accuracy estimate can be significantly

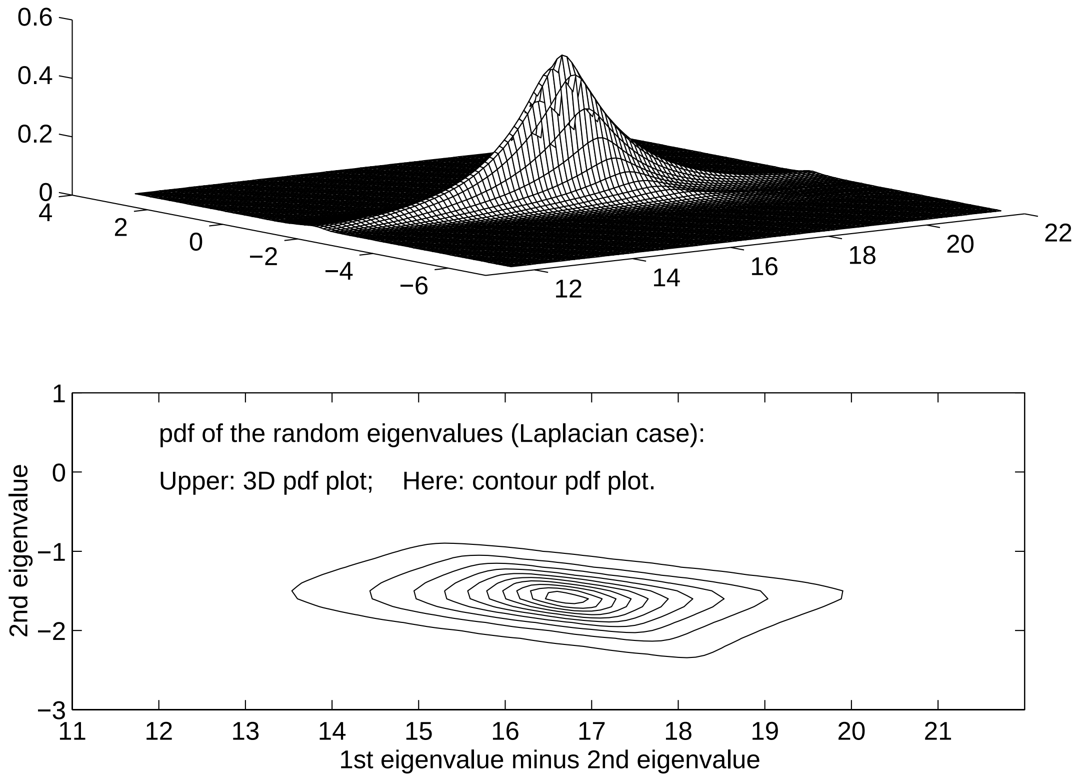

Figure 2: The probability density function of the two random principal stresses of the 3D deviatoric random stress tensor (after Xu 1999).

in error (Xu 1986). Xu & Grafarend (1996a, b) extended the first-order variance-covariance matrix to the second-order approximation. On the other hand, the one-to-one mapping between the (constrained or unconstrained) random tensor $\mathbf{T}$, the eigenvalues $\boldsymbol{\lambda}$ and rotations $\boldsymbol{\Phi}$ is nonlinear; thus biases of the estimated random eigenspectra are expected, the extent of which depends on the signal-to-noise ratio of the original random tensor $\mathbf{T}$. Surprinsingly, it seems that the bias issue of the random eigenspectra, although very important geophysically, has not been derived mathematically. Note, however, that some inequality results have been obtained by Cacoullos (1965), for example. Recently, the biases of the random eigenspectra with second order approximation were worked out by Xu (1996) and Xu & Grafarend (1996a, b).

In this section, we will further extend the statistical measures for 3D SRS random stress/strain tensors by Xu & Grafarend (1996a, b) to the unconstrained n-D case. The same technique can be applied to derive all the corresponding accuracy and biases of the random eigenspectra of 3D constrained random stress/strain tensors; this will not be discussed in this paper however. In principle, the accuracy and biases of the random eigenspectra can be derived using the estimation methods in nonlinear models (see *e.g.* Bates & Watts 1980, 1988; Beale 1960; Box 1971; Clarke 1980; Ratkowsky 1983; Seber & Wild 1989). The spectral decomposition (5) or (10) is very special, though nonlinear, in that there exist no redundant measurements. Thus we can directly derive the accuracy (of first- or second-order) and biases of the random eigenspectra on the basis of spectral decomposition with or without constraints.

