# The analysis of the Neumann and oblique derivative problem.Weak theory

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**Abstract:**
In this review paper the simple Neumann and oblique derivative problem is formulated for an exterior domain and mapped by the Kelvin-Raleigh transform, to an internal domain. The weak formulation of the two problems is subsequently studied and standard theorems of existence, uniqueness and well-posedness are proved.
The conditions of validity for such theorems have a clear interpretation for the geometric point of view.
An extensive Appendix, mostly without proofs, provides the relevant material on theory of distributions and Sobolev spaces.

## 1 Motivation and formulation of the problem

In geodesy some of the most fundamental problems of the gravity field determination from boundary observations are translated into exterior boundary value problems (BVP) for the Lapalce or Poisson equation (cfr. [Sansò, 1995], [Sansò, 1997]).
After suitable linearization and reductions-simplifications of various kinds we finally come out with a problem that can typically be formulated as follows:
*Given a simply connected bounded open domain $B$ with boundary $S$ and the exterior open domain $\Omega$, given some known distribution $f$ in $\Omega$ and a boundary datum $g$, to find a function (potential) $u$ in $\Omega$ such that*

$$\Delta u = f \; in \;\; \Omega \tag{1.1}$$

$$\underline{e} \cdot \nabla u + bu = g \; on \;\; S \tag{1.2}$$

$$u = 0\left(\frac{1}{r}\right) \qquad for \;\; r \to \infty \tag{1.3}$$

Please note that we have been purposly ambiguous in denoting $f$ as a distribution as it can be legitimately interpreted in both senses, physically as a mass distribution outside $S$ (this happens when classical geodetic reductions are applied so that $S$ lies partly inside the masses), or mathematically, in L. Schwarz sense. It has to be remarked that the physical situation in geodesy is such that $f$ has a bounded support in $\Omega$ although we shall go beyond this hypothesis for the sake of completeness in the analysis. As a byproduct of this remark we shall not insist on the condition (1.3), which derives from the requirement that $u$ is a regular harmonic function outside a sphere of sufficiently large radius, but we shall define in the next paragraph conditions of regularity at infinity suitable for the functional spaces we are going to work with.
As for the boundary condition (1.2) we remark that, excluding the case of mixed BVP's, $\underline{e}$ can be taken as a unit vector field on $S$. Typically in geodesy $\underline{e}$ (cfr. Fig. 1.1) is the direction of the normal gravity $\underline{\gamma}$, or its opposite. In turn $\underline{\gamma}$ is always directed fairly close to the radial direction $\underline{e}_r$, while the outer normal $\underline{\nu}$ to $S$ may be very distinct to $\underline{e}$, although this happens only on a small portion of the surface. In any event, we shall make the quite reasonable assumption that

$$\cos\alpha = \underline{e} \cdot \; \underline{\nu} \geq \alpha > 0 \text{ (on } S) \; ; \tag{1.4}$$

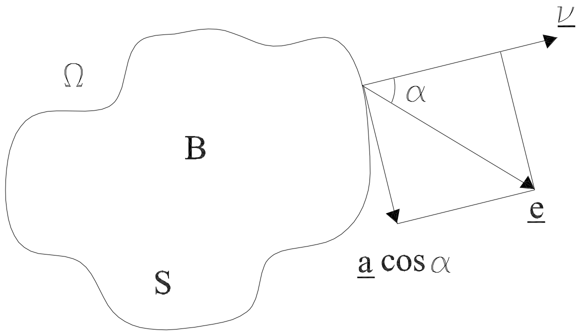


Figure 1.1: The geometry of BVP's analyzed.

this qualifies mathematically (1.1), (1.2), (1.3) as a regular oblique derivative problem. In this evenience, by dividing (1.2) by $\cos\alpha$ and rearranging the symbols in an obvious way, we can write

$$\underline{\nu} \cdot \nabla u + \underline{a} \cdot \nabla u + bu = g \tag{1.5}$$

or

$$\frac{\partial u}{\partial \nu} + \underline{a} \cdot \nabla_t u + bu = g \tag{1.6}$$

with $\underline{a}$ tangent to $S$,

$$\underline{a} \cdot \underline{\nu} = 0, (\text{on } S). \tag{1.7}$$

and $\nabla_t = \nabla - \underline{\nu}\partial_\nu$, the tangent component of the gradient. Let us remark that $S$ has to display some regularity and in this paper to fix the ideas we shall accept that $S$ is a $C^{2,\lambda}$ surface, i.e. it has $\lambda$ Holder continuous second derivatives in local coordinates.
In (1.6) $\underline{a}$ is small in the average (apart from mountainous areas) and $\underline{a} \cdot \nabla_t$ can be considered as a perturbation with respect to the main operator $\partial_\nu$ In one important instance, when gravity anomalies data are reduced to the ellipsoid for the determination of the geoid, we have identically $\underline{a} \equiv 0$.
Finally, the term in $b$ might or might not be present, depending on the problem considered: this is indeed not irrelevant to the mathematical analysis, since the uniqueness or non-uniqueness of the solution does depend on the sign of $b$. For instance, the simple Molodensky problem with boundary operator $B = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)$, is well known to have a null space of dimension 3. On the other hand, the operator $b \cdot u$ is much milder than $\partial_\nu \cdot u$ so that (1.6) can typically be turned into a Fredholm type equation, once the corresponding problem with $b = 0$ has been analyzed. Furthermore, $b = 0$ does correspond to the (linearized) fixed boundary gravimetric problem (cfr. [Sansò, 1997]), which is becoming a realistic problem with nowadays GPS observations, and which geodesy shares with another geoscience: geomagnetism.
Concluding, we will be considering the two problems

a) The exterior Neumann problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} & \text{on } S \\ u \to 0 & \text{at } \infty \end{cases} \tag{1.8}$$

b) The exterior regular oblique derivative problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \underline{a} \cdot \nabla_t u = g & \text{on } S \\ u \to 0 & \text{at } \infty \ . \end{cases} \tag{1.9}$$

In this paper we shall present the theory of existence, uniqueness and stability of the solution of the two problems in the framework of the weak concept, so called because the differential operators have to be interpreted in distribution sense. In a forthcoming paper we shall present the strong theory too, or the so-called analysis of the regularization problem, including its extension to the corresponding stochastic problems, where $u$ has to be interpreted as a generalized random field (cfr. [Sansò, 1997], [Sansò, 1995], [Sansò, Venuti, 1998]). Although there is a very large mathematical litterature for problems like (1.8), (1.9) and although the results we shall derive are not surprising, yet in the methods used for their proof and in particular in the use of a generalized Ladyzenkaya identity for the regularization, there is some new material analysis. Furthermore, the precise formulation of the condition for the existence and uniqueness of the solution of the pure oblique derivative problem is sufficiently simple to be interpreted in a geodetic sense. On the other hand, a number of remarks that could be considerably shortened for mathematicians, are nevertheless included into the paper to make it self-consistent for readers from a geodetic environment.

