Ellipsoidal corrections for the inverse Hotine/Stokes formulas

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ABSTRACT

The ellipsoidal corrections respectively for the spherical gravity disturbance (computed using the inverse Hotine formula) and the spherical gravity anomaly (computed using the inverse Stokes formula) are derived. The corrections consist of two parts: the simple analytical function part and the integral part. The input data are respectively the spherical gravity disturbances and the spherical gravity anomalies and the disturbing potentials, which are already available in some local areas and can be computed globally from the geopotential models such as EGM96. Further discussions on the integral part such as the singularity, the input data and the expansion into a series of spherical harmonics are included.

Keywords: Gravity; Disturbing potential; Inverse Stokes's formula; Inverse Hotine's formula; Ellipsoidal correction

1. Introduction

The satellite altimetry technique provides direct measurements of sea surface heights with respect to the reference ellipsoid, the geometrical reference surface of the Earth. Since 1973, a series of altimetry satellites such as SKYLAB, GEOS-3, SEASAT, GEOSAT, ERS-1 and TOPEX have been launched and have collected data over the oceans. Owing to instrument improvement, geophysical and environmental correction improvement and radial orbit error reduction, the precision of satellite altimetry measurements has improved from the 3-metre to the 2-centimetre level. The resolution of satellite altimeter data along the tracks has also come down from 70 km to 20 km (see Zhang 1993). Tremendous amounts of satellite altimeter data with very high precision have been collected since the advent of the satellite altimetry. After subtracting the dynamic sea surface topography, satellite altimetry can provide an estimation of the geoidal height N in ocean areas with a level of precision of about 10 cm [Rummel and Haagmans (1990)]. These geoidal height data can be used to recover the gravity disturbances and gravity anomalies over the oceans.

Papers reporting current results on recovering the gravity information from satellite altimeter data, and in some cases, a review of prior work, include those of Zhang and Blais (1995), Hwang and Parsons (1995), Olgiati et al. (1995), Sandwell and Smith (1996) and Kim (1996). The models employed for recovering the gravity information from the satellite altimeter data are mainly the spherical harmonic expansion of the disturbing potential, the Hotine/Stokes formulas and the inverse Hotine/Stokes formulas. The gravity disturbances/anomalies obtained via these models might as be called the spherical gravity disturbances/anomalies since these models are valid under the spherical approximation. In these models, the input and output data are supposed to be given on a sphere, the mean sphere. Unfortunately, the geoidal height N (disturbing potential T) from the altimetry and the gravity disturbance/anomaly $\delta g/\Delta g$ to be computed from N refer to the geoid which is very close to the reference ellipsoid S_e. They satisfy the following relations:

$$\Delta T(P) = 0 \qquad \text{(P is outside } S_e) \qquad (1.1)$$

$$T(P) = O(\frac{1}{r_P^3}) \qquad \text{(P is at infinity)} \qquad (1.2)$$

$$\frac{\partial}{\partial h_P} T(P) = -\delta g(P) \qquad \text{(P is on } S_e) \qquad (1.3)$$

$$\Delta g(P) = \delta g(P) + \frac{1}{\gamma_P} \frac{\partial \gamma_P}{\partial h_P} \qquad \text{(P is on } S_e) \qquad (1.4)$$

$$T(P) = \gamma_P N(P) \qquad \text{(P is on } S_e) \qquad (1.5)$$

where r_P is the radius of point P and $\frac{\partial}{\partial h_P}$ is the derivative along the ellipsoidal normal direction of P.

The maximum difference between S_e and the mean sphere is about 100 m. So we can treat the data given on the geoid as the data on the reference ellipsoid. The relative error caused by doing so is about the order of 10^{-4}. However, the relative error of substituting the reference ellipsoid by the mean sphercial surface is about the order of 3×10^{-3}. The effects of this error on the gravity anomaly and gravity disturbance, which are also called the effects of the Earth's flattening, may reach about 0.3 mGal. When the aim of the satellite altimetry is to recover the gravity information with accuracy less than 1 mGal, the effects of the Earth's flattening should be considered.

In order to reduce the effects of the Earth's flattening on the gravity anomaly, Wang (1999) proposed to add an ellipsoidal correction term to the spherical gravity anomaly recovered from the altimetry data via the inverse Stokes formula. The ellipsoidal correction is expressed by the integral formulas and in series of spherical harmonic expansions. In the integral formulas, an auxiliary function χ is needed for computing the ellipsoidal correction Δg^1 from the disturbing potential T, that is:

$$T \xrightarrow{\text{global integral}} \chi \xrightarrow{\text{global integral}} \Delta g^1$$

In this paper, we will derive new ellipsoidal correction formulas respectively to the spherical gravity disturbances and the spherical gravity anomalies. They consist of two parts: a simple analytical part and an integral part. The input data are respectively the spherical gravity disturbances and the spherical gravity anomalies and the disturbing potentials, which are already computed from altimetry data in some ocean areas with a high accuracy or are computed approximately from the Earth Models.