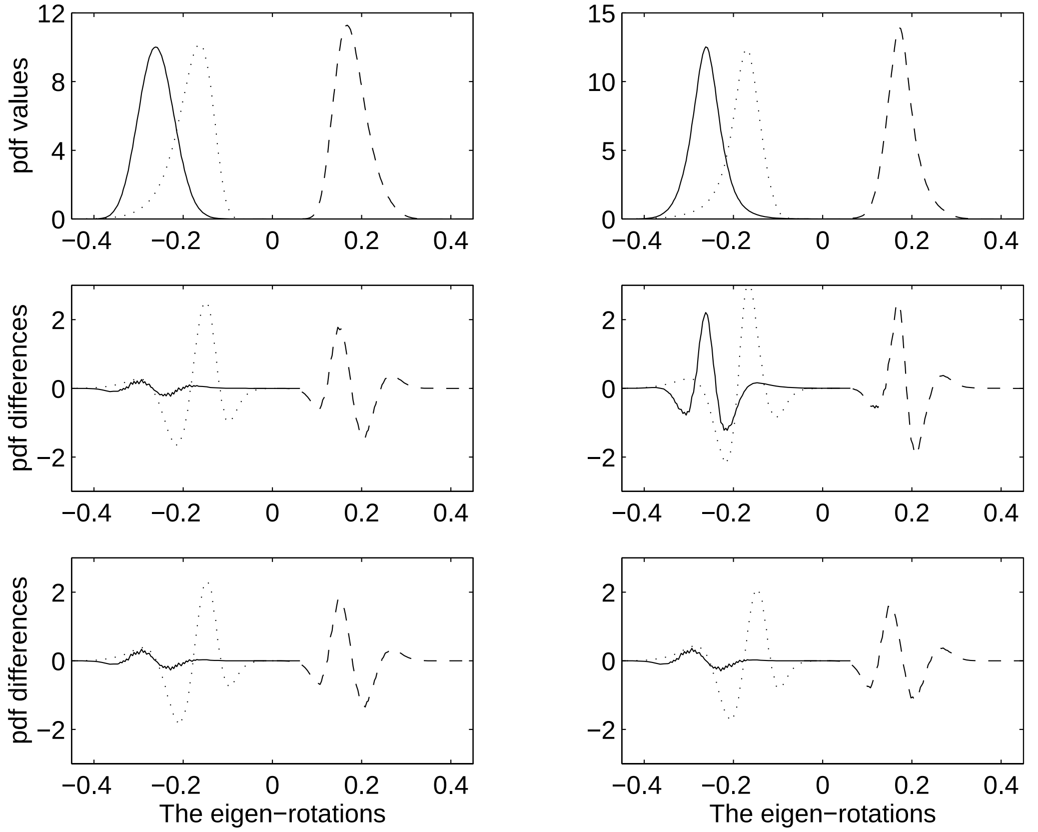

Figure 3: The marginal probability density functions of the three random rotations of the deviatoric random stress tensor (top two subplots), and the differences from their normal (middle two subplots) and Fisher's approximations (lower two subplots): solid line – θ_{21}; dashed line – θ_{31}; and dotted line – θ_{32}. The subplots on the left hand side are with the Gaussian model, and those on the right with the Laplacian model (after Xu 1999).

5.1 The biases of the random eigenspectra

Suppose that the n-D real SRS random tensor $\mathbf{T}$ has the mean $\mathbf{\Gamma}$ and the error $\boldsymbol{\epsilon}_T$, i.e.

$$\mathbf{T} = \mathbf{\Gamma} + \boldsymbol{\epsilon}_T. \tag{43}$$

The vector form of (43) is

$$v(\mathbf{T}) = v(\mathbf{\Gamma}) + \boldsymbol{\epsilon}, \tag{44}$$

where $\boldsymbol{\epsilon} = v(\boldsymbol{\epsilon}_T)$. It is further assumed that $\boldsymbol{\epsilon}$ has a vector of zero mean and a variance-covariance matrix $\mathbf{\Sigma}_\epsilon$. If the tensor $\mathbf{T}$ is constrained, then (44) only collects those functionally independent random components of $\mathbf{T}$. Let $\mathbf{T}$ have the random eigenvalues $\boldsymbol{\lambda}$ and random rotations $\mathbf{\Phi}$, which are functionally independent. Thus we have the following mapping:

$$\begin{bmatrix} \boldsymbol{\lambda} \\ \mathbf{\Phi} \end{bmatrix} = \rho\{v(\mathbf{T})\}, \tag{45}$$

where ρ maps $\mathbf{T}$ to $\boldsymbol{\lambda}$ and $\mathbf{\Phi}$.

Expanding (45) into a Taylor series and taking all the terms up to the second order approximation at the point $v(\mathbf{\Gamma})$, we obtain

$$\begin{bmatrix} \boldsymbol{\lambda} \\ \mathbf{\Phi} \end{bmatrix} = \rho\{v(\mathbf{\Gamma})\} + \dot{\rho}\{v(\mathbf{\Gamma})\}\boldsymbol{\epsilon} + \frac{1}{2}\mathbf{H}_\epsilon\boldsymbol{\epsilon}, \tag{46}$$