# 2 The Kelvin-Raleigh transform

This transform, also called the inverse radii transform, is useful here because it maps (1.8), (1.9) into BVP problems for an interior domain $\Omega$, which is in this case bounded.
The transform is simultaneously a transformation of coordinates and of the unknown function: more precisely, assume that $B$ is such as to cover the unit ball

$$B \supset B_1 \equiv \{r \leq 1\} \ ,$$

so that if in this paragraph we call $s$ the "exterior" radial variable, we have

$$s_P > 1 \ , \qquad \forall P \in \Omega \ ;$$

then we define a new variable $r$ and a new unknown function (potential) $v$, as

$$s = \frac{1}{r}(s > 1), \quad r = \frac{1}{s} \quad (r < 1) \tag{2.1}$$

$$v = (r, \sigma) = \frac{1}{r} u(\frac{1}{r}, \sigma) = s \ u(s, \sigma) \ . \tag{2.2}$$

As it is obvious, by (2.1) $\Omega$ is mapped into a set $\Omega' \subset B_1$.

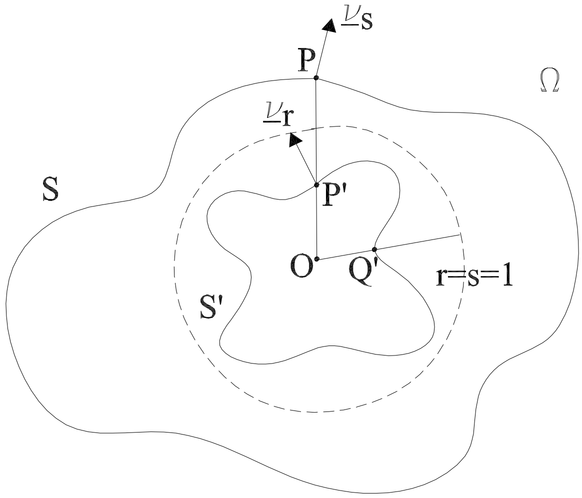


Figure 2.1: The geometry of the Kelvin-Raleigh transform.

so we have

$$\begin{aligned} &\Omega \text{ exterior to } \ S \\ &\Omega' \text{ interior to } \ S' \end{aligned}$$

Although not necessary, one can think that $S$ and $S'$ have (at least locally) equations

$$\begin{aligned} S \quad &\rightarrow \quad s = S(\sigma) \\ S' \quad &\rightarrow \quad r = R(\sigma) = \frac{1}{S(\sigma)} \ . \end{aligned} \tag{2.3}$$

A simple computation then shows that the two *exterior* normals $\underline{\nu}_s, \underline{\nu}_r$ are symmetrically placed with respect to the radial unit vector $\underline{e}_r = \underline{e}_s$. A straightforward computation shows that

$$\Delta_r v \equiv s^5 \Delta_s u \equiv s^5 f \ , \tag{2.4}$$

$$\frac{\partial v}{\partial \nu} + \frac{v}{\sqrt{R^2 + R_\vartheta^2 + \frac{1}{\sin_\vartheta^2} R_\lambda^2}} = -S^3 \frac{\partial u}{\partial \nu_s} = -S^3 g \ . \tag{2.5}$$

Accordingly we can map problem (1.8) into a problem of the form

$$\begin{cases} \Delta v = F & \text{in } \Omega' \\ \dfrac{\partial u}{\partial \nu} + bv = G & \text{on } S' \end{cases} \tag{2.6}$$

where

$$F(r, \sigma) = \frac{1}{r^5} f\left(\frac{1}{r}, \sigma\right) \ ; \ G(\sigma) \equiv -S^3(\sigma) g(\sigma) \tag{2.7}$$

and

$$\bar{b} \geq b(\sigma) = \frac{1}{\sqrt{R^2(\sigma) + R_\vartheta^2 + \frac{R_\lambda^2}{\sin_\vartheta^2}}} \geq b_0 = 1 \ . \tag{2.8}$$

As for the oblique derivative problem (1.9) one sees that it is transformed into

$$\begin{cases} \Delta v = F & \text{in } \ \Omega' \\ \dfrac{\partial v}{\partial \nu} + \underline{a} \cdot \nabla_t v + bv = G & \text{on } \ S' \end{cases} \tag{2.9}$$

where this time

$$\underline{a} = \frac{\underline{e}}{\underline{e} \cdot \underline{\nu}_r} - \underline{\nu}_r \ . \tag{2.10}$$

It is an important remark that if $\underline{e}$ is directed close to the radial direction, then $\underline{a}_r$ given by (2.10) and

$$\underline{a}_s = \frac{\underline{e}}{\underline{e} \cdot \underline{\nu}_s} - \underline{\nu}_s$$

must be close one another in size. The situation is illustrated in Fig. 2.2 when $\underline{e} \equiv \underline{e}_r$

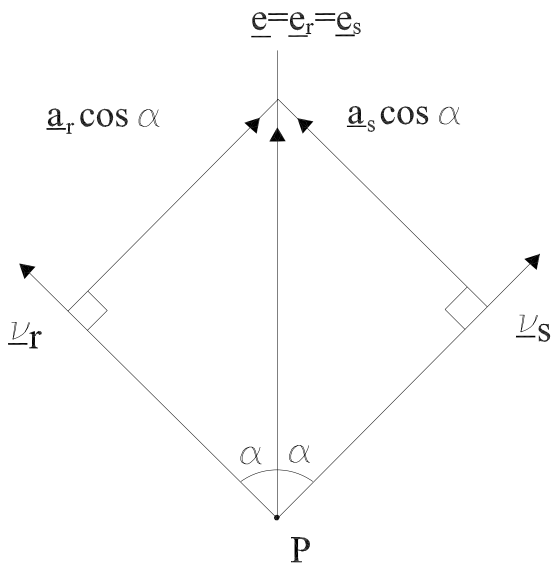


Figure 2.2: Geometry of the oblique derivative problem when $\underline{e} = \underline{e}_r = \underline{e}_s$.

Accordingly, when $\underline{a} \cdot \nabla_t$ can be considered as a perturbation in (1.9), the same is true in (2.9) and vice versa.

**Remark 2.1:** now that we have established the correspondence between external and internal problems at the level of notation we shall call again $B$ instead of $\Omega'$ the internal domain and $S$ instead of $S'$ its boundary.

