Map projections and boundary problems

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Abstract
This discussion of the transformation of the spherical or elliptical boundary value problems of physical geodesy into essentially plane problems was inspired by Grafarend and Krumm [?]. By transforming the Laplacian on the sphere or the ellipsoid under a conformal map projection, transformations may be found for the classical boundary problems. This leads to the idea of performing local geoid computations by variational methods, using perturbation methods for constructing suitable trial functions. As examples stereographic projections and projections of Mercator type are discussed.

1 Introduction

Among the many fields covered in the works of Erik W. Grafarend one is that of the boundary value problems of physical geodesy - some recent contributions are found in Grafarend and Krumm [?] and in Grafarend and Martinec [?]. Another field is that of map projections e.g. Grafarend [?], Grafarend and Syffus [?], [?]. However disparate they seem to be, there are connections between the two fields as may be inferred by the ideas of Grafarend and Krumm [?].

Conventionally, boundary value problems like the Stokes' problem on the sphere are solved in terms of geodetic coordinates (ϕ, λ), which are, in effect, nothing but the $(x, y)-$ coordinates of a plate carré map projection. In terms of this the Laplacian on the sphere of radius 1 may be expressed as

$$\Delta_S = \partial^2/\partial\phi^2 - \tan\phi\,\partial/\partial\phi + (\cos\phi)^{-2}\partial^2/\partial\lambda^2.$$

By way of this representation eigenvalues and eigenfunctions (surface harmonics) of Δ_S are computed and they may be used also for solving problems

$$q(\Delta_S)u = v$$

of pseudodifferential equation type as for instance the Stokes problem

$$R^{-1}\{(-\Delta_S + 1/4)^{1/2} - 3/2\}u = v,$$

where v is gravity anomaly and u geoidal height.

Then the question arises: If other map projections than the plate carré projection are used, how does Δ_S transform and is the result in any way useful for dealing with the problems of physical geodesy?

Concerning the first part of the question the answer, if we concentrate to conformal transformations, is

$$\Delta_S = s^2\Delta_P,$$

where s^2 is the local area scale, and where Δ_P is the Laplacian in the map plane. This formula is derived in section 2. The formula may be used to transform the classical formulae of physical geodesy to expressions in terms of the (x, y) map coordinates.

Of course the fact that s^2 is not constant for conformal transformations complicates the situation but still there might be some areas - e.g. working with heterogeneous data locally or regionally - where conformal map projections might be useful. The applicability is discussed in section 3. Sections 4-5 are devoted, in turn, to the stereographic projection and to Mercator type projections. Conformal conical projections and the Mercator projection of an ellipsoidal of rotation could be dealt with in the same way.

2 Transformation of the problems

Consider a map projection π mapping points on the Earth sphere S_R of radius R into the map plane P. We shall compute the transformations of pseudodifferential operators $q(\Delta_S)$ on the sphere to operators in the plane. It is natural to consider only conformal transformations π, since the invariance of the operators under transformations preserving the Riemann geometry should be recognized.

We start by recollecting some facts and introducing some notations relating to the conformal map projection π. The projection induces a linear mapping π_* from the tangent space T_a to $T_{\pi a}$ for any point $a \in S$. For T_a there is an orthonormal system $\boldsymbol{e}_\phi = R^{-1}\partial/\partial\phi \; \boldsymbol{e}_\lambda = R^{-1}(\cos\phi)^{-1}\partial/\partial\lambda$, where, for azimuthal or conical projections, ϕ is angular distance to the centre point on the sphere and λ azimuth from that point. For the Mercator projection ϕ is instead latitude and λ longitude. When dealing with transversal Mercator mappings, ϕ will be angular distance *to* and angle *along* the central meredian, respectively. For $T_{\pi a}$, which may be identified with the plane P, we use the standard orthonormal system $\boldsymbol{e}_x = \partial/\partial x$ and $\boldsymbol{e}_y = \partial/\partial y$. In terms of the ortonormal basis systems $\boldsymbol{e}_\phi, \boldsymbol{e}_\lambda$ and $\boldsymbol{e}_x, \boldsymbol{e}_y$ the linear mapping π_* is given by the matrix

$$A = R^{-1} \begin{bmatrix} \partial x/\partial\phi & \cos^{-1}\phi\partial x/\partial\lambda \\ \partial y/\partial\phi & \cos^{-1}\phi\partial y/\partial\lambda. \end{bmatrix}$$

