Ellipsoidal and topographical effects in the scalar free geodetic boundary value problem

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Abstract

In the formulation of the *scalar free boundary value problem* we solve for the gravity potential in the external space outside the earth's surface and for the vertical position of the boundary surface. After linearization the reduced boundary condition refers to the *Telluroid* $s \ni p$, and the new difference quantity δw, the disturbing potential, is introduced. To represent the unknown disturbing potential in the global basis of spherical harmonics a *harmonic analysis* has to be applied to the given boundary data. In this context the boundary data have to be *(downward) continued* from s to a reference surface which shows a rotational symmetry with respect to the earth's mean rotational axis. In general a sphere K or the surface E of an ellipsoid of revolution is selected.

After the analytical continuation of the evaluation operator E_s the boundary condition can be split in two parts. The main component is covered by the *isotropic term* which corresponds to the Stokes–problem. The residual part consists of the *ellipsoidal* and *topographical* components which are functionals of δw. Therefore an iterative solution strategy is appropriate. First numerical evaluations indicate that this iterative process converges for boundary data continued to an ellipsoid, but diverges if the boundary data is continued to a sphere.

Introduction

Geodetic boundary value problems represent idealized situations in geodetic data analysis. Neither the geodetic observations are continuous, nor they are given on the whole surface of the earth. Nevertheless the formulation in the framework of boundary value problems has two important aspects: First, in studying idealized problems in an "ideal" form, mathematical tools can be used which never can applied to real situations, providing deep results concerning the mathematical analysis, which can be generalized to more difficult situations. Second, for special data distributions analytical solutions can be derived which directly can be applied in data evaluation.

In recent years the challenge in the field of the geodetic boundary value problem has been directed to formulations approaching more and more the real data situation in Geodesy. The surface of the earth is now considered as a non–spherical surface, influenced by topography and ellipticity. Mixed boundary value problems have been investigated as well as problems induced by satellite and airborne data which will be available in near future. In this review paper we will restrict to "classical" boundary value problems on the background of geodetic data given on the earth's surface; also over–determined problems will not be considered here.

To determine both the external gravity potential W and the geometry of the earth's surface S, various *boundary value problems* (bvp) can be formulated. They depend on the utilized *observables* L and whether the boundary is supposed to be known or unknown. If the geometry of S is already determined by the classical terrestrial techniques or by methods of satellite geodesy, then the *fixed boundary value problem* (Klees, 1992, 1995) is under consideration. Otherwise the resulting bvp is of free type. In Grafarend et al. (1985) the *vectorial free bvp* is discussed which differs in many respects from the scalar free bvp, first of all tackled by Sacerdote and Sansò (1986). The relation between the unknowns W, S and the observables L is given by *boundary conditions*. Generally, they are of non–linear structure. In the following we want to focus on the *scalar free bvp*.

The scalar free boundary value problem

In the formulation of the *scalar free bvp* the horizontal position of each point P on the boundary $S \ni P$ is assumed to be given. So we have to introduce beside the external gravity potential W the vertical position P as a geometrical unknown. To solve for both unknowns two types of observables have to be given on the boundary surface in continuous form. We can assume, that the gravity potential $W(P)$ as well as gravity values $\Gamma(P) = |grad\, W(P)|$ on $S \ni P$ are given as boundary data. Assuming that the earth is rigid, rotating with the constant angular velocity ω and giving rise to the non–harmonic centrifugal potential Z, we can formulate the **non–linear scalar free bvp**: *Suppose the boundary data $W(P)$ and $\Gamma(P)$ to be given on $S \ni P$. The unknown gravity potential $W(\mathbf{x})$ has to fulfill the extended Laplace equation in the mass free space Ω_a outside S, and the gravitational potential V tends to zero if the geocentric distance $r = |\mathbf{x}|$ tends to infinity (radiation condition):*

$$\begin{array}{llll} Lap\, W(\mathbf{x}) & = & 2\omega^2\,, & \mathbf{x} \in \Omega_a \\ V(\mathbf{x}) & \sim & \frac{1}{r} + O\left(\frac{1}{r^3}\right)\,, & r \to \infty \\ W(P) & = & V(P) + Z(P)\,, & P \in S \\ \Gamma(P) & = & |grad\, W(P)|\,, & P \in S\,. \end{array} \tag{1}$$

To deal with small quantities, we have to introduce suitable approximations for the unknowns. The potential $W(r,\beta,\lambda)$ will be approximated through the analytical normal potential $w(r,\beta,\lambda)$. Here the gravity field of an equipotential reference ellipsoid (*Somigliana–Pizzetti* normal field) is often introduced. This potential is analytically easy to handle and shows a symmetry with respect to the earth's mean rotational axis:

$$\begin{array}{lll} w(r,\beta) & = & v(r,\beta) + z(r,\beta) \\ v(r,\beta) & = & \dfrac{\mu^v}{r}\left[1 - \sum\limits_k \left(\dfrac{a}{r}\right)^k J_k P_k(\sin\beta)\right]\,, \text{ with } \quad k \in \{2,4,6,\ldots,N^v\}\,. \end{array} \tag{2}$$

Alternatively we can combine the well known centrifugal potential

$$z(r,\beta) \;=\; Z(r,\beta) \;=\; \frac{1}{2}\omega^2 r^2 \cos^2\beta \;=\; \frac{1}{2}\frac{\mu^v}{r}\overline{m}\left(\frac{r}{a}\right)^3 \cos^2\beta\,, \tag{3}$$

containing the small constant $\overline{m} = \omega^2 a^3/\mu^v$, together with a truncated spherical harmonic expansion of the gravity potential V (for example up to degree and order $N^v = 20$). In the latter case the difference quantities become smaller, but the analytical amount of the whole problem increases.