**Remark 2.2:** with the inverse radii transform there is a certain modification of the functional properties of the known terms. However, since by hypothesis $S(\sigma) \in C^{2+\lambda}$ we do not have, in the context of this paper, significant differences between $g$ and $G$. On the contrary, since $F = s^5 f$ we see that $F \in L^2(\Omega')$, considering that $d\Omega' = \dfrac{d\Omega}{s^6}$, implies

$$\int_{\Omega'} F^2 d\Omega' = \int_\Omega s^4 f^2 d\Omega < +\infty \ ,$$

which imposes a well-defined asymptotic constraint on $f$.

# 3 Standard weak solutions of the Neumann and oblique derivative problems

We start from the Neumann problem

$$\begin{cases} \Delta v = F & \text{in } B \\ \dfrac{\partial v}{\partial \nu} + bv = G & \text{on } S \end{cases} \tag{3.1}$$
$$(\bar{b} \geq b|_S \geq b_0 > 0) \ .$$

Since we want to look for a solution less regular than $H^2$ in $B$ we have to find a way to write (3.1) in a form that is equivalent to it for classical solutions, $v \in H^2$, but which involves only first order derivatives, $v \in H^1$. This form is obtained by an application of a Green's identity, namely $\forall \varphi \in \Re_B \mathcal{D}$

$$\begin{aligned} \int_B \nabla v \cdot \nabla \varphi dB &= \int_S \frac{\partial v}{\partial \nu} \varphi dS - \int_B \Delta v \varphi dB = \\ = \ & - \int_S bv\varphi dS + \int_S G\varphi dS - \int_B F\varphi dB \end{aligned} \tag{3.2}$$

Let us recall that $\forall \varphi \in \Re_B \mathcal{D}$ means that $\forall \varphi \in \mathcal{D}$ ($C^\infty$ functions with compact support) we take its restriction to $B$ (open). Obviously if we take $v \in H^2$, then $F \in L^2(B)$ and $G \in H^{3/2}(S) \subset H^{1/2}(S)$ (cfr. Appendix) so that each single term of (3.2) is finite and meaningful $\forall \varphi \in \Re_B \mathcal{D}$; it is a basic issue that each term in (3.2) can be extended by continuity to the case that $v, \varphi \in H^1(B),\ F \in [H^1(B)]', G \in H^{-1/2}(S)$.

**Lemma 3.1:** let $v, \varphi \in H^1(B),\ F \in [H^1(B)]', G \in H^{-1/2}(S)$ then, interpreting the integrals by continuity as limits of sequences of regular functions,

$$\left| \int_B \nabla v \cdot \nabla \varphi dB \right| \leq \|v\|_{H^1} \cdot \|\varphi\|_{H^1} \tag{3.3}$$

$$\left| \int_S bv\varphi dS \right| \leq \text{const} \cdot \|v\|_{H^1(B)} \cdot \|\varphi\|_{H^1(B)} \tag{3.4}$$

$$\left| \int_S G\varphi dS \right| \leq \text{const} \cdot \|G\|_{H^{-1/2}(S)} \cdot \|\varphi\|_{H^{1/2}(S)} \tag{3.5}$$

$$\left| \int_B F\varphi dS \right| \leq \text{const} \cdot \|F\|_{(H^1)'} \cdot \|\varphi\|_{H^1(B)} \tag{3.6}$$

$\square$ (3.3) derives from Schwarz inequality and observing that $\{\int_B \nabla v^2 dB\}^{1/2} \leq |v|_{H'}$; (3.4) is again given by Schwarz inequality, recalling that $b \leq \bar{b}$, and trace theorems (see Appendix)

$$\begin{aligned} &\left| \int_s bv\varphi dS \right| \leq \text{const} \, \|v\|_{L^2(S)} \, \|\varphi\|_{L^2(S)} \leq \\ &\leq \text{const} \, \|v\|_{H^{1/2}(S)} \, \|\varphi\|_{H^{1/2}(S)} \leq \text{const} \, \|v\|_{H^1(B)} \, \|\varphi\|_{H^1(B)} \end{aligned}$$

(3.5) is a direct application of Lemma A.7 and of Sobolev trace Theorem. (3.6) is a direct application of Lemma A.8. $\square$

From Lemma 3.1 and Riesz representation theorem, of the dual of a Hilbert space with the same space, we see that (3.1) can be transformed into a simple equation from $H^1(B)$ into $H^1(B)$; in fact we can claim that

$$\begin{aligned} A(v,\varphi) &= \int_B \nabla v \cdot \nabla \varphi dB + \int_S bv\varphi dS \equiv < Av, \varphi >_{H^1(B)}, &(3.7)\\ &\int_S G\varphi dS = < \Gamma G, \varphi >_{H'} &(3.8)\\ &\int_B F\varphi dB = < CF, \varphi >_{H'} &(3.9)\end{aligned}$$

with, respectively

$$\|Av\|_{H^1(B)} \leq \overline{A}\, \|v\|_{H^1(B)} \tag{3.10}$$

$$\|\Gamma G\|_{H^1(B)} \leq \overline{\Gamma}\, \|G\|_{H^{-1/2}(S)} \tag{3.11}$$

$$\|CF\|_{H^1(B)} \leq \overline{C}\, \|F\|_{[H^1(B)]'} \ . \tag{3.12}$$

Accordingly (3.2), extended to the whole $H^1(B)$, becomes

$$\langle Av, \varphi \rangle_{H^1(B)} = \langle \Gamma G - CF, \varphi \rangle_{H^1(B)} \quad \forall \varphi \in H^1(B)$$

or

$$Av = \Gamma G - CF \ , \ v \in H^1(B) \ . \tag{3.13}$$

**Remark 3.1:** from the form of (3.7) one immediately realizes that $A$ is a selfadjoint operator in $H^1(B)$, in fact by symmetry

$$\langle Av, \varphi \rangle_{H'} = A(v,\varphi) = A(\varphi, v) = \langle A\varphi, v \rangle_{H^1} = \langle v, A\varphi \rangle_{H^1} \ . \tag{3.14}$$

Since for sure $\Gamma G - CF \in H^1(B)$ the study of equation (3.13) is reconducted to the question whether $A$ is an isomorphism (i.e. continuous, invertible and with continuous inverse) of $H^1(B)$ into itself.
To answer to that one can use a very basic lemma that we formulate here as Lemma 3.2 and prove in the Appendix as Lemma A.10.
**Lemma 3.2:** $\forall v \in H^1(B)$ the following inequality holds

$$\int_B v^2(P) dB \leq \text{const} \left\{ \int_B \nabla v^2 dB + \int_S v^2(P) dS_P \right\} \ . \tag{3.15}$$