2. Formulas for the ellipsoidal corrections to the spherical gravity disturbance and the spherical gravity anomaly

In this section, we will

(a) establish an integral equation, which shows the relation between the geoidal heights and the gravity disturbances on the reference ellipsoid;
(b) solve the integral equation to get the formula for the ellipsoidal correction to the inverse Hotine's formula (the spherical gravity disturbance);
(c) derive the formula for the ellipsoidal correction to the inverse Stokes formula (the spherical gravity anomaly) from the result of (b);

Establishment of the integral equation

It is easy to prove that for an arbitrarily point P_0 given inside S_e, the function

$$F(Q,P_0) \equiv \frac{r_Q^2 - r_{P_0}^2}{r_{P_0} l_{QP_0}^3} = 2\frac{\partial}{\partial r_Q}(\frac{1}{l_{PP_0}}) - \frac{1}{r_{P_0}}(\frac{1}{l_{QP_0}}) \tag{2.1}$$

satisfies

$$\begin{cases} \Delta F(Q,P_0) = 0 \qquad (Q \text{ is outside } S_e) \\ \lim\limits_{Q\to\infty} F(Q,P_0) = 0 \\ F(Q,P_0) \text{ is continuously differentiable on and outside } S_e \end{cases} \tag{2.2}$$

According to Green's second identity (Heiskanen and Moritz, 1962), we obtain that for an arbitrary function V that is harmonic and regular outside S_e and continuously differentiable on and outside S_e,

$$\int_{S_e} V(Q)\frac{\partial F(Q,P_0)}{\partial h_Q} dS_{eQ} = \int_{S_e} \frac{\partial}{\partial h_Q} V(Q) F(Q,P_0) dS_{eQ} \tag{2.3}$$

Let V in (2.3) be the disturbing potential T and V_a defined by (A1-1) in the Appendixes respectively. Then from (1.1), (1.2), (1.3), (A1-2) and (A1-9), we obtain

$$\int_{S_e} T(Q)\frac{\partial F(Q,P_0)}{\partial h_Q} dS_{eQ} = -\int_{S_e} \delta g(Q) F(Q,P_0) dS_{eQ} \tag{2.4}$$

$$\int_{S_e} \frac{\partial F(Q,P_0)}{\partial h_Q} dS_{eQ} = -\int_{S_e} \frac{1}{R}[1 + e^2(\frac{1}{6} - \frac{1}{2}\cos^2\theta_Q) + O(e^4)] F(Q,P_0) dS_{eQ} \tag{2.5}$$

For a given P on S_e, we obtain from (2.4) and (2.5) that

$$\int_{S_e} [T(Q) - T(P)]\frac{\partial F(Q,P_0)}{\partial h_Q} dS_{eQ}$$
$$= \int_{S_e} \{-\delta g(Q) + \frac{T(P)}{R}[1 + e^2(\frac{1}{6} - \frac{1}{2}\cos^2\theta_Q) + O(e^4)]\} F(Q,P_0) dS_{eQ} \tag{2.6}$$

According to (2.1) and the properties of the single-layer potential, we obtain by letting $P_0 \to P$ in (2.6) and neglecting the quantities of the order of $O(e^4)$ that

$$\int_{S_e} [T(Q) - T(P)] M(Q,P) dS_{eQ} = 4\pi\{-\delta g(P) + \frac{T(P)}{R}[1 + e^2(\frac{1}{6} - \frac{1}{2}\cos^2\theta_P)]\}\cos(r_P, h_P)$$
$$+ \int_{S_e} \{-\delta g(Q) + \frac{T(P)}{R}[1 + e^2(\frac{1}{6} - \frac{1}{2}\cos^2\theta_Q)]\} F(Q,P) dS_{eQ} \tag{2.7}$$

where

$$F(Q,P) \equiv \frac{r_Q^2 - r_P^2}{r_P l_{QP}^3} \tag{2.8}$$

and

$$M(Q,P) \equiv \frac{\partial}{\partial h_Q} F(Q,P) = \frac{2r_Q}{r_P l_{QP}^3} \frac{\partial r_Q}{\partial h_Q} - \frac{3(r_Q^2 - r_P^2)}{r_P l_{QP}^4} \frac{\partial l_{QP}}{\partial h_Q} \tag{2.9}$$

The kernel functions M(Q,P) and F(Q,P) are singular when Q→P. Their singularities for Q→P will be discussed in the Section 3.1.

The equation (2.7) is the integral equation from which the inverse Hotine formula and its ellipsoidal correction will be obtained.

2.2. Inverse Hotine' formula and its ellipsoidal correction

Denoting the projection of the surface element dS_{eQ} onto the unit sphere σ by $d\sigma_Q$, we have

$$dS_e = r_Q^2 \sec\beta_Q d\sigma_Q \tag{2.10}$$

where β_Q is the angle between the radius vector of Q and the surface normal of the surface S_e at point Q. With R the mean radius of the ellipsoid ($R = \sqrt[3]{a^2 b}$) and e the first eccentricity of the reference ellipsoid, and θ_P and θ_Q respectively the geocentric co-latitudes of the points P and Q on S_e, we have

$$r_P = R[1 + \frac{1}{2}e^2(\sin^2\theta_P - \frac{2}{3}) + O(e^4)] \tag{2.11a}$$

$$r_Q = R[1 + \frac{1}{2}e^2(\sin^2\theta_Q - \frac{2}{3}) + O(e^4)] \tag{2.11b}$$

$$l_{QP} = 2R\sin\frac{\psi_{QP}}{2}[1 + \frac{1}{4}e^2(\sin^2\theta_Q + \sin^2\theta_P - \frac{4}{3}) + O(e^4)] \tag{2.11c}$$

$$r_Q^2 \sec\beta_Q = R^2[1 + e^2(\sin^2\theta_Q - \frac{2}{3}) + O(e^4)] \tag{2.11d}$$