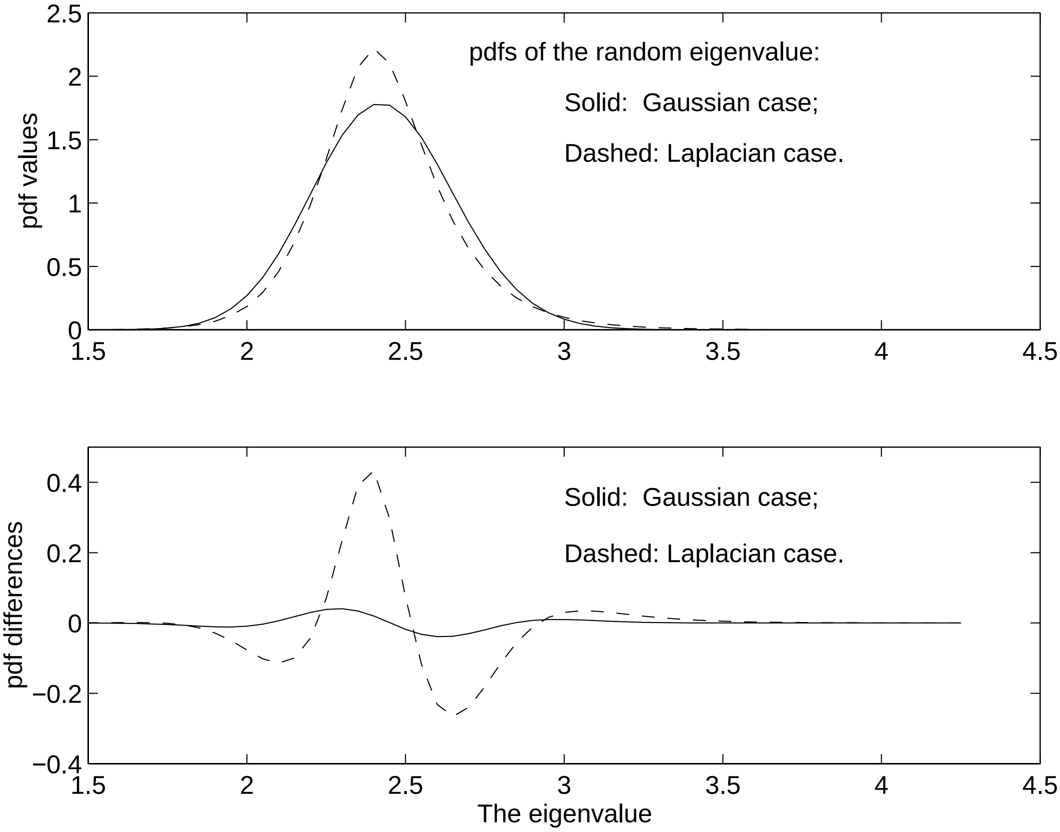

Figure 4: The pdfs of the random eigenvalue of the 3D pure shear random tensor and the differences from their normal approximations (after Xu 1999).

where

$$\dot{\rho}\{v(\mathbf{\Gamma})\} = \left.\frac{\partial(\boldsymbol{\lambda}^T, \mathbf{\Phi}^T)^T}{\partial[v(\mathbf{T})]^T}\right|_{\mathbf{T}=\mathbf{\Gamma}}, \tag{47}$$

$$\mathbf{H}_\epsilon = [\ddot{\mathbf{V}}(\lambda_1)\boldsymbol{\epsilon},\ ...,\ \ddot{\mathbf{V}}(\lambda_i)\boldsymbol{\epsilon},\ \ddot{\mathbf{V}}(\phi_{kl})\boldsymbol{\epsilon},\ ...,\ \ddot{\mathbf{V}}(\phi_{mn})\boldsymbol{\epsilon}]^T, \tag{48}$$

with

$$\ddot{\mathbf{V}}(\lambda_i) = \left.\frac{\partial^2\lambda_i}{\partial v(\mathbf{T})\partial[v(\mathbf{T})]^T}\right|_{\mathbf{T}=\mathbf{\Gamma}},$$

and

$$\ddot{\mathbf{V}}(\phi_{ij}) = \left.\frac{\partial^2\phi_{ij}}{\partial v(\mathbf{T})\partial[v(\mathbf{T})]^T}\right|_{\mathbf{T}=\mathbf{\Gamma}},$$

being the symmetric matrix of the second derivatives of λ_i and ϕ_{ij} with respect to the elements of $v(\mathbf{T})$, respectively, where the total number of ϕ_{ij} is equal to $[n(n+1)/2 - i - m_c]$. Thus the biases of $\boldsymbol{\lambda}$ and $\mathbf{\Phi}$ can be computed from (46) as follows:

$$\begin{aligned} bias\begin{bmatrix}\boldsymbol{\lambda}\\ \mathbf{\Phi}\end{bmatrix} &= E\begin{bmatrix}\boldsymbol{\lambda}\\ \mathbf{\Phi}\end{bmatrix} - \rho\{v(\mathbf{\Gamma})\} \\ &= \frac{1}{2}E\{\mathbf{H}_\epsilon\boldsymbol{\epsilon}\}. \end{aligned} \tag{49}$$

In order to calculate the biases of $\boldsymbol{\lambda}$ and $\mathbf{\Phi}$, we have to know the second derivatives $\dot{\rho}\{v(\mathbf{\Gamma})\}$, $\ddot{\mathbf{V}}(\lambda_i)$ and $\ddot{\mathbf{V}}(\phi_{ij})$. For a full n-D SRS random tensor $\mathbf{T}$, we have from (16) the first derivative