To approximate the boundary surface S a telluroid mapping $S \ni P \to s \ni p$ is defined. The most natural choice amongst several possible mappings (Hirvonen, 1960; Grafarend, 1978; Heck, 1986) is Molodensky's telluroid definition with the mapping equations:

$$\begin{array}{lll} B(p) & = & B(P) \\ L(p) & = & L(P) \\ w(p) - w(p_o) & = & W(P) - W(P_o)\,. \end{array} \tag{4}$$

The telluroid point p lies on the same ellipsoidal normal as P, parameterized through the geographical latitude $B(P)$ and the geographical longitude $L(P)$. It is fixed along this ellipsoidal normal $\mathbf{n}$ in such a way, that p has the same geopotential number in the normal field, as P in the actual field. P_o denotes a global height reference point, e.g. a tide gaugue, while p_o is the corresponding point on the ellipsoid.

Now we are able to formulate difference quantities, which act as new unknowns: The difference of the actual potential $W(Q)$ and the normal potential $w(Q)$ in the same point $Q \in \Omega_a$

$$\delta w \quad = \quad W(Q) - w(Q) \tag{5}$$

is called the *disturbing potential*. And the second (geometrical) unknown, the *height anomaly*

$$\Delta h \quad = \quad h(P) - h(p) \tag{6}$$

is the difference between the ellipsoidal height of P and the ellipsoidal height of the telluroid point p respectively. In the same way both boundary data are redefined: The *potential anomaly*

$$\Delta w \; = \; W(P) - w(p) \tag{7}$$

equals zero if the absolute potential W_o and w_o are selected in the framework of a proper datum definition (Heck and Rummel, 1990; Rummel and Ilk, 1995) in such a way that $\Delta w_o = W(P_o) - w(p_o) = 0$ holds. This fact becomes obvious if we compare equation (4). The scalar *gravity anomaly* $\Delta\gamma$ is defined as follows:

$$\Delta\gamma \; = \; \Gamma(P) - \gamma(p)\,. \tag{8}$$

After these considerations we can reformulate the still non–linear problem: *Suppose the boundary data Δw and $\Delta\gamma$ to be given on S. The unknown disturbing potential $\delta w(\mathbf{x})$ has to fulfill the Laplace equation in the mass free space Ω_a outside S, and the disturbing potential δw tends to zero if the geocentric distance r tends to infinity:*

$$\begin{array}{lcll} Lap\; \delta w(\mathbf{x}) & = & 0\,, & \mathbf{x} \in \Omega_a \\ \delta w(\mathbf{x}) & \sim & \frac{1}{r} + O\left(\frac{1}{r^3}\right)\,, & r \to \infty \\ \Delta w & = & W(P) - w(p) = \delta w(P) + w(P) - w(p) & \\ \Delta\gamma & = & \Gamma(P) - \gamma(p) = |grad\,[\delta w(P) + w(P)]\,| - \gamma(p)\,. & \end{array} \tag{9}$$

Linearization of the reduced boundary condition

In the formulation of the non–linear problem (9) the normal potential w, that occurs in the boundary condition, must be calculated in p and in the boundary point P. Since the vertical position of P is unknown, this formulation of the boundary conditions is unsuitable. Therefore a Taylor series expansion for the disturbing potential δw and the normal potential is set up in the known telluroid point p (Heck, 1988):

$$\begin{array}{lcl} \delta w(P) & = & \delta w(p) + (\partial_i \delta w \cdot n_i)\,\Delta h + \ldots \\ w(P) & = & w(p) + (\partial_i w \cdot n_i)\,\Delta h + \frac{1}{2}\,(\partial_{ij} w \cdot n_i \cdot n_j)\,(\Delta h)^2 + \ldots\,. \end{array} \tag{10}$$

The Taylorstep $\Delta h = h(P) - h(p)$ runs along the ellipsoidal normal $\mathbf{n}$ due to the used telluroid definition of Molodensky (4) and is equal to the height anomaly. The symbols n_i and ∂_i denote the Cartesian coordinates of the ellipsoidal unit normal vector $\mathbf{n}$ and the partial derivatives with respect to these coordinates, referring to an earth–fixed equatorial reference frame. The partial derivatives of second order $\partial_{ij} w$ can be understood as the elements of the Marussi matrix $\mathbf{M}$.

Substituting $\delta w(P)$ and $w(P)$ in the boundary condition Δw by its Taylor series (10) we get the expanded boundary condition, where the terms on the right hand side refer to the telluroid point p:

$$\Delta w \ = \ \delta w + < grad\ w, \mathbf{n} > \Delta h + \cdots . \tag{11}$$

Assuming that $\Delta w = 0$ holds, and neglecting the non–linear terms, we end up with a relationship for the height anomaly (Brun's formula):

$$\Delta h \quad = \quad -\tfrac{\delta w}{<grad\ w, \mathbf{n}>} . \tag{12}$$

Substitution of the vectorial gravity disturbance $\delta\boldsymbol{\gamma}(P) = grad\ \delta w(P)$ and the normal gravity vector $\boldsymbol{\gamma}(P) = grad\ w(P)$

$$\begin{aligned} \delta\boldsymbol{\gamma}(P) &= \delta\boldsymbol{\gamma}(p) + (\partial_i \delta\boldsymbol{\gamma} \cdot n_i)\, \Delta h + \ldots \\ \boldsymbol{\gamma}(P) &= \boldsymbol{\gamma}(p) + (\partial_i \boldsymbol{\gamma} \cdot n_i)\, \Delta h + \tfrac{1}{2} (\partial_{ij} \boldsymbol{\gamma} \cdot n_i \cdot n_j)\, (\Delta h)^2 + \ldots \end{aligned} \tag{13}$$

in the boundary condition for the gravity anomaly $\Delta\gamma$ results in the linearized boundary condition:

$$\Delta\gamma \ = < \frac{\boldsymbol{\gamma}}{\gamma}, grad\ \delta w > + < \frac{\boldsymbol{\gamma}}{\gamma}, \mathbf{M}\ \mathbf{n} > \Delta h + \ldots . \tag{14}$$

The right hand side of (14) is related to $p \in s$. Now the height anomaly Δh can be eliminated in (14) by use of (12). Neglecting the non–linear terms we end up with the linear reduced boundary condition for the scalar free bvp (Heck, 1989):

$$\Delta\gamma \ = < \frac{\boldsymbol{\gamma}}{\gamma}, grad\ \delta w > + < \frac{\boldsymbol{\gamma}}{\gamma}, \mathbf{M}\ \mathbf{n} > \frac{\delta w}{< \boldsymbol{\gamma}, \mathbf{n} >} . \tag{15}$$

In Heck and Seitz (1991, 1993, 1995) and Seitz et al. (1994) the effects of the non–linear terms have been studied. If a *Somigliana–Pizzetti* field is used as normal potential the effects on the vertical position due to the non–linear terms in the reduced boundary condition are less than 4mm in the vicinity of the earth's surface. They are decreasing to 2mm if a truncated spherical harmonic model (max degree 20) is used to model the normal gravitational potential.