**Remark 3.2:** from Lemma 3.2 we basically see that $A(v, v)$ is the square of an equivalent norm in $H^1(B)$. In fact from (3.10) we already know that

$$A(v,v) = \langle Av, v \rangle_{H^1(B)} \leq \overline{A}\, \|v\|^2_{H^1(B)} \tag{3.16}$$

while from (3.15) we clearly get

$$\begin{aligned} \|v\|^2_{H^1(B)} &= \int_B \nabla v^2 dB + \int_B v^2 dB \leq \widetilde{A} \left\{ \int_B \nabla v^2 dB + \int_S v^2 dS \right\} \leq \\ &\leq \widetilde{A}' \left\{ \int_B \nabla v^2 + \int_S bv^2 dS \right\} = \widetilde{A}' A(v,v) \ . &(3.17)\end{aligned}$$

so we have

$$\frac{1}{\widetilde{A}'}\, \|v\|^2_{H^1(B)} \leq A(v,v) \leq \overline{A}\, \|v\|^2_{H^1(B)} \ . \tag{3.18}$$

This of course proves at once that $A$ is an isomorphism of $H^1(B)$ onto itself. In fact by (3.17) $A$ is an invertible operator ($Av = 0 \to \|v\| = 0 \to v = 0$) and its range is closed in $H^2$. Moreover its range is dense in $H^1$, because by selfadjointness if $u \in H^1$ is such that

$$\forall v \in H^1, 0 = \langle Av, U\rangle = \langle v, Au\rangle$$

we must have as well

$$Au = 0 \quad \Rightarrow \quad u = 0.$$

Then the range of $A$ is the whole $H^1$ and we have just proved the following theorem.
**Theorem 3.1:** *$\forall F \in [H^1(B)]'$, $\forall G \in H^{-1/2}(S)$ the Neumann problem (3.1), translated into the weak form (3.2), has one and only one solution in $v \in H^1(B)$. Furthermore*

$$\|v\|_{H^2} \le C \left\{ \|F\|_{[H^1(B)]'} + \|G\|_{H^{-1/2}(S)} \right\} \ . \tag{3.19}$$

**Remark 3.3:** one might wonder why our Neumann problem has a unique solution while usually in analysis it is claimed to have a null space constituted by constant functions.
Indeed this would be the case if we had $b = 0$ in (3.1); however in our case $b \ge b_0 > 0$ exactly because we have inherited our BVP from an external formulation.
Since the external homogeneous Neumann problem is known to have only the zero solution, the same happens to its internal image (3.1).
We can come now to the oblique derivative problem, formulated as

$$\begin{cases} \Delta v = F & \text{in } \ B \\ \dfrac{\partial v}{\partial \nu} + \underline{a} \cdot \nabla_t v + bv = G & \text{on } \ S \ . \end{cases} \tag{3.20}$$

Following the same reasoning as for (3.1) we immediately come to the weak formulation

$$\begin{aligned} \forall \varphi \in H^1, \int_B \nabla v \cdot \nabla \varphi dB = & - \int_S \underline{a} \cdot \nabla_t v \varphi dS - \int_S b v \varphi dS + \\ + \quad & \int_S G \varphi dS - \int_B F \varphi dB \ . \end{aligned} \tag{3.21}$$

By using the same symbolism as before and putting

$$\alpha(v, \varphi) = \langle \alpha v, \varphi \rangle_{H_1} = \int (\underline{a} \cdot \nabla_t v) \varphi dS \tag{3.22}$$

we come straightforwardly to the formulation

$$\langle Av, \varphi \rangle_{H^1} + \langle \alpha v, \varphi \rangle_{H^1} = \langle \Gamma G - CF, \varphi \rangle_{H^1} \tag{3.23}$$

or

$$Av + \alpha v = \Gamma G - CF \ , \ v \in H^1 \ . \tag{3.24}$$

Since we have proved in (3.18) that

$$A \ge \frac{1}{\widetilde{\widetilde{A}}'} I$$

in $H^1(B)$, then (3.24) will have one and only one solution on condition that $\alpha$ is a bounded operator in $H^1(B)$ and that for instance

$$\|\alpha\| < \frac{1}{\widetilde{\widetilde{A}}'} \ .$$

Fortunately, following the famous theorem of Lax and Milgram (cfr. e.g. [Miranda, 1970]), we can find a milder condition for the existence and uniqueness of the solution of (3.23), which is summarized in the two requirements that

$$|A(v,\varphi)+\alpha(v,\varphi)| \le \text{const}\, \|v\|_{H^1}\, \|\varphi\|_{H^1} \tag{3.25}$$
$$A(v,v)+\alpha(v,v) \ge \text{const}\, \|v\|^2_{H^1} \ . \tag{3.26}$$

In fact from (3.26) we immediately see that

$$(A+\alpha)v = 0 \quad \Rightarrow \quad v=0 \tag{3.27}$$

i.e. the operator $(A+\alpha)^{-1}$ exists. Moreover $(A+\alpha)$ must have a dense range in $H^1$ since

$$\begin{aligned} u \in H^1, \langle (A+\alpha)v, u\rangle = 0 \quad \forall v, \rightarrow \langle (A+\alpha)u, u\rangle = 0 \\ \Rightarrow \quad u = 0 \end{aligned}$$

Finally, combining (3.26) and (3.25) one gets

$$\|v\|^2_{H^1} \le \text{const}\, \langle (A+\alpha)v, v\rangle \le \text{const}\, \|(A+\alpha)v\|_{H^1} \cdot \|v\|_{H^1}$$

entailing

$$\|v\|_{H^1(B)} \le \text{const}\, \|(A+\alpha)v\|_{H^1(B)} \ , \tag{3.28}$$

which means that $(A+\alpha)^{-1}$ is continuous, i.e. the range of $A+\alpha$ is closed and then it is the whole $H^1(B)$.
To prove (3.25) we need only to verify that

$$|\alpha(v,\varphi)| \le \text{const}\, \|\nabla_t v\|_{H^{-1/2}(S)}\, \|\underline{a}\varphi\|_{H^{1/2}(S)} \ . \tag{3.29}$$

Now assume that $\underline{a} \in C^\lambda(S), \lambda > 1/2$, then $\underline{a}$ is a multiplier in $H^{1/2}(S)$, i.e.

$$\|\underline{a}\varphi\|_{H^{1/2}(S)} \le \text{const}\, \|\varphi\|_{H^{1/2}(S)} \le \text{const}\, \|\varphi\|_{H^1(B)} \ . \tag{3.30}$$

