Strain in the Earth —
a Geodetic Perspective

W. Ian Reilly

Abstract

Recognition of the existence of horizontal displacement on faults in the Earth's crust became accepted only about a century ago, and with it came the theory of elastic rebound as the cause of earthquakes. This suggested the use of repeated geodetic measurements between widely distributed points on the Earth's surface to determine the accumulation or dissipation of elastic strain energy in the brittle crust. Such measurements can be modelled in terms of continuum mechanics, based on a three-dimensional vector field of particle velocity. The gradient of this vector field yields a set of invariants, or "estimable quantities", that characterise the rate of strain in the deforming medium. The use of the tensor calculus in formulating such continuum models concentrates attention on the underlying physical processes rather than on arbitrary coordinate systems. It also facilitates the use of higher-order spatial derivatives to describe the characteristics of heterogeneous strain — the bending of lines and the warping of surfaces — as well as providing for mathematically compact development of various interpolation schemes, including multi-dimensional polynomial expansions and least-squares collocation.

Key words: earth deformation, continuum mechanics, geodetic measurement.

Introduction

Omne tulit punctum qui miscuit utile dulci,
Lectorem delectando pariterque monendo.
(Full marks to him who combines profit with pleasure,
delighting the reader while instructing him — Horace, *Ars Poetica*)

There must be very few aspects of mathematical geodesy that have not been the subject of one of Professor Erik Grafarend's multifarious and erudite essays. His development with Burkhard Schaffrin of the concept of "estimable quantities" (Grafarend & Schaffrin 1974, 1976) continues to illuminate our understanding of what information can and cannot be extracted from a geodetic network. His work on the converse problem, the optimal design of geodetic networks, continues to be applied to deforming networks (*e.g.* Grafarend, 1986; Xu & Grafarend, 1995), and extended to the statistical analysis of the second-rank tensors that quantify strain and stress in the Earth (Xu & Grafarend, 1996a,b).

Much of the geodetic evidence of Earth deformation has come from pre-existing networks, and historic measurements, bereft of the advantage of advanced design. The present essay is concerned with an elementary continuum-mechanical interpretation of such measurements, extracting the strain invariants — the "estimable quantities" for the deforming medium. It acknowledges both the elucidative contribution of mathematical theory, and the dedication and professionalism of generations of surveyors who went into forest and desert and mountain and made better measurements than we could reasonably have demanded of them.

Origins

> "In no country, perhaps, where the English language is spoken, have earthquakes, or, to speak more correctly, the subterranean causes to which such movements are due, been so active in producing changes of geological interest as in New Zealand."
> (Lyell, 1872, vol. II: p. 82)

When Sir Charles Lyell wrote these words (with a fine chauvinistic flourish) in the Tenth Edition of his celebrated text *The Principles of Geology*, vertical displacement of the Earth's crust on faults was well known, and had been for many years: in 1802 Playfair had written

> "The greatest part of the facts relative to the fracture and dislocation of the strata, belongs to the history of veins…The frequency of these [slips], and their great extent, are well known wherever mines have been wrought." (Playfair, 1802, p 204)

In 1888 in North Canterbury, New Zealand, horizontal offsets of between 1.5 and 2.6 m were observed on the fault break that accompanied the earthquake of magnitude 7 that knocked the top off the Christchurch Cathedral spire. The Government geologist's report (McKay, 1890) was duly published and forgotten. This was but one example among many, from one small corner of the globe. Then, in 1906, the magnitude 8.3 earthquake that razed the city of San Francisco not only convinced the scientific world of the reality of horizontal displacement on faults — the San Andreas in this case — but also gave rise to Reid's theory of elastic rebound.

Geodetic evidence

It was soon realised that if earthquakes, and fault breaks, resulted from the sudden release of accumulated elastic strain in the crust, then the accumulation of strain between earthquakes might be manifest in the distortion of geodetic networks. Spurred by the Kwanto earthquake of 1923 in Japan, Terada & Miyabe (1929) derived and mapped the parameters of shear, rotation and dilatation from displacement vectors for each triangle of repeated surveys, and this type of study has since been continued. In New Zealand, H. W. Wellman, who had been a co-discoverer of the South Island's Alpine Fault, with its *ca*. 450 km dextral offset since the Cretaceous, determined the rate and orientation of shear strain from repeated triangulation in Marlborough (Wellman, 1955). This area, in the northern part of the South Island, together with the Alpine Fault itself, is now recognised as part of the obliquely convergent margin between the Pacific and the Australian lithospheric plates that passes through New Zealand.

The Marlborough region was further studied by H. M. Bibby, who developed a method of simultaneous reduction of repeated geodetic surveys, coincident in part or in whole, together with determination of deformation parameters (Bibby 1973, 1976, 1981). He found pervasive shear strain occurring over time spans of several decades in the absence of overt fault movement. This pioneering work has remained the basis for all subsequent analyses of geodetic data to determine earth deformation in New Zealand, and has had influence elsewhere.

What the geodetic evidence, then as now, leaves unresolved is the partitioning of observed shear strain between elastic (recoverable) and non-elastic (permanent) deformation. Does a low level of brittle failure indicate that strain energy is being dissipated in non-elastic creep, or simply that the observation period falls between major earthquakes, and that elastic strain is accumulating steadily? Information from more than purely geodetic measurements is needed even to discuss, much less resolve, such questions.