Now the *linearized scalar free bvp* can be formulated where the boundary condition refers to the telluroid $s \ni p$: *Suppose the boundary data* $\Delta\gamma$ *to be given. The unknown disturbing potential* $\delta w(\mathbf{x})$ *has to fulfill the Laplace equation in the mass free space* Ω_a *outside* s, *and the disturbing potential* δw *tends to zero if the geocentric distance* r *tends to infinity – this corresponds to the postulate of regularity at infinity:*

$$\begin{array}{llll} Lap\ \delta w(\mathbf{x}) & = & 0\,, & \mathbf{x} \in \Omega_a \\ \delta w(\mathbf{x}) & \sim & \frac{1}{r} + O\left(\frac{1}{r^3}\right)\,, & r \to \infty \\ \Delta\gamma & = & < \frac{\boldsymbol{\gamma}}{\gamma}, grad\ \delta w > + < \frac{\boldsymbol{\gamma}}{\gamma}, \mathbf{M}\ \mathbf{n} > \frac{\delta w}{<\boldsymbol{\gamma},\mathbf{n}>}\,, & \text{on } s\,. \end{array} \tag{16}$$

The boundary condition (15) in the formulation (16) can be expressed in operator style (Heck, 1991):

$$\Delta\gamma \quad = \quad B_s\{\delta w\} \quad = \quad E_s \circ D\{\delta w\} . \tag{17}$$

The boundary operator B_s maps the disturbing potential δw at the telluroid s onto the scalar free gravity anomaly $\Delta\gamma$. B_s is composed of the differential operator D and the evaluation operator E_s which are applied subsequently. The evaluation operator E_s relates the differential $D\{\delta w\}$ to the boundary surface s. The linear operator D, applied to the spatial function δw, can be represented by the identity operator I and the partial derivatives with respect to the spherical coordinates r, β and λ:

$$D \quad = \quad d_o I + d_r \tfrac{\partial}{\partial r} + d_\beta \tfrac{\partial}{\partial \beta} + d_\lambda \tfrac{\partial}{\partial \lambda} . \tag{18}$$

The coefficients d_o, d_r, d_β and d_λ of the differential operator are functionals of the normal field w:

$$\begin{aligned} d_o &= -<\frac{\boldsymbol{\gamma}}{\gamma}, \mathbf{M}\ \mathbf{n}> \frac{1}{<\boldsymbol{\gamma},\mathbf{n}>} \\ d_r &= \frac{\gamma_1}{\gamma} \\ d_\beta &= \frac{1}{r}\frac{\gamma_2}{\gamma} \\ d_\lambda &= \frac{1}{r\cos\beta}\frac{\gamma_3}{\gamma} \ \equiv \ 0\,, \ \text{for } w(r,\beta)\,. \end{aligned} \tag{19}$$

In (19), the components of the normal gravity vector $\boldsymbol{\gamma}(p)$ are denoted by γ_i and refer to the local orthonormal system $\{p; \mathbf{g}_i\}$. The basis vector $\mathbf{g}_1$ is parallel to the geocentric position vector $\mathbf{x}(p)$, $\mathbf{g}_2$ points to the north and $\mathbf{g}_3$ completes this right handed system. A survey about the linearization of boundary value problems is given in Heck (1997).

Series expansion of the differential operator

To obtain an analytical representation of the coefficients of the differential operator D, the normal potential w has to be described analytically. Therefore we introduce the following approximation w_a for the normal potential w:

$$w_a(r,\beta) \ = \ \frac{\mu^v}{r}\left[1 - J_2\left(\frac{a}{r}\right)^2 P_2(\sin\beta) - J_4\left(\frac{a}{r}\right)^4 P_4(\sin\beta) + \frac{1}{2}\overline{m}\left(\frac{r}{a}\right)^3\cos^2\beta\right]. \tag{20}$$

Under this approximation the coefficients d_o, d_r, d_β and d_λ are represented in the following analytical way. This new **second order approximation** of the differential operator D is reached after extensive analytical manipulations. For details see Seitz (1997).

$$\begin{aligned} d_o &= -\tfrac{2}{r}\Big\{1 - 3J_2P_2 + \tfrac{3}{2}\overline{m}\cos^2\beta \\ &\qquad - \tfrac{1}{4}e^2\sin^2\beta\left[2e^2\cos^2\beta - 12J_2\left(2-3\sin^2\beta\right) + 13\overline{m}\cos^2\beta\right] \\ &\qquad + \tfrac{3}{2}\tfrac{h}{a}\left(4J_2P_2 + 3\overline{m}\cos^2\beta\right) - \tfrac{9}{4}J_2^2\left(1 - 10\sin^2\beta + 13\sin^4\beta\right) \\ &\qquad -\tfrac{3}{4}J_2\overline{m}\cos^4\beta + \tfrac{3}{2}\overline{m}^2\left(1 - 3\sin^2\beta + 2\sin^4\beta\right) - 10J_4P_4 + O(e^6)\Big\} \\ d_r &= -1 + \tfrac{1}{2}\sin^2\beta\cos^2\beta\left(3J_2 + \overline{m}\right)^2 + O(e^6) \\ d_\beta &= -\tfrac{1}{r}\sin\beta\cos\beta\Big\{3J_2 + \overline{m} \\ &\qquad + 3\left(\tfrac{1}{2}e^2\sin^2\beta - \tfrac{h}{a}\right)\left(2J_2 - \overline{m}\right) + 9J_2^2P_2 \\ &\qquad +\tfrac{3}{2}J_2\overline{m}\left(1+\sin^2\beta\right) + \overline{m}^2\cos^2\beta - \tfrac{5}{2}J_4\left(3 - 7\sin^2\beta\right) + O(e^6)\Big\} \\ d_\lambda &= O(e^6), \quad [\equiv 0, \text{ if } w = w(r,\beta)]\,. \end{aligned} \tag{21}$$

The absolute error of the neglected terms in the boundary condition is less than $1\cdot 10^{-10}ms^{-2}$ if a *Somigliana–Pizzetti* normal field is used as reference field. If a truncated spherical harmonic expansion ($N^v = 20$) is used as normal gravitational potential v, the absolute error increases to $5\cdot 10^{-8}ms^{-2}$.