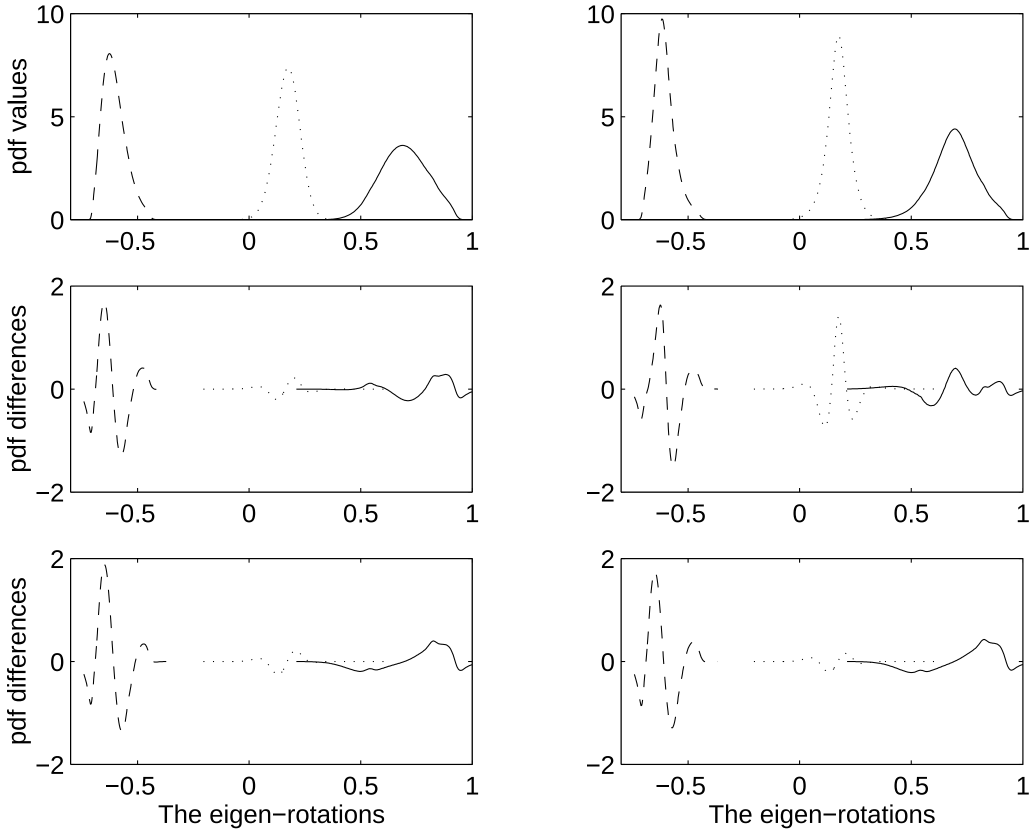

Figure 5: The marginal pdfs of the three random rotations of the 3D pure shear random tensor (top two subplots), and the differences from their normal (middle two subplots) and Fisher's approximations (lower two subplots): solid line – θ_{21}; dashed line – θ_{31}; and dotted line – θ_{32}. The subplots on the left hand side are with the Gaussian model and those on the right with the Laplacian model (after Xu 1999).

of $\boldsymbol{\lambda}$ and $\boldsymbol{\Phi}$ with respect to $v(\mathbf{T})$:

$$\dot{\rho}\{v(\boldsymbol{\Gamma})\} = \left.\frac{\partial(\boldsymbol{\lambda}^T, \boldsymbol{\Phi}^T)^T}{\partial[v(\mathbf{T})]^T}\right|_{\mathbf{T}=\boldsymbol{\Gamma}} = \mathbf{B}^{-1}[\mathbf{D}_n^+(\mathbf{G}\otimes\mathbf{G})\mathbf{D}_n]^T\Big|_{\mathbf{T}=\boldsymbol{\Gamma}}, \tag{50}$$

where $\mathbf{B}$ is given by

$$\mathbf{B} = [v(\mathbf{O}_1), v(\mathbf{O}_2), ..., v(\mathbf{O}_n), v(\mathbf{H}_{21}), ..., v(\mathbf{H}_{n1}), v(\mathbf{H}_{32}), ..., v(\mathbf{H}_{n(n-1)})],$$

with $\mathbf{O}_i$ being a zero matrix except for the ith diagonal element taking on the unit value. In the 3D case, the matrix $\mathbf{B}$ becomes:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2-\lambda_1 & 0 & (\lambda_2-\lambda_1)\sin\phi_{31} \\ 0 & 0 & 0 & 0 & (\lambda_3-\lambda_1)\cos\phi_{21} & (\lambda_1-\lambda_3)\sin\phi_{21}\cos\phi_{31} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\lambda_3-\lambda_2)\sin\phi_{21} & (\lambda_3-\lambda_2)\cos\phi_{21}\sin\phi_{31} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