On the other hand (cfr. the Appendix)

$$\|\nabla_t v\|_{H^{-1/2}(S)} \le \text{const}\, \|v\|_{H^{1/2}(S)} \ . \tag{3.31}$$

So (3.29) and then (3.26) is proved under the only condition $\underline{a} \in C^\lambda (\lambda > 1/2)$. As for (3.26) we first of all have

$$\begin{aligned} \alpha(v,v) &= \int_S (\underline{a}\cdot\nabla_t v) v dS = \frac{1}{2}\int_S \underline{a}\cdot\nabla_t(v^2) dS \\ &= -\frac{1}{2}\int_S (\nabla_t \cdot \underline{a}) v^2 dS \ . \end{aligned} \tag{3.32}$$

Therefore (recalling also (3.15), (3.17))

$$A(v,v)+\alpha(v,v) \equiv \int_B \nabla v^2 dB + \int_S \left[ b - \frac{1}{2}(\nabla_t \cdot \underline{a})\right] v^2 dS \ge \text{const}\, \|u\|^2_{H^1(B)} \ ,$$

if

$$b - \frac{1}{2}\nabla_t \cdot \underline{a} \equiv \beta_0 > 0 \ . \tag{3.33}$$

Therefore we have just proved the following theorem.

**Theorem 3.2:** $\forall F \in [H^1(B)]'$, $\quad \forall G \in H^{1/2}(S)$ *and for every* $\underline{a}$ *such that*

$$\underline{a} \in C^\lambda(S)(\lambda > 1/2), \qquad b - \frac{1}{2}\nabla_t \cdot \underline{a} \geq \beta_0 > 0 \tag{3.34}$$

*we have one and only one solution* $v$ *of (3.20) in* $H^1(B)$. *Moreover*

$$\|v\|_{H^1(B)} \leq const\left\{\|F\|_{[H^1(B)]'} + \|G\|_{H^{1/2}(S)}\right\} \ . \tag{3.35}$$

In this way we have accomplished the main analysis of the problem initially defined, (1.8), (1.9), in the light of the standard theory of weak solutions of BVP's for the Laplace operator.
From the mathematical point of view, to complete this analysis, one has to verify whether by adding regularity conditions to the data one gets a corresponding regularity improvement of the solution: for instance is it true that if we assume $F \in L^2(B)$ and $G \in H^{1/2}(S)$ (i.e. data one order of derivation more regular) we have also for the solution $v \in H^2(B)$? This question will be answered in a paper to follow this one.
**Remark 3.4:** s a last comment, let us observe that the condition (3.34) has a simple rough interpretation from the geometric point of view, in fact since

$$\underline{a} = \frac{\underline{\nu}_e}{\cos I} - \underline{\nu} \ ,$$

where $\underline{\nu}_e$ is the normal to the ellipsoid through the point, $\underline{\nu}$ is the normal to $S$ and $I$ the inclination of $S$ with respect to the normal vertical, for regions where $I \sim 0, \cos I \sim 1$

$$\nabla_t \cdot \underline{a} \cong 2(c_e - c_s) \tag{3.36}$$

where $c_e$ is the mean curvature of the ellipsoid and $c_s$ is the mean curvature of $S$. Going through the reciprocal radii transformation, mean curvatures become mean curvature radii $\rho_e, \rho_s$, while $b \leq \frac{1}{R}$ becomes just the radial distance of the point $P$ on the surface from the origin; so (3.34) with (3.36) transformed becomes just

$$r_P \geq (\rho_e - \rho_s)_P$$

which is certainly a reasonable assumption. A closer look to this relation should be given when the surface $S$ becomes rougher and the inclination $I$ plays a major role.

# A Appendix

In this appendix we shall try to summarize, mostly without proofs, the theory of Sobolev spaces and a few facts about functional analysis which have been used throughout the paper.
**Definition A.1:** $\mathcal{D}$ is the space of functions $\varphi \in C^{\infty}(R^3)$[1] endowed with the notion of limit

$$\left\{\varphi_n \underset{\mathcal{D}}{\to} \varphi\right\} \quad \Leftrightarrow \quad \{\text{Supp } \varphi_n, \text{Supp } \varphi \subset K \text{compact fixed set,} \\ \varphi_n^{(K)} \to \varphi^{(K)} \text{uniformly on } K\Big\} \tag{A.1}$$

We note that indeed in (A.1) (A.1) can change from sequence to sequence but has to be fixed with respect to $n$. We recall also that([$A$ indicating the closure of $A$)

$$\text{Supp } \varphi \equiv [\{x; \varphi(x) \neq 0\}] \ .$$

**Definition A.2:** $\mathcal{D}(B), B$ open, is the subspace of $\mathcal{D}$

$$\mathcal{D}(B) \equiv \{\varphi \in \mathcal{D} \ ; \ \text{Supp } \varphi \subset B\} \tag{A.2}$$

**Lemma A.1:** $\mathcal{D}(B)$ is a closed subspace of $\mathcal{D}$.
**Definition A.3:** $\mathcal{D}'$ is the topological vector space of distributions, $T$, in $R^3$, i.e. of linear continuous functionals on $\mathcal{D}$

$$T, \qquad \langle T, \varphi \rangle \in R \ , \ |\langle T, \varphi \rangle| < +\infty \qquad\qquad \forall \varphi \in \mathcal{D} \tag{A.3}$$

$$\langle T, \lambda\varphi + \mu\psi \rangle = \lambda \langle T, \varphi \rangle + \mu \langle T, \psi \rangle \tag{A.4}$$

$$\{\varphi_n \underset{\mathcal{D}}{\to} \varphi\} \Rightarrow \langle T, \varphi_n \rangle \to \langle T, \varphi \rangle \ . \tag{A.5}$$

**Lemma A.2:** $\mathcal{D}'$ is a complete topological vector space, with the weak dual topology

$$\{T_n \to T\} \Leftrightarrow \langle T_n, \varphi \rangle \to \langle T, \varphi \rangle \ , \quad \forall \varphi \in \mathcal{D} \tag{A.6}$$

**Remark A.1:** let $f$ be a measurable function $f \in L^2_{loc}$ (i.e. $\int_{(r \leq R)} f^2 dB < +\infty \ \ \forall R$); then

$$\langle T_f, \varphi \rangle \equiv \int_{R^3} f\varphi dx \tag{A.7}$$

is a distribution $T_f$ which we identify with the function $f$

$$T_f = f$$

In particular $T = 0$ can be made to coincide with any measurable function equal to zero almost everywhere, since

$$\langle T, \varphi \rangle = \int f\varphi dx = 0 \quad \forall \varphi \in \mathcal{D}$$

implies

$$f = 0 \quad a.e.$$

**Definition A.4:** $\mathcal{D}'(B)$ is the topological dual of $\mathcal{D}(B)$.
**Definition A.5:** let $\Omega$ be the largest open set such that

$$\langle T, \varphi \rangle = 0 \quad \forall \varphi, \text{Supp } \varphi \subset \Omega \ ;$$