The coefficient d_o can be split off into a dominant term, caused by the isotropic part μ^v/r of the normal potential, and the so–called *ellipsoidal terms*. The anisotropy of v and the influence of the centrifugal potential z contribute to the *ellipsoidal term* δd_o:

$$d_o \ = \ -\tfrac{2}{r} + \delta d_o\,. \tag{22}$$

The same decomposition can be applied for the radial term

$$d_r \ = \ -1 + \delta d_r\,. \tag{23}$$

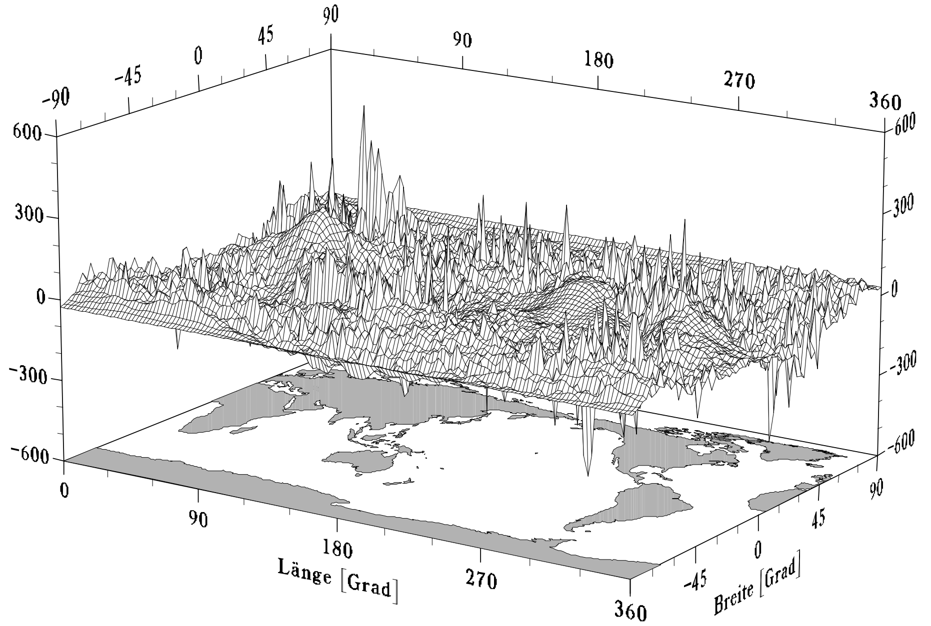


Figure 1: Ellipsoidal terms $[10^{-8}ms^{-2}]$ in the reduced boundary condition of the scalar free bvp. V: OSU91a1f, w: GRS80.

The linear boundary condition (15), (17) can now be written as

$$\Delta\gamma \quad = \quad E_s\left\{-\tfrac{2}{r}\delta w - \tfrac{\partial \delta w}{\partial r}\right\} + E_s \circ \delta D\left\{\delta w\right\} \, . \tag{24}$$

The linear differential operator D is here decomposed in an isotropic part and the ellipsoidal increments:

$$\begin{aligned} D &= -\tfrac{2}{r}I - \tfrac{\partial}{\partial r} + \delta d_o I + \delta d_r \tfrac{\partial}{\partial r} + d_\beta \tfrac{\partial}{\partial \beta} + d_\lambda \tfrac{\partial}{\partial \lambda} \\ &= -\tfrac{2}{r}I - \tfrac{\partial}{\partial r} + \delta D \, . \end{aligned} \tag{25}$$

If only the isotropic term μ^v/r is considered and – pay attention to this fact – the centrifugal potential is omitted, one deals with the **isotropic** or **radial approximation** of the differential operator D. This leads to the *fundamental equation of Physical Geodesy* (Heiskanen and Moritz, 1967):

$$\Delta\gamma \quad = \quad -\tfrac{2}{r}\delta w - \tfrac{\partial \delta w}{\partial r} \, . \tag{26}$$

In this rough approximation of the boundary condition the ellipsoidal terms δd_o, δd_r, d_β and d_λ are neglected. The ellipsoidal terms in the linear reduced boundary condition of the scalar free bvp are given with (25) through the expression:

$$E_s \circ \delta D\left\{\delta w\right\} \; = \; E_s\left\{\delta d_o \delta w + \delta d_r \frac{\partial \delta w}{\partial r} + d_\beta \frac{\partial \delta w}{\partial \beta} + d_\lambda \frac{\partial \delta w}{\partial \lambda}\right\} \, . \tag{27}$$

With the spherical harmonic model OSU91a1f from the Ohio State University (Rapp et al., 1991) representing the actual gravitational field of the earth and the Geodetic Reference System 1980 (GRS80) as reference field, the total ellipsoidal terms (27) are illustrated in figure 1. The maximum values are in the range of $\pm 600 \cdot 10^{-8}ms^{-2}$.