By differentiating (50) and after some lengthy derivation, we obtain the second derivatives of $\boldsymbol{\lambda}$:

$$\ddot{\mathbf{V}}(\lambda_i) = \ddot{\mathbf{V}}_{\rho\lambda_i}\mathbf{B}^{-1}[\mathbf{D}_n^+(\mathbf{G}\otimes\mathbf{G})\mathbf{D}_n]^{-1}\mid_{\mathbf{T}=\boldsymbol{\Gamma}}, \tag{51}$$

and the second derivatives of $\mathbf{\Phi}$:

$$\ddot{\mathbf{V}}(\phi_{ij}) = \ddot{\mathbf{V}}_{\rho\phi_{ij}}\mathbf{B}^{-1}[\mathbf{D}_n^{+}(\mathbf{G}\otimes\mathbf{G})\mathbf{D}_n]^{-1}\mid_{\mathbf{T}=\mathbf{\Gamma}}, \tag{52}$$

where

$$\begin{aligned} & \{\ddot{\mathbf{V}}_{\rho\lambda_1},\ ...,\ \ddot{\mathbf{V}}_{\rho\lambda_n},\ \ddot{\mathbf{V}}_{\rho\theta_{21}},\ ...,\ \ddot{\mathbf{V}}_{\rho\theta_{n1}},\ \ddot{\mathbf{V}}_{\rho\theta_{32}},\ ...,\ \ddot{\mathbf{V}}_{\rho\theta_{n(n-1)}}\}^T \\ = & \{vec[\ddot{\mathbf{V}}_1^T],\ ...,\ vec[\ddot{\mathbf{V}}_n^T],\ vec[\ddot{\mathbf{V}}_{21}^T],\ ...,\ vec[\ddot{\mathbf{V}}_{n1}^T],\ vec[\ddot{\mathbf{V}}_{32}^T],\ ..., vec[\ddot{\mathbf{V}}_{n(n-1)}^T]\}, \end{aligned}$$

$$\ddot{\mathbf{V}}_i = -\mathbf{B}^{-1}\frac{\partial\mathbf{B}}{\partial\lambda_i}\mathbf{B}^{-1}[\mathbf{D}_n^{+}(\mathbf{G}\otimes\mathbf{G})\mathbf{D}_n]^{-1}, \quad i=1,2,...,n$$

$$\begin{aligned} \ddot{\mathbf{V}}_{ij} = & -\mathbf{B}^{-1}\frac{\partial\mathbf{B}}{\partial\phi_{ij}}\dot{\rho}\{v(\mathbf{T})\} \\ & -\dot{\rho}\{v(\mathbf{T})\}\mathbf{D}_n^{+}[(\frac{\partial\mathbf{G}}{\partial\phi_{ij}}\otimes\mathbf{G})+(\mathbf{G}\otimes\frac{\partial\mathbf{G}}{\partial\phi_{ij}})]\mathbf{D}_n[\mathbf{D}_n^{+}(\mathbf{G}\otimes\mathbf{G})\mathbf{D}_n]^{-1}, \quad i>j. \end{aligned}$$

Substituting (51) and (52) into (49), we can readily compute the biases of the random eigenvalues and random rotations. For more details in the 3D case, the reader is referred to Xu & Grafarend (1996b). In the same manner and using the differential relations in Section 3, we can first derive all the required second derivatives and then compute the biases of the random eigenvalues and random rotations of the 3D random deviatoric stress/strain and pure shear tensors; these are omitted here.

5.2 The accuracy of the random eigenspectra

Omitting the second order term of $\boldsymbol{\epsilon}$ from (46) and then applying the error propagation law to it, we obtain the variance-covariance matrix of $\boldsymbol{\lambda}$ and $\mathbf{\Phi}$ as follows:

$$D\begin{bmatrix}\boldsymbol{\lambda}\\ \mathbf{\Phi}\end{bmatrix} = \dot{\rho}\{v(\mathbf{\Gamma})\}\mathbf{\Sigma}_\epsilon\dot{\rho}^T\{v(\mathbf{\Gamma})\}, \tag{53}$$

which is of a first order approximation (see also Angelier et al. 1982; Soler & van Gelder 1991; Xu & Grafarend 1996b).

Since the ratio of signal to noise of random stress/strain tensors in the Earth Sciences may be small, the accuracy of higher order approximation may be necessary. In what follows we will derive the second order accuracy estimate. Applying the definition of variance-covariance to (46), we have

$$D\begin{bmatrix}\boldsymbol{\lambda}\\ \mathbf{\Phi}\end{bmatrix} = \dot{\rho}\{v(\mathbf{\Gamma})\}\mathbf{\Sigma}_\epsilon\dot{\rho}^T\{v(\mathbf{\Gamma})\} + \frac{1}{4}E\{\mathbf{H}_\epsilon\boldsymbol{\epsilon} - E(\mathbf{H}_\epsilon\boldsymbol{\epsilon})\}\{\mathbf{H}_\epsilon\boldsymbol{\epsilon} - E(\mathbf{H}_\epsilon\boldsymbol{\epsilon})\}^T. \tag{54}$$