[1]The notions given here are all valid in $R^n$ but we limit ourselves to the case of $R^3$, relevant to geodetic problems.

then the support of $T$ is the closed set

$$\text{Supp } T = \Omega^C \tag{A.8}$$

**Remark A.2:** $\mathcal{D}'(B)$ is isomorphic to the closed subspace of $\mathcal{D}'$ of all $T$ such that Supp $T \subset B$.
**Definition A.6:** first we observe that any differential monomial

$$D^s = D_1^{s_1} D_2^{s_2} D_3^{s_3} \ , \quad |s| = s_1 + s_2 + s_3 \ ,$$

is a continuous linear operator such that

$$D^s \varphi = \psi \in \mathcal{D} \ , \quad \forall \varphi \in \mathcal{D} \ ,$$

because Supp $\psi \subseteq$ Supp $\varphi$; then we define

$$D^s T = U \Leftrightarrow \langle U, \varphi \rangle \equiv (-1)^{|s|} \langle T, D^s \varphi \rangle \ , \quad \forall \varphi \in \mathcal{D} \tag{A.9}$$

**Remark A.3:** any measurable function considered as a distribution has distributional derivatives of any order.
In particular two functions $f, g$ coinciding almost everywhere have the same derivatives, since $u = f - g = 0$ a.e. so that

$$\langle D_i u, \varphi \rangle = - \langle u, D_i \varphi \rangle = - \int u D_i \varphi dx = 0 \ .$$

Note that, with Definition A.6 for any function $f$ with continuous derivatives $D_i f$, the distributional derivativs are the same functions.
**Definition A.7:** the Sobolev space $H^k$ ($k$ integer $\geq 0$) is defined as the linear subspace of $\mathcal{D}'$ of functions $f$ such that

$$\left( D^j = D_1^{j_1} D_2^{j_2} D_3^{j_3} \right) , \int \sum_{|j|=0}^{k} (D^j f)^2 dx < +\infty \ , \tag{A.10}$$

for instance for $f \in H^1$

$$\int \left[ f^2 + \sum_{j=1}^{3} (D_j f)^2 \right] dx < +\infty \ .$$

Let us observe that (A.10) is the square of a norm derived from a scalar product, then $H^k$ is at least a pre-Hilbert space.
**Remark A.4:** by using the Fourier transform $\widehat{f}$ of $f$ and using the Parseval's identity, we see that (A.10) is equivalent to

$$\int \left( \sum_{|j|=0}^{k} \xi^{2j} \right) \left| \widehat{f} \right|^2 d\xi < +\infty \ . \tag{A.11}$$

where $\xi^{2j} = \xi_1^{2j_1} \xi_2^{2j_2} \xi_3^{2j_3}$.
Since clearly the polynomial

$$\sum_{|j|=0}^{k} \xi^{2j} = 1 + \xi_1^2 + \xi_2^2 + \xi_3^2 + \ldots \geq 1$$

is strictly positive, we see from (A.11) that $\widehat{f} \in L^2$, i.e. $f \in L^2$ and more precisely that

$$f \in H^k \quad \Leftrightarrow \quad \widehat{f} = \left( \sum_{|j|=0}^{k} \xi^{2j} \right)^{-1/2} \widehat{g} \ ; \ \widehat{g} \in L^2 \tag{A.12}$$

**Lemma A.3:** as a consequence of (A.12) we see that $H^k$ is complete, i.e. it is a Hilbert space.
**Remark A.5:** since we have, $\forall\xi$

$$a\left(1+|\xi|^2\right)^k \leq \sum_{|j|=2}^{k} \xi^{2j} \leq b\left(1+|\xi|^2\right)^k, (a,b>0),$$

we see that condition (A.11) is equivalent to

$$\int (1+|\xi|^2)^k \left|\widehat{f}\right|^2 d\xi < +\infty \tag{A.13}$$

which defines an equivalent norm in $H^k$.
**Definition A.8:** for any real $s \geq 0$ we define $H^s$ as the space of $f \in L^2$ such that

$$\int (1+|\xi|^2)^s \left|\widehat{f}\right|^2 d\xi \ < +\infty \tag{A.14}$$

In this way we can define as well fractionary Sobolev spaces like $H^{1/2}, H^{3/2}$ etc.
We note explicitely that (A.14) makes sense also $\forall s$ real, $s < 0$; so we can introduce as well Sobolev spaces with negative order.
**Definition A.9:** let $\Re_B$ be the operator of restriction to $B$ of a function $f$ defined in $R^3$; then

$$H^s(B) \equiv \{\Re_B f \ ; \ f \in H^s\} \ . \tag{A.15}$$

**Lemma A.4:** $H^s(B)$ is a Hilbert space and, when $s = k$ integer,

$$\|f\|^2_{H^k(B)} = \int_B \left(\sum_{|j|=0}^{k} D^j f\right)^2 dB \tag{A.16}$$

**Lemma A.5:** for any real $s, s'$, we have the embedding chain

$$\mathcal{D} \subset H^s \subset H^{s'} \qquad (s' \leq s) \ ; \tag{A.17}$$

meaning that

$$f \in \mathcal{D} \Rightarrow f \in H^s \ , \ f \in H^s \Rightarrow f \in H^{s'} \qquad (s' \leq s) \ ;$$

the embedding operator $J$

$$J : H^s \to H^{s'} \qquad Jf \equiv f$$

is dense and, when $s' < s$, compact.
This means that the image of $\mathcal{D}$ in any $H^s$ is dense and that given a sequence $\{f_n\}$ bounded in $H^s$ ($\|f_u\|_{H^s} <$ const), it has at least an accumulation point $\overline{f}$ in $H^{s'}$.
The same holds true for $\mathcal{D}(B), H^s(B), H^{s'}(B)$.
We underline that the above statements are valid for positive as well as for negative $s$.
**Remark A.6:** with the help of local coordinates systems one can extended the concept of Sobolev spaces to surfaces. To make it simple let's assume that $S$ is a surface with finite, continuous curvature so that we can introduce local systems of coordinates

$$\underline{\sigma} \equiv (\vartheta, \lambda), \ \sigma \in Q \equiv \{\vartheta_1 \leq \vartheta \leq \vartheta_2, \ \lambda_1 \leq \lambda \leq \lambda_2\}$$

with orthogonal coordinate lines along the principal curvature directions.
Then $\forall f$ sufficiently smooth

$$\nabla_t f = \underline{e}_\vartheta \frac{1}{\rho_\vartheta} \frac{\partial f}{\partial \vartheta} + \underline{e}_\lambda \frac{1}{\rho_\lambda} \frac{\partial f}{\partial \lambda} \tag{A.18}$$