In the geodetic literature since Jekeli (1981), Cruz (1986) or Pavlis(1988) the ellipsoidal correction terms ε_γ and ε_h are customary. They are applied to the boundary data. The corrected gravity anomalies are now related to the isotropic approximation of the boundary condition

(24). The correction terms ε_γ and ε_h (Jekeli, 1981; there was a misprint in the sign for ε_h) are proportional to the coefficients δd_o and d_β of the differential operator:

$$\begin{array}{lclcl} \varepsilon_\gamma & = & \delta d_o \, \delta w & = & \frac{1}{r}\left[6J_2\left(\frac{a}{r}\right)^2 P_2(\sin\beta) - \frac{3\omega^2 r^3}{\mu^v}\cos^2\beta\right]\delta w \\ \varepsilon_h & = & d_\beta \, \frac{\partial \delta w}{\partial \beta} & = & -\frac{1}{r}e^2 \sin\beta\cos\beta \, \frac{\partial \delta w}{\partial\beta} \, . \end{array} \tag{28}$$

The ellipsoidal term $E_s\{\delta d_o \delta w\}$ has a very smooth behaviour which can be seen in figure 2. The ellipsoidal term $E_s\{d_\beta \frac{\partial \delta w}{\partial \beta}\}$ is depicted in figure 3.

The usually applied correction terms ε_γ and ε_h are a **first order approximation**. The term ε_γ describes the influence of the difference between the isotropic field and the exact normal field in the boundary condition. One can assess that $|\varepsilon_\gamma| \leq 230 \cdot 10^{-8} ms^{-2}$ holds in the vicinity of the earth's surface. In the first order approximation the term ε_h corrects for the fact, that the partial derivative with respect to the geocentric distance r instead of the derivative in direction to the ellipsoidal normal is applied to the disturbing potential δw in the boundary condition. The terms ε_γ and ε_h (28) are often further simplified by taking advantage of the relations (Heiskanen and Moritz, 1967, p78):

$$\begin{array}{lcl} J_2 & = & \frac{1}{3}\left(e^2 - \overline{m}\right) + \cdots \\ \overline{m} & = & \frac{e^2}{2} + \cdots \, , \end{array} \tag{29}$$

which are first order approximations. It should be noticed that these relations (29) are only valid when a normal field of *Somigliana–Pizzetti*–type is used! With these assumptions the representation

$$\begin{array}{lcl} \varepsilon_\gamma & \approx & -\frac{1}{r}e^2\left(2 - 3\sin^2\beta\right)\delta w \\ \varepsilon_h & \approx & -\frac{1}{r}e^2 \sin\beta\cos\beta \, \frac{\partial \delta w}{\partial \beta} \end{array} \tag{30}$$

results that is mostly referred to in geodetic literature in the context of ellipsoidal corrections (Lelgemann, 1970; Pellinen, 1982; Martinec, 1995).

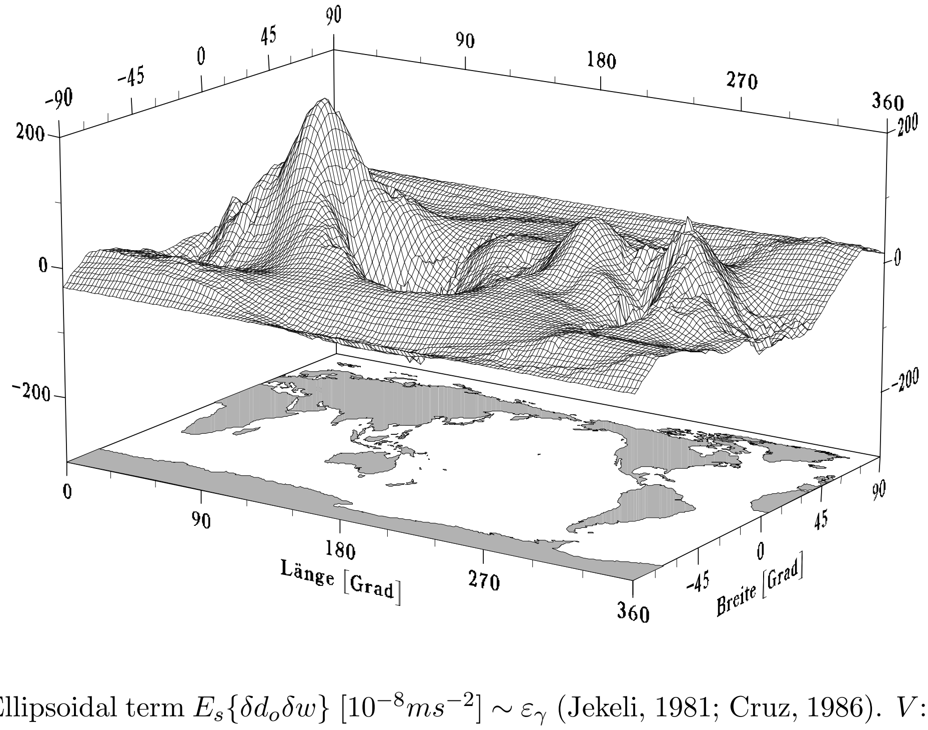


Figure 2: Ellipsoidal term $E_s\{\delta d_o \delta w\}$ $[10^{-8} ms^{-2}] \sim \varepsilon_\gamma$ (Jekeli, 1981; Cruz, 1986). V: OSU91a1f, w: GRS80.