The second term of (54) can be derived using results of the fourth order moments of quadratic statistics (Searle 1971; Rao & Kleffe 1988; Searle et al. 1992), and is given by

$$E\{\mathbf{H}_\epsilon\boldsymbol{\epsilon} - E(\mathbf{H}_\epsilon\boldsymbol{\epsilon})\}\{\mathbf{H}_\epsilon\boldsymbol{\epsilon} - E(\mathbf{H}_\epsilon\boldsymbol{\epsilon})\}^T = 2\mathbf{M}_{H_\epsilon\epsilon}, \tag{55}$$

where

$$\mathbf{M}_{H_\epsilon\epsilon} = \begin{bmatrix}\mathbf{M}_\lambda & \mathbf{M}_{\lambda\Phi}\\ \mathbf{M}_{\lambda\Phi}^T & \mathbf{M}_\Phi\end{bmatrix}, \tag{56}$$

with

$$\mathbf{M}_\lambda = [tr\{\ddot{\mathbf{V}}(\lambda_i)\mathbf{\Sigma}_\epsilon\ddot{\mathbf{V}}(\lambda_j)\mathbf{\Sigma}_\epsilon\}]_{n\times n},$$

$$\mathbf{M}_{\Phi} = [tr\{\ddot{\mathbf{V}}(\theta_{ij})\mathbf{\Sigma}_{\epsilon}\ddot{\mathbf{V}}(\theta_{ij})\mathbf{\Sigma}_{\epsilon}\}]_{n\times n},$$

$$\mathbf{M}_{\lambda\Phi} = [tr\{\ddot{\mathbf{V}}(\lambda_i)\mathbf{\Sigma}_{\epsilon}\ddot{\mathbf{V}}(\theta_{ij})\mathbf{\Sigma}_{\epsilon}\}]_{n\times n}.$$

Substituting (56) into (54), we obtain the variance-covariance matrix of $\boldsymbol{\lambda}$ and $\boldsymbol{\Phi}$ with the second order approximation as follows:

$$D\begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\Phi} \end{bmatrix} = \dot{\rho}\{v(\mathbf{\Gamma})\}\mathbf{\Sigma}_{\epsilon}\dot{\rho}^T\{v(\mathbf{\Gamma})\} + \frac{1}{2}\mathbf{M}_{H_{\epsilon}\epsilon}. \tag{57}$$

The accuracy of the random principal stress/strain components and random rotations of a full 3D random stress/strain tensor can be found in Xu & Grafarend (1996b). Using the same techniques as in the above, one can also obtain the first- and second-order accuracies of the random eigenvalues and random rotations of the 3D random deviatoric stress/strain and pure shear tensors; they are also omitted here however.

We finally summarize the bias and accuracy results of the random spectra in the following theorem.

Theorem 4 *Let* $\mathbf{T}$ *be a real n-D SRS random tensor, whose random eigenvalues and random rotation parameters are respectively denoted by* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n)$ *with its elements satisfying* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ *and* $\boldsymbol{\Phi} = (\phi_{21}, \phi_{31}, ..., \phi_{n(n-1)})$. *Assume that* $\mathbf{T}$ *(or more specifically,* $v(\mathbf{T})$*) has the mean* $v(\mathbf{\Gamma})$ *and the variance-covariance matrix* $\mathbf{\Sigma}_{\epsilon}$. *Then the biases and variance-covariance matrix of the second-order approximation of the random eigenvalues and random rotations are given by*

$$bias\begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\Phi} \end{bmatrix} = \frac{1}{2}E\{\mathbf{H}_{\epsilon}\boldsymbol{\epsilon}\}, \tag{58}$$

and

$$D\begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\Phi} \end{bmatrix} = \dot{\rho}\{v(\mathbf{\Gamma})\}\mathbf{\Sigma}_{\epsilon}\dot{\rho}^T\{v(\mathbf{\Gamma})\} + \frac{1}{2}\mathbf{M}_{H_{\epsilon}\epsilon}, \tag{59}$$

respectively, where the matrix $\mathbf{H}_{\epsilon}$ *of (58) is given in Subsection 5.1.*

6 References

Aki, K. & Richards, P.G., 1980. *Quantitative Seismology*, W.H. Freeman, San Francisco.

Altmann, S.L., 1986. *Rotations, Quaternions, and Double Groups*, Clarendon Press, Oxford.

Amadei, B. & Stephansson, O., 1997. *Rock Stress and its Measurement*, Chapman & Hall, London.

Anderson, T.W., 1958. *An Introduction to Multivariate Statistical Analysis*, John Wiley & Sons, New York.

Angelier, J., 1984. Tectonic analysis of fault slip data sets, *J. geophys. Res.,* **B89**, 5835-5848.

Angelier, J., 1989. From orientations to magnitudes in paleostress determinations using fault slip data, *J. struct. Geol.,* **11**, 37-50.

Angelier, J., Tarantola, A., Vallete, B. & Manoussis S., 1982. Inversion of field data in fault tectonics to obtain the regional stress — I. single phase fault populations: a new method of computing the stress tensor, *Geophys. J. R. astr. Soc.*, **69**, 607-621.