Since we assume the map projection to be conformal we have

$$A = sU,$$

where s is the local (length) scale of the map and U is an orthogonal matrix. The area scale is, consequently, s^2.

Now, turning to the problem of transforming the Laplacian Δ_S on the sphere, we shall relate it to the Laplacian Δ_P in the plane. For that purpose it is convenient to use exterior algebra (see e.g. Flanders [**?**]). For a general Riemann manifold M - we shall indeed restrict ourselves to an orientable manifold of dimension 2 or, even more specific, to a sphere or an ellipsoid of rotation - the Laplacian of a function f is defined by

$$\Delta f = *d * df,$$

where d is exterior differentiation, and $*$ the Hodge star operator. For a conformal map projection $(x, y) = \boldsymbol{x} = \boldsymbol{x}(\phi, \lambda)$ of a sphere or an ellipsoid of rotation into the plane

$$\begin{aligned} \boldsymbol{e}_\phi &= \frac{1}{s}\frac{\partial \boldsymbol{x}}{\partial\phi} \\ \boldsymbol{e}_\lambda &= \frac{1}{s\cos\phi}\frac{\partial \boldsymbol{x}}{\partial\lambda} \end{aligned}$$

form (cf. the matrix $A = sU$) an orthogonal moving frame for the plane. Hence

$$d\boldsymbol{x} = sd\phi\boldsymbol{e}_\phi + s\cos\phi d\lambda\boldsymbol{e}_\lambda$$

and we find an orthogonal frame for one-forms in the plane

$$\begin{aligned}\sigma_1 &= s d\phi \\ \sigma_2 &= s\cos\phi d\lambda\end{aligned}$$

with proper orientation so that

$$\begin{aligned}*\sigma_1 &= \sigma_2 \\ *\sigma_2 &= -\sigma_1 \\ *(\sigma_1\sigma_2) &= 1.\end{aligned}$$

Now we can compute the Laplacian

$$\begin{aligned}\Delta_P f &= *d*(s^{-1}f'_\phi\sigma_1 + (s\cos\phi)^{-1}f'_\lambda\sigma_2) \\ &= *d(s^{-1}f'_\phi\sigma_2 - (s\cos\phi)^{-1}f'_\lambda\sigma_1) \\ &= *d(\cos\phi f'_\phi d\lambda - (\cos\phi)^{-1}f'_\lambda d\phi) \\ &= *((\cos\phi f'_\phi)'_\phi + (\cos\phi)^{-1}f''_{\lambda\lambda})d\phi d\lambda) \\ &= *(s^{-2}(\Delta_S f)s^2\cos\phi\, d\phi d\lambda) \\ &= *((s^{-2}\Delta_S f\, dxdy) \\ &= s^{-2}\Delta_S f\end{aligned}$$

We formulate this as a theorem.

Theorem 1 *If π is a conformal map projection of the sphere S_R into the plane P, the corresponding relation between the Laplacians is*

$$\Delta_S = s^2\Delta_P,$$

where s is the length scale of the projection.

Now it is easy to see how invariant pseudodifferential operators on the sphere transform.