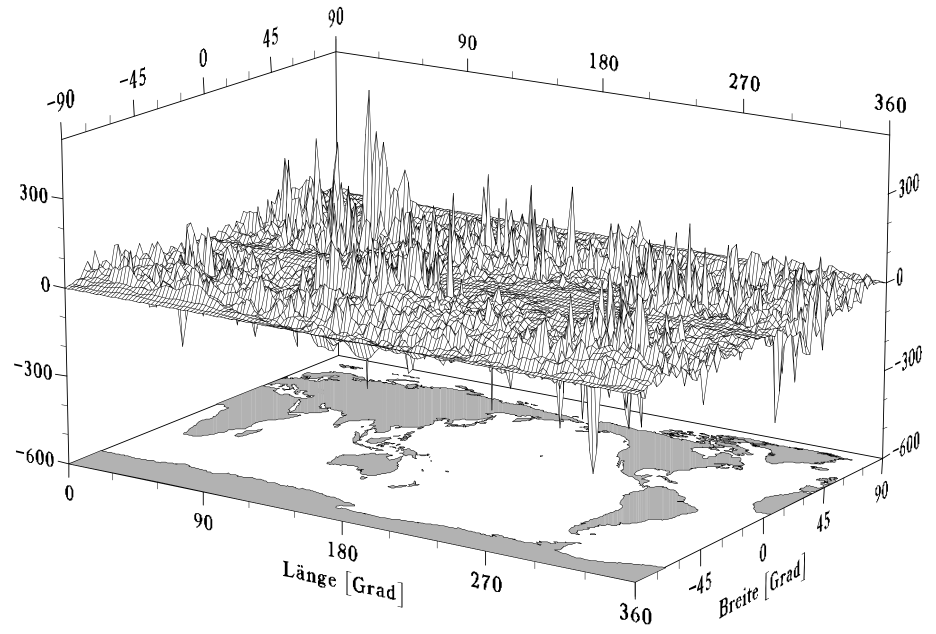


Figure 3: Ellipsoidal term $E_s\{d_\beta \frac{\partial \delta w}{\partial \beta}\}$ $[10^{-8} ms^{-2}] \sim \varepsilon_h$ (Jekeli, 1981; Cruz, 1986). V: OSU91a1f, w: GRS80.

Formulation of an isotropic problem

The object of our efforts is the determination of the harmonic coefficients $\overline{C}_{nm}^{\delta w}$, $\overline{S}_{nm}^{\delta w}$ which represent the disturbing potential δw. This task can be performed for example by harmonic analysis of boundary data that meets an isotropic boundary condition on a surface, axi–symmetric with respect to the earth's mean rotational axis. The simplest choice of such a surface is a sphere of radius a. On a sphere $K \ni k$ the corresponding boundary condition reads:

$$\Delta\gamma_k \;=\; B_k\{\delta w\} \;=\; E_k \circ D_i\{\delta w\} \;:=\; \Delta\gamma - \Delta_k\,. \tag{31}$$

The linear differential operator D_i consists of the identity operator I and the partial derivative with respect to the geocentric distance r. The differential $D_i\{\delta w\}$ is restricted to the surface of the sphere by applying the evaluation operator E_k. The resulting spherical bvp

$$\begin{array}{lcll} Lap\;\delta w(\mathbf{x}) & = & 0\,, & \mathbf{x} \in \Omega_a \\ \delta w(\mathbf{x}) & \sim & \frac{1}{r} + O\left(\frac{1}{r^3}\right)\,, & r \to \infty \\ \Delta\gamma_k & := & \Delta\gamma - \Delta_k \;=\; -\frac{2}{a} E_k\left\{\delta w\right\} - E_k\left\{\frac{\partial \delta w}{\partial r}\right\} & \end{array} \tag{32}$$

is formally the third boundary value problem on a sphere. The boundary data we have to deal with is the original boundary data $\Delta\gamma$ (computed with the full normal field) reduced by the **ellipsoidal and topographic correction** term Δ_k.

In a similar way we can formulate the boundary condition on the surface of an ellipsoid of revolution $E \ni e$:

$$\Delta\gamma_e \;=\; B_e\left\{\delta w\right\} \;=\; E_e \circ D_i\{\delta w\} \;:=\; \Delta\gamma - \Delta_e\,. \tag{33}$$

The terms Δ_k or Δ_e correct for the anisotropy of the normal field and the difference between the telluroid s and the surface of a sphere or an ellipsoid respectively. In the following we will restrict ourselves to the isotropic boundary value problem on the surface of an ellipsoid:

$$\begin{array}{lcll} Lap\ \delta w(\mathbf{x}) & = & 0\,, & \mathbf{x} \in \Omega_a \\ \delta w(\mathbf{x}) & \sim & \frac{1}{r} + O\left(\frac{1}{r^3}\right)\,, & r \to \infty \\ \Delta\gamma_e & := & \Delta\gamma - \Delta_e \ = \ -E_e\left\{\frac{2}{r}\delta w\right\} - E_e\left\{\frac{\partial \delta w}{\partial r}\right\}\,. & \end{array} \tag{34}$$

The unknown disturbing potential δw still has to fulfill the requirements (16). The reduced boundary data $\Delta\gamma_e$ on the surface of an ellipsoid $E \ni e$ has to fulfill the isotropic boundary condition on E. It is obvious, that the ellipsoidal and topographic correction terms Δ_e are functionals of δw which we solve for. Therefore an iterative procedure is required.

Analytical continuation of the boundary condition onto an ellipsoid of revolution

To come up with a representation of the boundary condition on the surface of an ellipsoid (33) we have to analytically continue the boundary condition from the telluroid onto E. This is done by a formal Taylor series expansion of the evaluation operator E_s. To that end we select the Taylorstep in radial direction:

$$r(p) - r(e) \ = \ h\left(1 + \frac{1}{2}e^4 \sin^2\beta \cos^2\beta\right) + a \cdot O(e^8)\,. \tag{35}$$

If (35) is divided by a we have the representation

$$\frac{r(p) - r(e)}{a} \ = \ \frac{h}{a}\left(1 + \frac{1}{2}e^4 \sin^2\beta \cos^2\beta\right) + O(e^8)\,, \tag{36}$$

where the expression h/a is of the order of e^2. The formal Taylor expansion of E_s or E_e between the telluroid point $p \in s$ and the corresponding point $e \in E$ on the surface of the ellipsoid can be performed with the Taylorpoint situated either on E or s:

If we set up the Taylor expansion for E_e, which we need for a formulation like (34), in $p \in s$ we get from (36) the representation

$$E_e \ = \ E_s + \sum_{n=1}^{\infty} \frac{a^n}{n!}\left(\frac{r(e) - r(p)}{a}\right)^n E_s \circ \frac{\partial^n}{\partial r^n}\,. \tag{37}$$

Rearranging (37) with respect to the zero order term E_s and substituting this representation from (24) we have the new representation of the boundary condition

$$\Delta\gamma \ = \ B_s\{\delta w\} \ = \ \underbrace{E_e\left\{-\frac{2}{r}\delta w - \frac{\partial \delta w}{\partial r}\right\}}_{\text{isotropic term}} + \Delta_e\,. \tag{38}$$

The whole ellipsoidal and topographical components are included in the term

$$\Delta_e \ = \ c^{se}_{r_0\beta_1} E_s\left\{\frac{\partial \delta w}{\partial \beta}\right\} + \sum_{i=0}^{8} c^{se}_{r_i\beta_0} E_s\left\{\frac{\partial^i \delta w}{\partial r^i}\right\}\,. \tag{39}$$

The partial derivatives of the disturbing potential have to be computed at the telluroid. The coefficients $c^{se}_{r_i\beta_j}$ have been derived in Seitz (1997).