Furthermore, from Molodensky et al. (1962), we have

$$\frac{\partial r_Q}{\partial h_Q} = \cos(r_Q, h_Q) = 1 + O(e^4) \tag{2.12a}$$

$$\frac{\partial l_{QP}}{\partial h_Q} = \sin\frac{\psi_{QP}}{2}[1 - \frac{1}{4}e^2(3\cos^2\theta_Q + \cos^2\theta_P - \frac{(\cos\theta_Q - \cos\theta_P)^2}{\sin^2\frac{\psi_{QP}}{2}}) + O(e^4)] \tag{2.12b}$$

$$-\frac{1}{\gamma_Q}\frac{\partial \gamma_Q}{\partial h_Q} = \frac{2}{R}[1 + e^2(\cos^2\theta_Q - \frac{1}{6}) + O(e^4)] \tag{2.12c}$$

It then follows from (2.8) and (2.9) that

$$F(Q,P) r_Q^2 \sec\beta_Q = f(\psi_{QP}, \theta_Q, \theta_P)[e^2 + O(e^4)] \tag{2.13}$$

$$M(Q,P) r_Q^2 \sec\beta_Q = \frac{M(\psi_{QP})}{R}[1 + e^2(\frac{1}{2}\cos^2\theta_P - \frac{1}{6}) + O(e^4)] \tag{2.14}$$

where

$$f(\psi_{QP},\theta_Q,\theta_P)=\frac{1}{8}\frac{\sin^2\theta_Q-\sin^2\theta_P}{\sin^3\frac{\psi_{QP}}{2}} \tag{2.15a}$$

$$M(\psi_{QP})=\frac{1}{4\sin^3\frac{\psi_{QP}}{2}} \tag{2.15b}$$

Let

$$\delta g(Q)=\delta g^0(Q)+\delta g^1(Q)e^2+O(e^4) \tag{2.16}$$

Inserting (2.10), (2.13), (2.14) and (2.16) into (2.7) and neglecting the quantities of order of $O(e^4)$, we obtain

$$\begin{aligned}&\int_\sigma [T(Q)-T(P)]\frac{M(\psi_{QP})}{R}[1+e^2(\frac{1}{2}\cos^2\theta_P-\frac{1}{6})]d\sigma_Q\\ &=4\pi\{-\delta g^0(P)+\frac{T(P)}{R}+e^2[-\delta g^1(P)+\frac{T(P)}{R}(\frac{1}{6}-\frac{\cos^2\theta_P}{2})]\}\\ &-e^2\int_\sigma[\delta g^0(Q)-\frac{T(P)}{R}]f(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q\end{aligned} \tag{2.17}$$

Noting (A2-4) in the Appendix and (2.15), it follows that

$$\delta g^0(P)=\frac{T(P)}{R}-\frac{1}{4\pi R}\int_\sigma[T(Q)-T(P)]M(\psi_{QP})d\sigma_Q \tag{2.18}$$

$$\delta g^1(P)=\delta g_1^1(P)+\delta g_2^1(P) \tag{2.19}$$

where

$$\delta g_1^1(P)=(\frac{\cos^2\theta_P}{2}-\frac{1}{6})\delta g^0(P) \tag{2.19a}$$

$$\delta g_2^1(P)=-\frac{1}{4\pi}\int_\sigma\delta g^0(Q)f(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q \tag{2.19b}$$

The formula (2.18) is the inverse Hotine formula, from which the spherical gravity disturbance is computed, and (2.19) is the ellipsoidal correction for the inverse Hotine formula.

2.2. Inverse Stokes' formula and its ellipsoidal correction

According to (1.4) and noting (2.12c), (2.18) and (2.19), we have that

$$\begin{aligned}\Delta g(P)&=\delta g(P)+(\frac{1}{\gamma_P}\frac{\partial\gamma_P}{\partial h_P})T(P)\\ &=\delta g^0(P)-\frac{2T(P)}{R}+e^2[\delta g^1(P)-\frac{2T(P)}{R}(\cos^2\theta_P-\frac{1}{6})]+O(e^4)\end{aligned} \tag{2.20}$$

Let

$$\Delta g(P)=\Delta g^0(P)+\Delta g^1(P)e^2+O(e^4) \tag{2.21}$$

then

$$\Delta g^0(P) = \delta g^0(P) - \frac{2T(P)}{R}$$
$$= -\frac{T(P)}{R} - \frac{1}{4\pi R}\int_\sigma [T(Q) - T(P)]M(\psi_{QP})d\sigma_Q \tag{2.22}$$

$$\Delta g^1(P) = \delta g^1(P) - \frac{2T(P)}{R}(\cos^2\theta_P - \frac{1}{6})$$
$$= (\frac{\cos^2\theta_P}{2} - \frac{1}{6})\delta g^0(P) - \frac{1}{4\pi}\int_\sigma \delta g^0(Q)f(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q - \frac{2T(P)}{R}(\cos^2\theta_P - \frac{1}{6})$$
$$= \Delta g_1^1(P) + \Delta g_2^1(P) \tag{2.23}$$

where

$$\Delta g_1^1(P) = -\frac{T(P)\cos^2\theta_P}{R} + (\frac{\cos^2\theta_P}{2} - \frac{1}{6})\Delta g^0(P) \tag{2.23a}$$

$$\Delta g_2^1(P) = -\frac{1}{2\pi R}\int_\sigma T(Q)f(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q - \frac{1}{4\pi}\int_\sigma \Delta g^0(Q)f(\psi_{QP},\theta_Q,\theta_P)d\sigma_Q \tag{2.23b}$$

The formula (2.22) is the inverse Stokes formula, from which the spherical gravity anomaly is computed, and (2.23) is the ellipsoidal correction for the inverse Stokes formula.