Corollary 1 *If $p = q(\Delta_S)$ is an invariant pseudodifferential operator on the sphere S, it is transformed, by a conformal map projection π with local area scale s^2, to the operator $p = q(s^2\Delta_P)$ in the plane.*

3 Application to physical geodesy

The Stokes' problem on the sphere may be written - see Svensson [**?**] - as

$$R^{-1}\{(-\Delta_S + 1/4)^{1/2} - 3/2\}u = v$$

and hence, according to Theorem **??** in terms of the map coordinates

$$R^{-1}\{(-s^2\Delta_P + 1/4)^{1/2} - 3/2\}u = v.$$

For the fixed boundary value problem the corresponding equations are

$$R^{-1}\{(-\Delta_S + 1/4)^{1/2} + 1/2\}u = v \tag{1}$$

and

$$R^{-1}\{(-s^2\Delta_P + 1/4)^{1/2} + 1/2\}u = v.$$

Globally there does not seem to be much of a point in using map coordinates, since obviously we will get the eigenvalues and eigenfunctions just by taking them from the sphere and transforming

them by the projection. However, locally or regionally, the situation may be different. We are going to discuss mainly local problems, which may be solved by a perturbation technique.
Hence, consider the problem

$$
\begin{aligned}
R^{-1}\{(-s^2\Delta_P+1/4)^{1/2}-3/2\}u &= v \text{ in } \Omega & (2)\\
u &= 0 \text{ on } \delta\Omega & (3)
\end{aligned}
$$

Here $H_{1/2}$ denotes the Sobolev-Slobodeckiĭ space of all u such that $(-\Delta_P+1)^{1/4}u$ is square integrable, Ω is a domain in the map plane, and $H^{\circ}_{1/2}(\Omega)$ is the subspace of u vanishing outside Ω
One approach is to work with eigenvalues and eigenfunctions for Dirichlet problems

$$
\begin{aligned}
pu=-s^2\Delta_P u &= \lambda u \text{ in } \Omega\\
u &= 0 \text{ on } \delta\Omega.
\end{aligned}
$$

Such eigenfunctions u satisfy

$$
\begin{aligned}
q(s^2\Delta_p)u &= q(\lambda)u\\
u &\in H^{\circ}_{1/2}(\Omega).
\end{aligned}
$$

For the approximate case s =constant, eigenfunctions for many important classes of domains are very well known. Hence, for a circular disc of radius ρ the eigenvalues are $\lambda=(\alpha_{nk}/\rho)^2$, where α_{nk} is the k:th zero of the Bessel function J_n and the eigenfunctions are $u=J_0(r/\rho)$ if $n=0$ and $u=J_n(r/\rho)\sin(n\theta)$ or $u=J_n(r/\rho)\cos(n\theta)$ if $n>0$.For a rectangle with sides parallell to the $x,y-$ directions and of lengths a,b the eigenvalues are $\lambda_{jk}=((j/a)^2+(k/b)^2)^{1/2},\ j,k\geq 0$ with eigenfunctions $(\sin j(x-x_0)\pi/a)\cdot(\sin k(y-y_0)\pi/b)$, (x_0,y_0) being the lower left corner of the rectangle.
In order to compute corresponding eigenvalues and eigenfunctions for the perturbed system we recall briefly the technique in the simplest case. Hence, let p be a formally selfadjoint operator, densely defined on a Hilbert space H and assume that λ is an eigenvalue with a single eigenfunction u. Let $p+\delta p$ also be selfadjoint and assume that δp is, in some sense small. Then one might hope that there is an eigenvalue $\lambda+\delta\lambda$ and an eigenfunction $u+\delta u$ in some sense close to λ and u respectively. The computation is iterative. First we formulate the equation

$$(p+\delta p)(u+\delta u)=(\lambda+\delta\lambda)(u+\delta u)$$

and rewrite it as

$$(p-\lambda)\delta u=-(\delta p-\delta\lambda)u+\delta\lambda\delta u-\delta p\delta u$$

or, with $\delta u_0=0,\ \delta\lambda_0=0,$

$$(p-\lambda)\delta u_{n+1}=-(\delta p-\delta\lambda_{n+1})u+\delta\lambda_n\delta u_n-\delta p\delta u_n \tag{4}$$