Table 1. Error $[10^{-5}ms^{-2}]$ in the analytical approximation of Δ_k depending on the order of the Taylor series. V: OSU91a1f, w: GRS80, $w_a(J_0, J_2, J_4, \overline{m})$.

max. order of $\frac{\partial^i}{\partial r^i}$	Extreme values of the approximation error	
8	−0.001	0.001
7	−0.008	0.010
6	−0.053	0.073
5	−0.326	0.463
4	−1.922	2.562
3	−9.865	11.974
2	−41.280	45.350
1	−133.227	129.352

Table 2. Error $[10^{-5}ms^{-2}]$ in the analytical approximation of Δ_e depending on the order of the Taylor series. V: OSU91a1f, w: GRS80, $w_a(J_0, J_2, J_4, \overline{m})$.

max. order of $\frac{\partial^i}{\partial r^i}$	Extreme values of the approximation error	
8	−0.001	0.001
7	−0.001	0.001
6	−0.001	0.001
5	−0.001	0.001
4	−0.005	0.005
3	−0.127	0.153
2	−2.523	1.435
1	−16.819	28.744

As an alternative to this procedure, related to the question raised by Sansò and Sona (1995), Sansò (1995) about the correct choice of the Taylorpoint we also expanded the ellipsoidal and topographical terms with the Taylorpoint in $e \in E$. Here we get directly the representation of the evaluation operator E_s which we have to substitute from (24)

$$E_s \quad = \quad E_e + \sum_{n=1}^{\infty} \frac{a^n}{n!} \left(\frac{r(p) - r(e)}{a} \right)^n E_e \circ \frac{\partial^n}{\partial r^n} \, . \tag{40}$$

The resulting ellipsoidal and topographical terms

$$\Delta_e \;=\; \sum_{i=0}^{4} c^{es}_{r_i \beta_1} E_e \left\{ \frac{\partial^{i+1} \delta w}{\partial r^i \partial \beta} \right\} + \sum_{i=0}^{8} c^{es}_{r_i \beta_0} E_e \left\{ \frac{\partial^i \delta w}{\partial r^i} \right\} . \tag{41}$$

are now related to the surface of the ellipsoid. It is obvious that the coefficients $c^{es}_{r_i \beta_j}$ are different from $c^{se}_{r_i \beta_j}$, also in there signs. The upper limits of the Taylor series in the alternative developments (38), (39) and (40), (41), respectively, have been chosen such that the same absolute error level of $\pm 1 \cdot 10^{-8} ms^{-2}$ is achieved, which was verified by numerical calculations on the basis of OSU91a1f in Seitz (1997). The ellipsoidal and topographical terms Δ_e are in the range of $\pm 20 \cdot 10^{-5} ms^{-2}$ as can be seen in figure 4. The effect of neglecting the ellipsoidal and topographical terms – using the boundary data $\Delta\gamma$ without applying a correction for the anisotropy of the normal potential and for the geometrical distance between the telluroid and the ellipsoid – on the vertical position of equipotential surfaces in the vicinity of the earth's surface is plotted in figure 5. The total effect can amount up to nearly $2m$.
In a similar way the evaluation operator E_s is continued in Seitz (1997) onto a sphere. The resulting approximation errors for Δ_k and Δ_e are listed in the tables 1 and 2 respectively. They

are also given for different maximum orders of the partial derivative in radial direction. To achieve a maximum error of $\pm 1 \cdot 10^{-8} ms^{-2}$ in case of the continuation to a sphere one has to perform the Taylor expansion up to the order $k = 8$. The Taylor expansion for the evaluation operator can be truncated after the 5th order in case of the ellipsoidal boundary.

A flow chart of the whole process starting from the non–linear boundary condition, the linearization, the different levels of approximation for the differential operator and the analytical continuation of the boundary condition onto the surface of an ellipsoid is given in table 3.

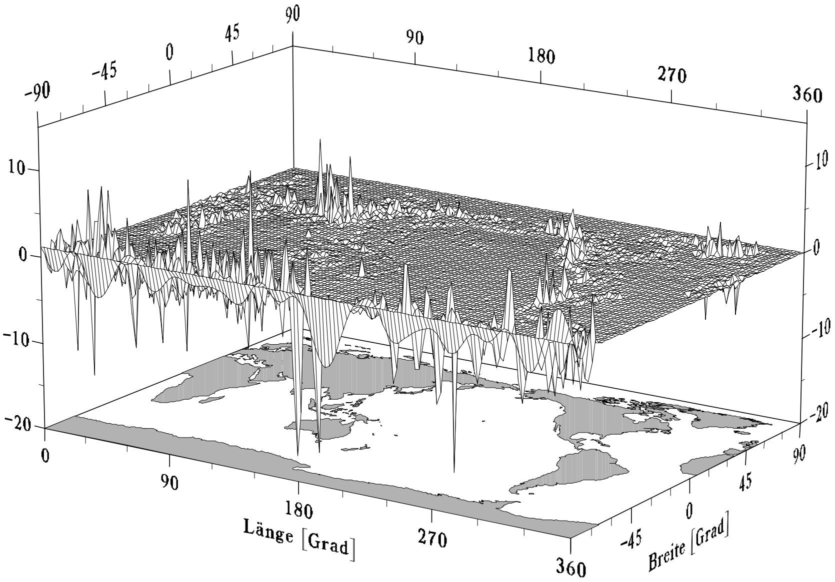


Figure 4: Ellipsoidal and topographical terms Δ_e $[10^{-5} ms^{-2}]$. V: OSU91a1f, w: GRS80.