3. Practical considerations for the integrals in the formulas

In the above section, we obtained the closed formulas (2.19) and (2.23) of the ellipsoidal corrections δg^1 and Δg^1 respectively to the inverse Hotine formula (2.18) (the spherical gravity disturbance δg^0) and the inverse Stokes formula (2.22) (the spherical gravity anomaly Δg^0) from the basic integral equation (2.7). The formula (2.19) (formula (2.23)) is expressed as a sum of a simple analytical function and an integral about δg^0 (Δg^0 and T). Obviously, the first part of δg^1 (Δg^1) is easy to compute from δg^0 (Δg^0 and T). In the following, we will discuss the integral parts (2.19b) and (2.23b).

3.1. Singularities

The integrals in the formulas (2.7), (2.18), (2.19b), (2.22) and (2.23b) are singular because their kernel functions M(Q, P), F(Q, P) and $M(\psi_{QP})$, $f(\psi_{QP},\theta_Q,\theta_P)$ are singular when Q→P or $\psi_{QP} \to 0$.

The singularity of the integral in the inverse Stokes (or Hotine) formula (2.22) (or (2.18)) has been discussed in many references such as Heiskanen and Moritz (1967) and Zhang (1993). Here we discuss the singularities of the integrals in (2.7), (2.19b) and (2.23b).

According to (2.13), we know that the integral in the left side of (2.7) and the integrals in the inverse Stokes formula (2.18) and the inverse Stokes formula (2.22) have the same form. So the integral in the left side of (2.7) can be treated with the same method used in processing the inverse Stokes (Hotine) formula.

Similarly according to (2.14), the integral in the left side of (2.7) and the integrals in (2.19b) and (2.23b) have the same form. So in the following, we only discuss the method to treat the singularity of the integral in (2.19b).

Obviously, we only need to consider the integral in the innermost spherical cap area σ_0 with the center at the computation point P and the radius ψ_0, which is so small that the spherical cap area can be treated as a plane. That is we discuss the following integral

$$\overline{\delta g}(P) = \frac{1}{4\pi}\int_{\sigma_0} \delta g^0(Q) f(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q \tag{3.1}$$

From (A2-2) in the Appendix and (2.15a), and noting that σ is a unit sphere, we have

$$\begin{aligned}\overline{\delta g}(P) = &\frac{1}{4\pi}\int_{\psi_{QP}=0}^{\psi_0}\int_{\alpha_{QP}=0}^{2\pi} \delta g^0(Q)\frac{1}{8\sin^3\frac{\psi_{QP}}{2}}\{4\sin^2\frac{\psi_{QP}}{2}\cos^2\frac{\psi_{QP}}{2}[\cos^2\theta_P - \sin^2\theta_P\cos^2\alpha_{QP}]\\ &- 2\cos\theta_P\cos\psi_{QP}\sin\theta_P\sin\psi_{QP}\cos\alpha_{QP}\}\sin\psi_{QP}d\alpha_{QP}d\psi_{QP}\\ = &\frac{1}{4\pi}\int_{l_{QP}=0}^{l_0}\int_{\alpha_{QP}=0}^{2\pi}\delta g^0(Q)\{(1-\frac{l_{QP}^2}{4})[\cos^2\theta_P - \sin^2\theta_P\cos^2\alpha_{QP}]\\ &-\frac{1}{l_{QP}}(1-\frac{l_{QP}^2}{2})(1-\frac{l_{QP}^2}{4})^{\frac{1}{2}}\sin 2\theta_P\cos\alpha_{QP}\}d\alpha_{QP}dl_{QP}\end{aligned} \tag{3.2}$$

where

$$l_0 = 2\sin\frac{\psi_0}{2}. \tag{3.3}$$

For Q in σ_0, we expand $\delta g^0(Q)$ into a Taylor series at the computation point P:

$$\delta g^0(Q) = \delta g^0(P) + x\delta g_x^0(P) + y\delta g_y^0(P) + \cdots \tag{3.4}$$

where the rectangular coordinates x, y are defined by

$$x = l_{QP}\cos\alpha_{QP}; \quad y = l_{QP}\sin\alpha_{QP} \tag{3.5}$$

so that the x-axis points North, and

$$\delta g_x^0(P) = \frac{\partial\delta g^0}{\partial x}(P); \ \delta g_y^0(P) = \frac{\partial\delta g^0}{\partial y}(P) \tag{3.6}$$

The Taylor series (3.4) may also be written as

$$\delta g^0(Q) = \delta g^0(P) + [\delta g_x^0(P)\cos\alpha_{QP} + \delta g_y^0(P)\sin\alpha_{QP}]l_{QP} + \cdots \tag{3.7}$$

Inserting this into (3.2), performing the integral with respect to α_{QP} first, noting (A2-3) in the Appendix and neglecting the quantities of $O(l_0^2)$, we have

$$\overline{\delta g}(P) = \frac{l_0}{4}[\delta g^0(P)(3\cos^2\theta_P - 1) + \delta g_x^0(P)] \tag{3.8}$$

We see that the effect of the innermost spherical cap area on the integral (2.19b) depends, to a first approximation, on $\delta g^0(P)$ and $\delta g^0_x(P)$. The value of $\delta g^0_x(P)$ can be obtained from the map of δg^0. It is the inclination of North-South profile through P.