Bates, D.M. & Watts, D.G., 1980. Relative curvature measures of nonlinearity (with discussions), *J.R. Statist. Soc.*, **B42**, 1-25

Bates, D.M. & Watts, D.G., 1988. *Nonlinear Regression Analysis and its Applications*, John Wiley & Sons, New York

Beale, E.M.L., 1960. Confidence regions in non-linear estimation (with discussions), *J.R. Statist. Soc.*, **B22**, 41-88

Boutet de Monvel, A., Khorunzhy, A. & Vasilchuk, V., 1996. Limiting eigenvalue distribution of random matrices with correlated entries, *Markov Proc. and Related Fields*, **2**, 607-636.

Box, M.J., 1971. Bias in nonlinear estimation (with discussions), *J.R. Statist. Soc.*, **B33**, 171-201

Brekhovskikh, L.M. & Goncharov, V., 1994. *Mechanics of Continua and Wave Dynamics*, 2nd ed., Springer Verlag, Berlin.

Cacoullos, T., 1965. On the bias of functions of characteristic roots of a random matrix, *Biometrica*, **52**, 87-94.

Cichowicz, A., 1993. An automatic S-phase picker, *Bull. seis. Soc. Amer.*, **83**, 180-189.

Clarke, G.P.Y., 1980. Moments of the least squares estimators in a non-linear regression model, *J.R. Statist. Soc.*, **B42**, 227-237

Dai, H. & MacBeth, C., 1997. Application of back-propagation neural networks to identification of seismic arrival types, *Phys. Earth planet. Interior*, **101**, 177-188.

Fang, K.T., Kotz, S. & Ng, K.W., 1990. *Symmetric Multivariate and Related Distributions*, John Wiley & Sons, New York.

Fisher, N.I., Lewis, T. & Embleton, B.J.J., 1993. *Statistical Analysis of Spherical Data*, Cambridge University Press, Cambridge.

Fisher, R.A., 1953. Dispersion on a sphere, *Proc. R. Soc.*, **A217**, 295-305.

Frank, F.C., 1966. Deduction of earth strains from survey data, *Bull. seismol. Soc. Am.*, **56**, 35-42.

Furstenberg, H. & Kesten, H., 1960. Products of random matrices, *Ann. math. Statist.*, **31**, 457-469.

Gephart, J. & Forsyth, D., 1984. An improved method for determining the regional stress tensor using earthquake focal mechanism data: application to the San Fernando earthquake sequence, *J. geophys. Res.*, **B89**, 9305-9320.

Girko, V.L., 1979. Distribution of eigenvalues and eigenvectors of hermitian stochastic matrices, *Ukrain. Math. J.*, **31**, 421-424

Girko, V.L., 1985. Spectral theory of random matrices, *Russian math. Surveys*, **40**, 77-120.

Girko, V.L., 1989. Asymptotic of the distribution of the spectrum of random matrices, *Russian math. Surveys*, **44**, 3-36.

Girko, V.L., 1990a. Limit theorems for distributions of eigenvalues of random symmetrical matrices, *Theory Prob. math. Statist.*, **41**, 25-31.

Girko, V.L., 1990b. *Theory of Random Determinants*, Kluwer Academic Publishers, Amsterdam.

Girko, V.L., 1992. Asymptotic behavior of the eigenvalues of empirical covariance matrices, (part 1), *Theory Prob. math. Statist.*, **44**, 37-44.

Girko, V.L., 1993. Asymptotic behavior of the eigenvalues of empirical covariance matrices, (part 2), *Theory Prob. math. Statist.*, **46**, 33-37.

Girko, V.L., 1997. Strong law for the singular values and eigenvectors of random matrices, *Random Oper. Stoch. Equ.*, **5**, 81-104.

Grafarend, E., 1986. Three-dimensional deformation analysis: Global vector spherical harmonic and local element representation, *Tectonophysics*, **130**, 337-359.

Hansen, K. & Mount, V., 1990. Smoothing and extrapolation of crustal stress orientation measurements, *J. geophys. Res.*, **B95**, 1155-1165.

Helffrich, G.R., 1997. How good are routinely determined focal mechanisms? Empirical statistics based on a comparison of Harvard, USGS and ERI moment tensors, *Geophys. J. Int.*, **131**, 741-750.

Horiuchi, S., Rocco, G. & Hasegawa, A., 1995. Discrimination of fault planes from auxiliary planes based on simultaneous determination of stress tensor and a large number of fault plane solutions, *J. geophys. Res.,* **B100**, 8327-8338.

Johnson, N.L. & Kotz, S., 1970. *Distributions in Statistics: Continuous Univariate Distributions*, John Wiley & Sons, New York.

Johnson, N.L. & Kotz, S., 1972. *Distributions in Statistics: Continuous Multivariate Distributions*, John Wiley & Sons, New York.

Kagan, Y.Y., 1987. Point sources of elastic deformation: elementary sources, static displacements, *Geophys. J. R. astr. Soc.*, **90**, 1-34.

Kagan, Y.Y., 1990. Random stress and earthquake statistics: spatial dependence, *Geophys. J. Int.*, **102**, 573-583.

Kagan, Y.Y., 1991. Seismic moment distribution, *Geophys. J. Int.*, **106**, 123-134.

Kagan, Y.Y., 1992a. On the geometry of an earthquake fault system, *Phys. Earth planet. Interior*, **71**, 15-35.