the condition for solvability of the equation is the orthogonality condition

$$((\delta p-\delta\lambda_{n+1})u+\delta\lambda_n\delta u_n-\delta p\delta u_n,u)=0,$$

which yields

$$\delta\lambda_{n+1}=(\delta pu+\delta\lambda_n\delta u_n-\delta p\delta u_n,u)/(u,u)$$

and $\delta u_{n+1}s$ as the solution (preferrably chosen orthogonal to u in order to get uniqueness) of equation (**??**).
The mixed problem (**??**) may be solved by variational methods: Minimize $(pu-v,u),\ u\in M_0$, where the standard scalar product in the plane is used and M_0 is a trial function space of finite dimension. Here we must assume that p is a positive operator on $H^{\circ}_{1/2}(\Omega)$. This is the case if

the diameter of Ω on the sphere is less than 149° - see Svensson [?]. For the fixed boundary value problem (**??**) p is positive for any Ω.
Assume that $\{u_j\}_{j=1}^m$ is a basis for M_0 and that each u_j is an eigenfunction for p i.e. $pu_j = \lambda_j u_j$. Then the variational problem ends up with the problem of solving the system

$$\sum_{k=1}^{m}(pu_j, u_k)x_k = (v, u_j),\ j = 1, 2, \dots m$$

or

$$\sum_{k=1}^{m}\lambda_j(u_j, u_k)x_k = (v, u_j),\ j = 1, 2, \dots, m$$

and putting

$$u = \sum_{j=1}^{m} x_k u_k.$$

Hence, eigenfunctions computed by the perturbation technique may be used favourably as trial functions. One may use for example rectangular or triangular grids and eigenfunctions vanishing outside individual rectangles or triangles.
Another approach is an iterative minimization, where the trial functions used are eigenfunctions but now of the approximate operator

$$\tilde{p} = q(s_0^2\Delta),$$

where s_0 is an approximate value for s in the region. We put $u_0 = 0$ and minimize for $n = 0, 1, 2, \dots$

$$(\tilde{p}u_{n+1} + (p - \tilde{p})u_n - v, u_{n+1}).$$

The question of the convergence of the iteration schemes, wether in the direct or indirect eigenvalue approach, is rather technical and we leave it for the time being.

4 Stereographic projection

Among the azimuthal projections, the stereographic is unique in that it is perfectly conformal. The projection is given by

$$(x, y) = (c \sin\phi/(1 + \cos\phi)(\cos\lambda, \sin\lambda).$$

where c/R is the length scale of the map at the centre, and

$$s = c/(R(1 + \cos\phi))$$

the local length scale varying with the distance to the centre. Expressing this in terms of the map coordinates we get

$$s = \frac{c}{2R}(1 + (\frac{r}{c})^2),$$

where $r = \sqrt{x^2 + y^2}$.

5 The Mercator type projections

Among the cylindrical map projections we only consider the one which is conformal, the Mercator projection. It is given by

$$(x, y) = c(\ln(\tan(\pi/4 + \phi/2), \lambda)$$

with

$$s = c/(R\cos\phi).$$

Here ϕ is angular distance to the central meredian and λ arc length measured along the central meredian. Expressed in map coordinates we get

$$s = (c/R)(\cosh x/c)^{-1}$$

Hence

$$\Delta_S = (c/R)^2(\cosh x/c)^{-2}(\partial^2/\partial x^2 + \partial^2/\partial y^2)$$

It should be observed that, since the scale factor here depends upon x only, computation of eigenfunctions, vanishing outside rectangles, is faciliated by simplifying the problem essentially to a problem in one dimension. The eigenfunctions then have the form

$$u = w(x) \cdot \sin k\pi(y - y_0)/b,$$

where

$$-s^2(x)(w''(x) - (k\pi/b)^2) = \lambda w.$$

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