3.2. Input data

In (2.19b) and (2.23b), the input data are respectively δg^0, and Δg^0 and T. These data are available only in some ocean areas. Here we give a little modification on the input data.

According to (2.16) and (2.21), we have

$$\delta g^0(Q) = \delta g(Q) - \delta g^1(Q)e^2 + O(e^4) \tag{3.9}$$

$$\Delta g^0(Q) = \Delta g(Q) - \Delta g^1(Q)e^2 + O(e^4) \tag{3.10}$$

In addition, the disturbing potential T(P) on the reference ellipsoid can be expressed as

$$T(P) = T^0(P) + e^2 T^1(P) \tag{3.11}$$

where $T^0(P)$ is the spherical approximation of T(P). Since δg^1 should be multiplied by e^2 before it is added to δg^0, we obtain by inserting respectively above formulas into the integrals in (2.19) and (2.22) and neglecting the quantities of order of $O(e^2)$ that

$$\delta g^1_2(P) = \frac{1}{4\pi}\int_\sigma \delta g(Q) f(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q \tag{3.12}$$

$$\Delta g^1_2(P) = \frac{1}{4\pi}\int_\sigma [\Delta g(Q) + \frac{2T^0(Q)}{R}] f(\psi_{QP}, \theta_Q, \theta_P) d\sigma_Q \tag{3.13}$$

where δg is the gravity disturbance which can be computed approximately from the global geopotential models, Δg and T^0 are respectively the gravity anomaly and the spherical disturbing potential which are already available globally with the resolutions of less than 1 degree and the accuracy of a few metres and locally with higher resolutions and higher accuracy.

3.3. Spherical harmonic expansions of the integrals

In the following, we will expand $\delta g^1_2(P)$ and $\Delta g^1_2(P)$ into series of spherical harmonics so that they can be computed from the global geopotential models.

According to Chapter 2-14 of Heiskanen and Moritz (1967), under the spherical approximation, we have

$$\delta g(\theta, \lambda) = \frac{1}{R}\sum_{n=2}^{\infty}(n+1)T_n(\theta, \lambda) \tag{3.14}$$

where $T_n(\theta, \lambda)$ is Laplace's surface harmonics of the disturbing potential T:

$$T_n(\theta,\lambda) = \sum_{m=0}^{n}[c_{nm}R_{nm}(\theta,\lambda) + d_{nm}S_{nm}(\theta,\lambda)] \tag{3.15}$$

Let

$$\delta g(\theta,\lambda)\cos^2\theta = \frac{1}{R}\sum_{n=2}^{\infty}(n+1)X_n(\theta,\lambda) \tag{3.16}$$

From (2.22) and the definitions of $\delta g_2^1(P)$ and $\Delta g_2^1(P)$, we know that these two integrals are equal. So according to (1-102) of Heiskanen and Moritz (1967), we have from (3.12) that

$$\Delta g_2^1(P) = \delta g_2^1(P) = \frac{1}{4\pi}\int_\sigma \frac{\delta g(Q)\cos^2\theta_Q - \delta g(P)\cos^2\theta_P}{8\sin^3\frac{\psi_{QP}}{2}}d\sigma_Q - \frac{\cos^2\theta_P}{4\pi}\int_\sigma \frac{\delta g(Q) - \delta g(P)}{8\sin^3\frac{\psi_{QP}}{2}}d\sigma_Q$$

$$= \frac{1}{2R}[-\sum_{n=2}^{\infty}n(n+1)X_n(\theta_P,\lambda_P) + \sum_{n=2}^{\infty}n(n+1)T_n(\theta_P,\lambda_P)\cos^2\theta_P] \tag{3.17}$$

According to (A3-4) and (A3-5),

$$X_n(\theta,\lambda) = \sum_{m=0}^{n}[\delta E_{nm}R_{nm}(\theta,\lambda) + \delta F_{nm}S_{nm}(\theta,\lambda)] \tag{3.18}$$

$$\sum_{n=2}^{\infty}n(n+1)T_n(\theta,\lambda)\cos^2\theta = \sum_{n=2}^{\infty}n(n+1)\sum_{m=0}^{n}[\delta G_{nm}R_{nm}(\theta,\lambda) + \delta H_{nm}S_{nm}(\theta,\lambda)] \tag{3.19}$$

where $\{\delta E_{nm}, \delta F_{nm}\}$ and $\{\delta G_{nm}, \delta H_{nm}\}$ are defined as follows

$$\begin{Bmatrix}\delta E_{nm}\\ \delta F_{nm}\end{Bmatrix} = \frac{n-1}{n+1}\alpha_{n-2}^m\begin{Bmatrix}c_{n-2m}\\ d_{n-2m}\end{Bmatrix} + \beta_n^m\begin{Bmatrix}c_{nm}\\ d_{nm}\end{Bmatrix} + \frac{n+3}{n+1}\gamma_{n+2}^m\begin{Bmatrix}c_{n+2m}\\ d_{n+2m}\end{Bmatrix} \tag{3.20}$$