($\rho_\vartheta, \rho_\lambda$ curvature radii).
Assume now that $f \equiv 0$ outside the patch $A_{\underline{\sigma}} \equiv \underline{x}(\underline{\sigma}), (\underline{\sigma} \in Q)$ where the local system $\underline{\sigma}$ is defined above, then, considering that $dS = \rho_\vartheta \rho_\lambda d\vartheta d\lambda$, we can set

$$\int_S (\nabla_t f)^2 dS = \int_{A_{\underline{\sigma}}} (\nabla_t f)^2 dS = \int_\vartheta \left[ \frac{\rho_\lambda}{\rho_\vartheta} \left( \frac{\partial f}{\partial \vartheta} \right)^2 + \frac{\rho_\vartheta}{\rho_\lambda} \left( \frac{\partial f}{\partial \lambda} \right)^2 \right] d\vartheta d\lambda$$

showing that, if $a \leq \frac{\rho_\lambda}{\rho_\vartheta} \leq b$, the two conditions

$$\int_S |\nabla_t f|^2 \, dS < +\infty \ , \qquad \int_Q |\nabla_\sigma f|^2 \, dQ < +\infty \tag{A.19}$$

are equivalent.
Since we can split $S$ into a finite number of regular overlapping patches, we see that the case $f \in H^1(S)$ can be defined through coordinates transformations, stretching $S$ on $\Re^2$.
**Definition A.10:** let $S$ be a surface with a parametric representation, $\underline{x}(t_1, t_2)$, continuous up to $k$-th derivatives, then we define $H^k$ as the closure of $\{\Re_S \varphi \ , \ \varphi \in \mathcal{D}\}$ in the norm

$$\int_S \sum_{s_1+s_2=0}^{k} \left( \frac{\partial^{s_1}}{\partial t_1} \frac{\partial^{s_2}}{\partial t_2} f \right)^2 dS = \|f\|^2_{H^k(S)} \ ; \tag{A.20}$$

in particular

$$\int_S \left( f^2 + |\nabla_t f|^2 \right) dS = \|f\|^2_{H^1(S)} \ ;$$

moreover, by stretching $S$ on $\Re^2$ one can define as well $H^s(S)$ for any real $s \geq 0$.
**Lemma A.6:** the operator of multiplication of $f \in H^{1/2}(S)$ by a function $\alpha \in C^\lambda$ $(\lambda > 1/2)$ is bounded in $H^{1/2}(S)$

$$\|\alpha f\|_{H^{1/2}(S)} \leq \text{const} \cdot \|f\|_{H^{1/2}(S)} \ . \tag{A.21}$$

This is easy to understand by using an equivalent definition of the $H^{1/2}(S)$ norm

$$\|f\|^2_{H^{1/2}(S)} = \int_S f^2 dS + \int_S dS_y \int_S dS_x \frac{|f(x) - f(y)|^2}{|x-y|^4} \ ,$$

given by Gagliardo (cfr. [Lions, Magenes, 1968]).
**Lemma A.7:** let $f \in H^s(B), \ s > 1/2$; then, if we call $\Re_S f$ the trace of $f$ on $S$, we have

$$\Re_S f \in H^{s-1/2}(S) \ ; \ \|\Re_S f\|_{H^{s-1/2}(S)} \leq C \, \|f\|_{H^s(B)} \tag{A.22}$$

**Remark A.7:** let us note explicitely that $H^s(B)$, when $s > 1/2$, is a space of functions which on $S$ can have a trace $\neq 0$. Therefore one cannot say that $\mathcal{D}(B)$ is dense in $H^s(B)$. We call

$$(s > 1/2), H^s_0(B) \equiv [\mathcal{D}(B)]_{H^s(B)} \ ; \tag{A.23}$$

$H^s_0(B)$ is a proper closed subspace of $H^s(B)$. On the contrary, when $s < 1/2$, it is not possible to define a continuous operator of trace $\Re_S$, therefore we have

$$(s < 1/2) \quad H^s(B) \equiv [\mathcal{D}(B)] \ . \tag{A.24}$$

(A.24) holds for negative values of $s$, too. On the other hand, as claimed in Lemma A.5, when $B \equiv \Re^3, \mathcal{D}$ is dense in all the $H^s$.

**Remark A.8:** let us observe that $\forall \varphi, \psi \in \mathcal{D}$ the following inequalities hold

$$\left| \int \varphi\psi dx \right|^2 \equiv \left| \int \widehat{\varphi}^* \widehat{\psi} d\xi \right|^2 \leq \int |\widehat{\varphi}|^2 (1+|\xi|^2)^{-s} d\xi \cdot \tag{A.25}$$
$$\cdot \int \left|\widehat{\psi}\right|^2 (1+|\xi|^2)^s d\xi \equiv \|\varphi\|^2_{H^{-s}} \|\psi\|^2_{H^s} \ ,$$

where the integrals refer to two whole $R^3$.
Since in $R^3\mathcal{D}$ is dense in both $H^{-s}$ and $H^s$ we see that if we take

$$\varphi_n \in \mathcal{D} \qquad \varphi_n \underset{H^{-s}}{\rightarrow} f$$

$$\psi_n \in \mathcal{D} \qquad \psi_n \underset{H^{s}}{\rightarrow} g$$

we can extend the symbol $\int \varphi\psi dx$ to

$$\int fg dx \equiv \lim_{n,m\to\infty} \int \varphi_n \psi_m dx \ ; \tag{A.26}$$

furthermore we have

$$\forall f \in H^{-s}, \ \forall g \in H^s; \left| \int fg dx \right| \leq \|f\|_{H^{-s}} \|g\|_{H^s} \ . \tag{A.27}$$

If we repeat the same reasoning for $s > 1/2$ and $\varphi_n \in \mathcal{D}(B), \psi_n \in \mathcal{D}(B)$, since $\mathcal{D}(B)$ is dense in $H^s_0(B)$ but not in $H(B)$ we see that

$$\forall f \in H^{-s}(B), \quad \forall g \in H^s_0(B) \ ; \ \left| \int_B fg dB \right| \leq \|f\|_{H^{-s}(B)} \|g\|_{H^s(B)} \tag{A.28}$$

On the other hand when we take a closed surface $S$, since by coordinate transformation we map it onto $R^2$, we have, like in (A.27),

$$\forall f \in H^{-s}(S), \ \forall g \in H^s(S) \ ; \left| \int_S fg dS \right| \leq \|f\|_{H^s(S)} \leq \|g\|_{H^{-s}(S)} \ . \tag{A.29}$$

Therefore we claim that, indicating by $X'$ the dual of a space $X$,

$$H^{-s} \equiv (H^s)' \ , \quad H^{-s}(B) \equiv (H^s_0(B))' \ , \quad H^{-s}(S) = (H^s(S))' \tag{A.30}$$

and by these identifications we mean that if $F \in (H^s)'$ then there is $f \in H^{-s}$ such that

$$F(g) \equiv \int fg dx, \quad \forall g \in H^s$$

and so forth.