Kagan, Y.Y., 1992b. Correlations of earthquake focal mechanisms, *Geophys. J. Int.*, **110**, 305-320.

Kagan, Y.Y., 1994. Observational evidence from earthquakes as a nonlinear dynamic process, *Physica D*, **77**, 160-192.

Kagan, Y.Y. & Knopoff, L., 1985a. The first order statistical moment of the seismic moment tensor, *Geophys. J. R. astr. Soc.*, **81**, 429-444.

Kagan, Y.Y. & Knopoff, L., 1985b. The two-point correlation of the seismic moment tensor, *Geophys. J. R. astr. Soc.*, **83**, 637-656.

Kostrov, B.V. & Das, S., 1988. *Principles of Earthquakes Source Mechanics*, Cambridge University Press, Cambridge.

Larson, H.L. & Shubert, B.O., 1979. *Probabilistic Models in Engineering Sciences*, John Wiley & Sons, New York.

Lay, T. & Wallace, T.C., 1995. *Modern Global Seismology*, Academic Press, New York.

Lu, Z., Wyss, M & Pulpan, H., 1997. Details of stress directions in the Alaska subduction zone from fault plane solutions, *J. geophys. Res.,* **B102**, 5385-5402.

Magnus, J. & Neudecker, H., 1988. *Matrix Differential Calculus with Applications in Statistics and Econometrics*, John Wiley & Sons, New York.

Mehta, M.L., 1990. *Random Matrices and the Statistical Theory of Energy Levels, (extended edition)*, Academic Press, New York.

McKenzie, D.P, 1969. The relation between fault plane solutions for earthquakes and the directions of the principal stresses, *Bull. seis. Soc. Amer.*, **59**, 591-601.

Mood, A., Graybill, F. & Boes, D., 1974. *Introduction to the Theory of Statistics*, McGRAW-Hill, Tokyo.

Oravecz, F. & Petz, D., 1997. On the eigenvalue distribution of some symmetric random matrices, *Acta Sci. Math.*, **63**, 383-395.

Prescott, W.H., 1981. The determination of displacement fields from geodetic data along a strike slip fault, *J. geophys. Res.*, **B86**, 6067-6072.

Rao, C.R. & Kleffe, J., 1988. *Estimation of Variance Components and Applications*, North Holland, Amsterdam

Ratkowsky, D.A., 1983. *Nonlinear Regression Modeling: A Unified Practical Approach*, Marcel Dekker, Inc., New York

Rocco, G., Horiuchi, S. & Hasegawa, A., 1997. Study on the state of stress around earthquake swarm areas using P-wave polarity data: Nikko-Ashio and Teishi Knoll areas, *J. Phys. Earth,* **45**, 29-40.

Samson, J.C., 1977. Matrix and Stokes vector representations of detectors for polarized waveforms: theory, with some applications to teleseismic waves, *Geophys. J. R. astr. Soc.*, **51**, 583-603.

Searle, S.R., 1971. *Linear Models*, John Wiley & Sons, New York

Savage, J.C. & Burford, R.O., 1970. Accumulation of tectonic strain in California, *Bull. seismol. Soc. Am.*, **60**, 1877-1896.

Searle, S.R., Casella, G. & McCullach, C., 1992. *Variance Components*, John Wiley & Sons, New York

Seber, G. & Wild, C., 1989. *Nonlinear Regression*, John Wiley & Sons, New York

Soler T. & van Gelder B., 1991. On covariances of eigenvalues and eigenvectors of second rank symmetric tensors, *Geophys. J. Int.*, **105**, 537-546.

Stroud, A.H., 1971. *Approximate Calculation of Multiple Integrals*, Prentice-Hall, New Jersey.

Wyss, M., Liang, B.Y., Tanagawa, W. & Wu, X., 1992. Comparison of orientations of stress and strain tensors based on fault plane solutions in Kaoiki, Hawaii, *J. geophys. Res.*, **B97**, 4769-4790.

Xu, P.L., 1986. Variance-covariance propagation for a nonlinear function, *J. Wuhan Techn. Uni. Surv. Mapping*, **11**, No.2, 92-99 (in Chinese with English abstract).

Xu, P.L., 1996. Statistical aspects of 2-D symmetric random tensors, Proc. FIG int. Symp. Deformation Measurement, Hong Kong, pp.203-210.

Xu, P.L., 1999. Spectral theory of constrained second-rank symmetric random tensors, *Geophys. J. Int.*, **138**, 1-24.

Xu, P.L. & Grafarend, E., 1996a. Probability distribution of eigenspectra and eigendirections of a two-dimensional, symmetric rank two random tensor, *J. Geod.*, **70**, 419-430.

Xu, P.L. & Grafarend, E., 1996b. Statistics and geometry of the eigenspectra of 3-D second-rank symmetric random tensors, *Geophys. J. Int.*, **127**, 744-756.

Xu, P.L. & Rummel, R., 1994. A simulation study of smoothness methods in recovery of regional gravity fields, *Geophys. J. Int.*, **117**, 472-486.

Xu, P.L. & Shimada, S., 1997. Inferring the relative measure of principal stress components, *Geophys. J. Int.*, **130**, F1-F4.