$$\begin{Bmatrix}\delta G_{nm}\\ \delta H_{nm}\end{Bmatrix} = \frac{(n-2)(n-1)}{n(n+1)}\alpha_{n-2}^m\begin{Bmatrix}c_{n-2m}\\ d_{n-2m}\end{Bmatrix} + \beta_n^m\begin{Bmatrix}c_{nm}\\ d_{nm}\end{Bmatrix} + \frac{(n+2)(n+3)}{n(n+1)}\gamma_{n+2}^m\begin{Bmatrix}c_{n+2m}\\ d_{n+2m}\end{Bmatrix} \tag{3.21}$$

with

$$\alpha_n^m = \frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)} \tag{3.22}$$

$$\beta_n^m = \frac{2n^2 - 2m^2 + 2n - 1}{(2n+3)(2n-1)} \tag{3.23}$$

$$\gamma_n^m = \frac{(n+m)(n+m-1)}{(2n+1)(2n-1)} \tag{3.24}$$

So we obtain from (3.17) that

$$\Delta g_2^1(P) = \delta g_2^1(P) = \frac{1}{R}\sum_{n=2}^{\infty}\sum_{m=0}^{n}[\delta A_{nm}R_{nm}(\theta_P,\lambda_P) + \delta B_{nm}S_{nm}(\theta_P,\lambda_P)] \tag{3.25}$$

where

$$\begin{Bmatrix} \delta A_{nm} \\ \delta B_{nm} \end{Bmatrix} = -\frac{(n-1)(n-m-1)(n-m)}{(2n-1)(2n-3)} \begin{Bmatrix} c_{n-2m} \\ d_{n-2m} \end{Bmatrix} + \frac{(n+3)(n+m+1)(n+m+2)}{(2n+5)(2n+3)} \begin{Bmatrix} c_{n+2m} \\ d_{n+2m} \end{Bmatrix} \tag{3.26}$$

Thus we express $\delta g_2^1(P)$ and $\Delta g_2^1(P)$ by a series of spherical harmonics. The input data $\{c_{nm}, d_{nm}\}$ are the spherical harmonic coefficients of the disturbing potential.

5. Conclusions

This paper gives the ellipsoidal corrections $\delta g^1(P)$ and $\Delta g^1(P)$ for the inverse Hotine formula, the spherical gravity disturbance $\delta g^0(P)$, and the inverse Stokes formula, the spherical gravity anomaly $\Delta g^0(P)$, respectively.

- By adding the ellipsoidal corrections to their spherical solutions, the error of the gravity disturbance and the gravity anomaly decrease from $O(e^2)$ to $O(e^4)$, which is sufficient for most practical purposes.

- $\delta g^1(P)$ is expressed as a sum of a simple analytical function of $\delta g^0(P)$ and an integral in terms of δg^0. In the practical computation of the integral, the input data δg^0 can be substituted by the gravity disturbance δg, which can be approximately computed from the global geopotential models. The integral part of $\delta g^1(P)$ can also be computed directly from the global geopotential models via the formula (3.25).

- $\Delta g^1(P)$ is expressed as a sum of a simple analytical function of $\Delta g^0(P)$ and T(P) and an integral in terms of Δg^0 and T. In the practical evaluation of the integral, the input data Δg^0 and T(P) can be substituted respectively by the gravity anomaly Δg and the spherical disturbing potential T^0, which are already available globally with resolutions better than 1 degree and accuracy within a few metres, and locally with higher resolutions and higher accuracy. The integral part of $\Delta g^1(P)$ can also be computed directly from the global geopotential models via the formula (3.25).

- Comparing to the ellipsoidal correction to gravity anomaly given in Wang (1999), $\Delta g^1(P)$ is simpler not only in the formulation but also in the auxiliary data Δg^0 (or Δg), in comparison to the auxiliary data χ used in Wang (1999).

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Appendix

1. Denote respectively the ellipsoidal coordinates and the spherical coordinates of a point P by (u_P, β_p, λ_P) and (r_P, θ_P, λ_P). According to Chapter 1-20 of Heiskanen and Moritz (1967), we know that

$$V_a(P) \equiv \frac{Q_0(i\frac{u_P}{E})}{Q_0(i\frac{b}{E})} \tag{A1-1}$$

is harmonic and regular outside S_e and continuously differentiable on and outside S_e, and for Q on S_e,

$$V_a(Q)=1 \tag{A1-2}$$

From Chapter 2-7, 2-8 and 2-9 of Heiskanen and Moritz (1967), we have

$$Q_0(i\frac{u_P}{E}) = -i\tan^{-1}\frac{E}{u_P} \tag{A1-3}$$

$$\frac{\partial V_a}{\partial h_P}(P) = -\sqrt{\frac{u_P^2+E^2}{u_P^2+E^2\sin^2\beta_P}}\frac{\partial V_a}{\partial u_P}(P) \tag{A1-4}$$

and

$$\tan^{-1}\frac{E}{b} = \frac{E}{b}[1-\frac{1}{3}(\frac{E}{b})^2+O((\frac{E}{b})^4)] = e'[1-\frac{1}{3}e'^2+O(e'^4)] \tag{A1-5}$$