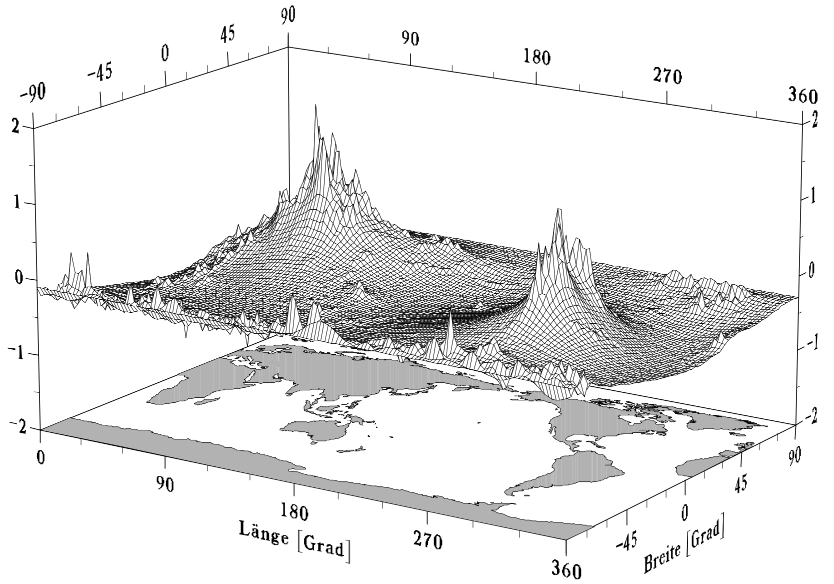
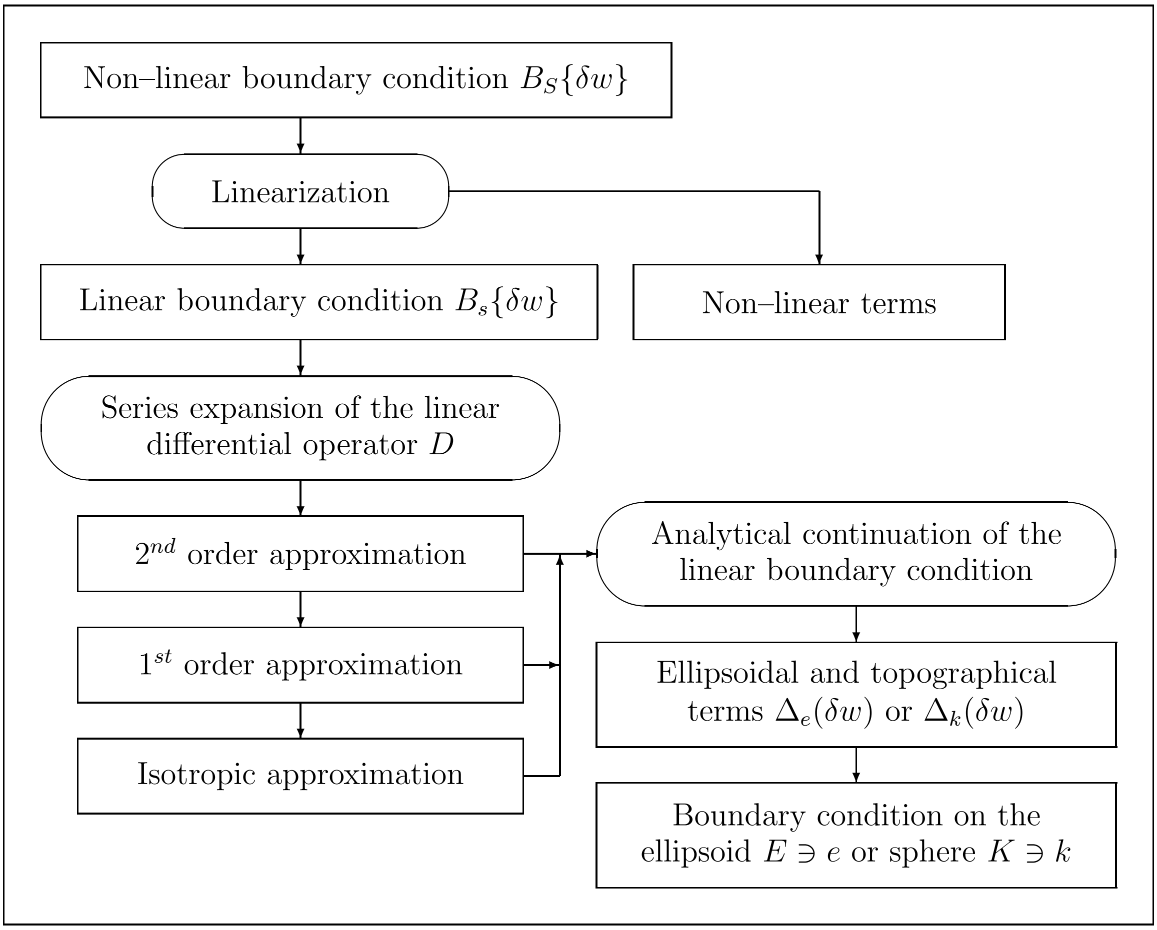


Figure 5: Effect $[m]$ due to Δ_e on the vertical position of equipotential surfaces. V: OSU91a1f, w: GRS80.

Table 3. Flow chart for the analytical evaluation of the boundary condition for the scalar free bvp.



An iterative solution

As already mentioned the ellipsoidal and topographical correction terms ((39) or (41)) are functionals of the unknown disturbing potential which we solve for. If we try to solve the bvp by a harmonic analysis of the reduced boundary data (33) on the surface of an ellipsoid we have to set up an iterative procedure. First numerical tests indicated that the whole procedure diverges if we continue the data onto a sphere. It converges when we use a surface of an ellipsoid, on which we perform the harmonic analysis. Further results will be given in Seitz and Heck (1999).

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