The above Remark is a particular case of a more general result.
**Lemma A.8:** let $X, Y$ be two Hilbert spaces with $X \subset Y$, the embedding being dense and continuous; then if we identify $Y' \equiv Y$ via the Riesz theorem we can write

$$X \subset Y \equiv Y' \subset X' \tag{A.31}$$

with $Y' \equiv Y$ continuously and densely embedded in $X'$, so that $\forall x' \in X'$ we can write

$$x'(x) \equiv (x', x)_Y \tag{A.32}$$

understanding the scalar product as a limit of $(y_n, x)_Y$ with $y_n \in Y$ and $y_n \underset{X'}{\rightarrow} x'$.

**Lemma A.9:** if $D^k$ is any differential operator of order $|k|$, then $D^k$ is a continuous linear operator of $H^s$ into $H^{s-|k|}$, i.e.

$$\left\| D^k f \right\|_{H^{s'}} \leq \text{const} \, \|f\|_{H^{s'+|k|}} \ ; \tag{A.33}$$

moreover if $A$ is any continuous linear operator $A : H^s \to H^{s'}$ $(s' > s)$, then $A$ is a compact operator of $H^s$ into itself.

**Lemma A.10:** we want to sketch the proof of the following fundamental inequality (Rellich): $\forall f \in H^1(B)$ we have

$$\int_B f^2 dB \leq \text{const} \left\{ \int_S f^2 dB + \int_B |\nabla f|^2 \, dB \right\} \ . \tag{A.34}$$

□ It is enough to prove (A.34) $\forall \Re_B f, f \in \mathcal{D}$. Let us consider the Green's function $G(\underline{x}, \underline{y})$ of the Laplacian in the domain $B$ and put

$$w(\underline{x}) \equiv G(\underline{x}, 0) \ .$$

The function $w(\underline{x})$ is harmonic in $B \backslash \{0\}, w = 0$ on $S$ while $w = 0\left(\frac{1}{r}\right)$ when $r \to 0$; moreover the surfaces $S_{\overline{w}} \equiv \{\underline{x}; w(\underline{x}) = \overline{w}\}$ are interior to one another while $\overline{w} \to \infty$, so that if $B_{\overline{w}} \equiv$ interior $\{S_{\overline{w}}\}$, one has

$$B_0 = B, \ B_{w_1} \subset B_{w_2} \text{ iff } \ w_1 \supset w_2 \ , \tag{A.35}$$

in addition the vector $-\nabla w$ is such that

$$|\nabla w| \neq 0 \text{ in } B \backslash \{0\} \ , \ |\nabla w| = 0 \left( \frac{1}{r^2} \right) \ \text{ for } r \to 0 \tag{A.36}$$

so that

$$|\nabla w| \geq G_0 > 0 \ ; \tag{A.37}$$

furthermore the force lines of $-\nabla w$ never cross, while they have a focus in 0 and if we introduce a curvilinear coordinate $\ell_Q$ as in Fig. A.1, $\forall Q \neq 0$ we have a couple $\{P, \ell\}$ such that $P \in S, 0 \leq \ell \leq L$ which identifies univocally $Q$.

We also observe that $-\nabla w$ is directed as the exterior normal to $S_w$ and that

$$|\nabla w| = \frac{\partial u}{\partial \ell} \ ; \tag{A.38}$$

furthermore, from Gauss theorem, we know that

$$\left( \frac{\partial w}{\partial \ell} \right)_Q dS_Q \equiv \left( \frac{\partial w}{\partial \ell} \right)_P dS_P \ , \tag{A.39}$$

implying that

$$\forall Q; P \in B \backslash \{0\}, \quad dS_Q \leq C dS_P \ ; \tag{A.40}$$

likewise $\forall Q'$ with $\ell_{Q'} \leq \ell_Q$ we have

$$dS_{Q'} \leq J dS_Q \ . \tag{A.41}$$

Then, for $f \in \mathcal{D}$, we write

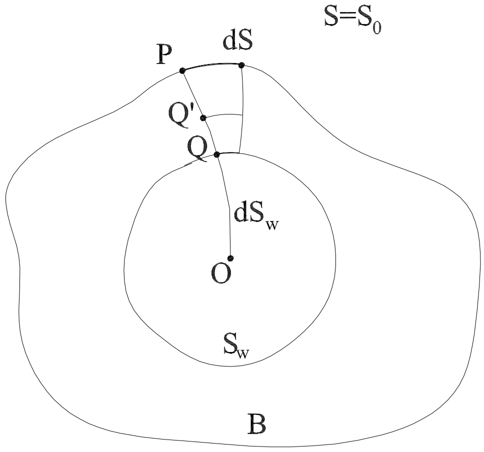


Figure A.1: Green's coordinates in $B$.

$$f(Q) = f(P) + \int_P^Q \left(\frac{\partial f}{\partial \ell}\right) d\ell \quad \Rightarrow \quad |f(Q)|^2 \leq 2\left[f^2(P) + L\int_P^Q \left(\frac{\partial f}{\partial \ell}\right)^2 d\ell\right]$$

so that, using (A.40) and (A.41),

$$|f(Q)|^2 \, dS_Q \leq \text{const}\left[f^2(P) dS_P + \int_P^Q \left(\frac{\partial f}{\partial \ell}\right)^2 dS_{Q'} \cdot d\ell\right] .$$

Observing that $dS_Q d\ell = dB_Q$ and extending the integration from $Q$ up to the origin $O$ along the force line $L_P$, we get

$$\int_{L_P} |f(Q)|^2 \, dB_Q \leq \text{const} \cdot L \left[f^2(P) dS_P + \int_{L_P} \left(\frac{\partial f}{\partial \ell}\right)^2 dB_{Q'}\right]$$

which finally integrated over all $P \in S_0$ proves (A.34). □

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