So for Q on S_e,

$$\frac{\partial V_a}{\partial h_Q}(Q) = -\sqrt{\frac{b^2+E^2}{b^2+E^2\sin^2\beta_Q}}\frac{\frac{E}{b^2+E^2}}{\tan^{-1}\frac{E}{b}} = -\frac{1}{a\sqrt{1+e'^2\sin^2\beta_Q}}\frac{1}{1-\frac{1}{3}e'^2+O(e'^4)}$$
$$= -\frac{1}{a}[1+e'^2(\frac{1}{3}-\frac{1}{2}\sin^2\beta_Q)+O(e'^4)] \tag{A1-6}$$

Since

$$a = R(1+\frac{1}{6}e^2);\ e'^2 = e^2 + O(e^4) \tag{A1-7}$$

and

$$\sin^2\beta_Q = \frac{\tan^2\beta_Q}{1+\tan^2\beta_Q} = \frac{a^2\cos^2\theta_Q}{b^2\sin^2\theta_Q + a^2\cos^2\theta_Q} = \cos^2\theta_Q[1+e^2\sin^2\theta_Q + O(e^4)]$$
$$= \cos^2\theta_Q + O(e^2) \tag{A1-8}$$

Equation (A-6) can be rewritten as

$$\frac{\partial V_a}{\partial h_Q}(Q) = -\frac{1}{R}[1+e^2(\frac{1}{6}-\frac{1}{2}\cos^2\theta_Q) + O(e^4)] \tag{A1-9}$$

2. From the spherical triangle of Figure 1, we have

$$\cos\theta_Q = \cos\psi_{QP}\cos\theta_P + \sin\psi_{QP}\sin\theta_P\cos\alpha_{QP} \tag{A2-1}$$

and hence

$$\begin{aligned}\sin^2\theta_Q - \sin^2\theta_P &= \cos^2\theta_P - \cos^2\theta_P\cos^2\psi_{QP} - \sin^2\theta_P\sin^2\psi_{QP}\cos^2\alpha_{QP}\\ &\quad - 2\cos\theta_P\cos\psi_{QP}\sin\theta_P\sin\psi_{QP}\cos\alpha_{QP}\\ &= 4\sin^2\frac{\psi_{QP}}{2}\cos^2\frac{\psi_{QP}}{2}[\cos^2\theta_P - \sin^2\theta_P\cos^2\alpha_{QP}]\\ &\quad - 2\cos\theta_P\cos\psi_{QP}\sin\theta_P\sin\psi_{QP}\cos\alpha_{QP}\end{aligned} \tag{A2-2}$$

Noting that

$$\int_0^{2\pi} d\alpha_{QP} = 2\pi;\ \int_0^{2\pi}\cos\alpha_{QP}d\alpha_{QP} = 0;\ \int_0^{2\pi}\cos^2\alpha_{QP}d\alpha_{QP} = \pi \tag{A2-3}$$

we obtain that

$$\begin{aligned}\int_\sigma \frac{\sin^2\theta_Q - \sin^2\theta_P}{\sin^3\frac{\psi_{QP}}{2}}d\sigma &= \int_0^\pi\int_0^{2\pi}\frac{\sin^2\theta_Q - \sin^2\theta_P}{\sin^3\frac{\psi_{QP}}{2}}4\sin\frac{\psi_{QP}}{2}d\sin\frac{\psi_{QP}}{2}d\alpha_{QP}\\ &= 16\pi(2\cos^2\theta_P - \sin^2\theta_P)\int_0^1(1-x^2)dx\\ &= 16\pi(2\cos^2\theta_P - \frac{2}{3})\end{aligned} \tag{A2-4}$$

3. According to Chapter 2-14 of Heiskanen and Moritz (1967), under the spherical approximation, we have

$$\delta g(\theta,\lambda) = \frac{1}{R}\sum_{n=2}^{\infty}(n+1)T_n(\theta,\lambda) \tag{A3-1}$$

and

$$\Delta g(\theta,\lambda) = \frac{1}{R}\sum_{n=2}^{\infty}(n-1)T_n(\theta,\lambda) \tag{A3-2}$$

where $T_n(\theta, \lambda)$ is Laplace's surface harmonics of the disturbing potential T:

$$T_n(\theta,\lambda) = \sum_{m=0}^{n}[c_{nm}R_{nm}(\theta,\lambda) + d_{nm}S_{nm}(\theta,\lambda)] \tag{A3-3}$$

Let

$$\sum_{n=2}^{\infty}(n+1)T_n(\theta,\lambda)\cos^2\theta \equiv \sum_{n=2}^{\infty}(n+1)\sum_{m=0}^{n}[\delta E_{nm}R_{nm}(\theta,\lambda) + \delta F_{nm}S_{nm}(\theta,\lambda)] \tag{A3-4}$$

$$\sum_{n=2}^{\infty}n(n+1)T_n(\theta,\lambda)\cos^2\theta \equiv \sum_{n=2}^{\infty}n(n+1)\sum_{m=0}^{n}[\delta G_{nm}R_{nm}(\theta,\lambda) + \delta H_{nm}S_{nm}(\theta,\lambda)] \tag{A3-5}$$

From (A11) of Wang (1999) (Note: there is a printing error in that formula) and (A3-3), we know that

$$\begin{aligned} T_n(\theta,\lambda)\cos^2\theta = \sum_{m=0}^{n}\{&c_{nm}[\alpha_n^m R_{n+2m}(\theta,\lambda) + \beta_n^m R_{nm}(\theta,\lambda) + \gamma_n^m R_{n-2m}(\theta,\lambda)] \\ &+ d_{nm}[\alpha_n^m S_{n+2m}(\theta,\lambda) + \beta_n^m S_{nm}(\theta,\lambda) + \gamma_n^m S_{n-2m}(\theta,\lambda)]\} \end{aligned} \tag{A3-6}$$

where

$$\alpha_n^m = \frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)} \tag{A3-7}$$

$$\beta_n^m = \frac{2n^2-2m^2+2n-1}{(2n+1)(2n-1)} \tag{A3-8}$$

$$\gamma_n^m = \frac{(n+m)(n+m-1)}{(2n+1)(2n-1)} \tag{A3-9}$$

So

$$\begin{aligned} \begin{Bmatrix}\delta E_{nm}\\ \delta F_{nm}\end{Bmatrix} &= \sum_{k=2}^{\infty}\frac{k+1}{n+1}N_n^m\int_{\sigma_0}T_k(\theta,\lambda)\cos^2\theta\begin{Bmatrix}R_{nm}(\theta,\lambda)\\ S_{nm}(\theta,\lambda)\end{Bmatrix}d\sigma_0 \\ &= \frac{n-1}{n+1}\alpha_{n-2}^m\begin{Bmatrix}c_{n-2m}\\ d_{n-2m}\end{Bmatrix} + \beta_n^m\begin{Bmatrix}c_{nm}\\ d_{nm}\end{Bmatrix} + \frac{n+3}{n+1}\gamma_{n+2}^m\begin{Bmatrix}c_{n+2m}\\ d_{n+2m}\end{Bmatrix} \end{aligned} \tag{A3-10}$$

$$\begin{aligned} \begin{Bmatrix}\delta G_{nm}\\ \delta H_{nm}\end{Bmatrix} &= \sum_{k=2}^{\infty}\frac{k(k+1)}{n(n+1)}N_n^m\int_{\sigma_0}T_k(\theta,\lambda)\cos^2\theta\begin{Bmatrix}R_{nm}(\theta,\lambda)\\ S_{nm}(\theta,\lambda)\end{Bmatrix}d\sigma_0 \\ &= \frac{(n-2)(n-1)}{n(n+1)}\alpha_{n-2}^m\begin{Bmatrix}c_{n-2m}\\ d_{n-2m}\end{Bmatrix} + \beta_n^m\begin{Bmatrix}c_{nm}\\ d_{nm}\end{Bmatrix} + \frac{(n+2)(n+3)}{n(n+1)}\gamma_{n+2}^m\begin{Bmatrix}c_{n+2m}\\ d_{n+2m}\end{Bmatrix} \end{aligned} \tag{A3-11}$$

Similarly letting

$$\sum_{n=2}^{\infty}(n-1)T_n(\theta,\lambda)\cos^2\theta \equiv \sum_{n=2}^{\infty}(n-1)\sum_{m=0}^{n}[\Delta E_{nm}R_{nm}(\theta,\lambda) + \Delta F_{nm}S_{nm}(\theta,\lambda)] \tag{A3-12}$$

$$\sum_{n=2}^{\infty}n(n-1)T_n(\theta,\lambda)\cos^2\theta \equiv \sum_{n=2}^{\infty}n(n-1)\sum_{m=0}^{n}[\Delta G_{nm}R_{nm}(\theta,\lambda) + \Delta H_{nm}S_{nm}(\theta,\lambda)] \tag{A3-13}$$

then

$$\begin{Bmatrix} \Delta E_{nm} \\ \Delta F_{nm} \end{Bmatrix} = \frac{n-3}{n-1}\alpha_{n-2}^{m}\begin{Bmatrix} c_{n-2m} \\ d_{n-2m} \end{Bmatrix} + \beta_{n}^{m}\begin{Bmatrix} c_{nm} \\ d_{nm} \end{Bmatrix} + \frac{n+1}{n-1}\gamma_{n+2}^{m}\begin{Bmatrix} c_{n+2m} \\ d_{n+2m} \end{Bmatrix} \tag{A3-14}$$

$$\begin{Bmatrix} \Delta G_{nm} \\ \Delta H_{nm} \end{Bmatrix} = \frac{(n-2)(n-3)}{n(n-1)}\alpha_{n-2}^{m}\begin{Bmatrix} c_{n-2m} \\ d_{n-2m} \end{Bmatrix} + \beta_{n}^{m}\begin{Bmatrix} c_{nm} \\ d_{nm} \end{Bmatrix} + \frac{(n+2)(n+1)}{n(n-1)}\gamma_{n+2}^{m}\begin{Bmatrix} c_{n+2m} \\ d_{n+2m} \end{Bmatrix} \tag{A3-15}$$

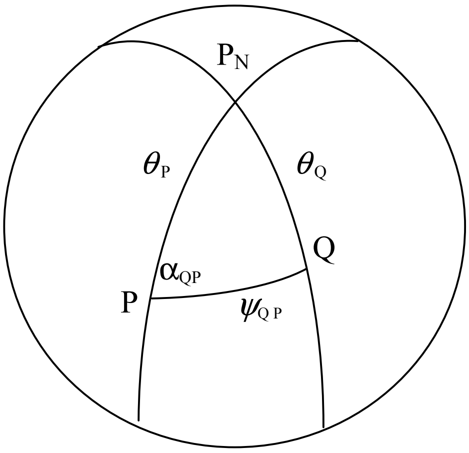

Figure 1. Spherical triangle