Continuum mechanics

A geodetic network is a discrete measuring system. The Earth's crust is riven by faults — discrete fractures. What, then, is the justification for describing deformation in terms of continuum mechanics? In essence, it is a question of scale. Even if deformation occurs by slip on a sequence of faults that separate rigid blocks, it can be treated as continuous if the fault separation is a sufficiently small fraction of the station spacing of the geodetic network. In the absence of obvious fault movement within the observing period, the continuum model is the most general and unbiased, and can always be superimposed on a discrete faulting model.

The problem of describing the deformation of the Earth's crust has much in common with fluid dynamics, and can use the methods of vector and tensor analysis made familiar in geodesy by Hotine (1969). There are two principal differences, however. The first is one of scale: the relative velocity between two lithospheric plates may be of the order of 50 mm/yr, or 1.5×10^{-9} m s^{-1}, compared, say, with the velocity of sound at sea level of about 300 m s^{-1}: a ratio of 2×10^{11}. The second is one of the experimental environment: the experimenter in fluid dynamics is at no loss for a fixed frame of reference provided by the apparatus around, over, and through which his fluids flow; but the student of earth deformation can depend on no such reliable framework — it is as if he were floating on a broad river, with neither shore nor bottom discernible. This is no disadvantage if his object is to study the intrinsic deformation of the crust, and to rely on direct, local and differential, physical measurements: distinctions between *Lagrangian* and *Eulerian* coordinates can be cast aside, and tedious arguments about datums become irrelevant.

A primary advantage of using the methods of the tensor calculus in continuum mechanics resides in the immediate identification of vector and tensor quantities with objective physical fields, such as the particle velocity field of the deforming medium, independently of arbitrary coordinate systems. A second advantage is the ease with which covariant differentiation of such spatially variable velocity fields opens the way to the calculation of higher-order invariants — the "estimable quantities" of continuum mechanics — and the clarity with which these can be seen to be independent of particular coordinate systems.

Deformation in three dimensions

If the velocity of a material particle *P* is represented by the vector $\mathbf{u}^{\mathbf{r}}$, then the deformation in the vicinity of *P* is given by the gradient of the velocity vector, and represented by the covariant derivative of $\mathbf{u}^{\mathbf{r}}$, *viz.* $\mathbf{u}^{\mathbf{r}}_{\mathbf{s}}$. The tensor $\mathbf{u}^{\mathbf{r}}_{\mathbf{s}}$ has in general nine independent coefficients; it can be decomposed into symmetric and antisymmetric parts

$$\mathbf{u}^{\mathbf{r}}_{\mathbf{s}} = \boldsymbol{\sigma}^{\mathbf{r}}_{\mathbf{s}} + \boldsymbol{\tau}^{\mathbf{r}}_{\mathbf{s}} \tag{1}$$

The symmetric tensor $\boldsymbol{\sigma}^{\mathbf{r}}_{\mathbf{s}}$, the strain rate tensor of six independent terms, describes the intrinsic deformation of the medium. An idealised test apparatus would comprise a regular tetrahedron embedded in the medium, where any one of its six sides could shorten or lengthen independently of the other five. The use of such an apparatus is unlikely, although a geodetic approximation could be attained by a suitable array of benchmarks on the floor and flanks of a deep valley, interconnected by distance measurements.

The antisymmetric tensor $\boldsymbol{\tau}^{\mathbf{r}}_{\mathbf{s}}$, the rotation rate tensor of three independent terms, describes the mean rotation of the small volume about *P* with respect to some external reference frame. It can be represented by its equivalent rotation-rate vector $\mathbf{T}^{\mathbf{r}}$, defined by

$$\mathbf{T}^{\mathbf{r}} = -\tfrac{1}{2}\, \boldsymbol{\varepsilon}^{\mathbf{rst}}\, \mathbf{a}_{\mathbf{su}}\, \boldsymbol{\tau}^{\mathbf{u}}_{\mathbf{t}} = -\tfrac{1}{2}\, \boldsymbol{\varepsilon}^{\mathbf{rst}}\, \mathbf{a}_{\mathbf{su}}\, \mathbf{u}^{\mathbf{u}}_{\mathbf{t}} \tag{2}$$

where $\boldsymbol{\varepsilon}^{\mathbf{rst}}$ is the alternating tensor, and $\mathbf{a}_{\mathbf{su}}$ the metric tensor in three dimensions.

Inverting (2) and substituting in (1), we have

$$\mathbf{u}^{\mathbf{r}}_{\mathbf{s}} = \boldsymbol{\sigma}^{\mathbf{r}}_{\mathbf{s}} + \mathbf{a}^{\mathbf{rp}}\, \boldsymbol{\varepsilon}_{\mathbf{pqs}}\, \mathbf{T}^{\mathbf{q}} \tag{3}$$

Dilatation and shear

The symmetric strain rate tensor $\boldsymbol{\sigma}^{\mathbf{r}}_{\mathbf{s}}$ can be subjected to a principal axis decomposition. Let the principal axes be denoted by the set of orthogonal unit vectors ($\mathbf{i}^{\mathbf{r}}, \mathbf{j}^{\mathbf{r}}, \mathbf{k}^{\mathbf{r}}$), ordered such that the eigenvalues resulting from the decomposition of $\boldsymbol{\sigma}^{\mathbf{r}}_{\mathbf{s}}$ are ranked as

$$\boldsymbol{\sigma}^{r}{}_{s}\, \mathbf{i}_{r}\, \mathbf{i}^{s} \;\geq\; \boldsymbol{\sigma}^{r}{}_{s}\, \mathbf{j}_{r}\, \mathbf{j}^{s} \;\geq\; \boldsymbol{\sigma}^{r}{}_{s}\, \mathbf{k}_{r}\, \mathbf{k}^{s} \tag{4}$$

These eigenvalues represent the rates of extensional strain in the principal directions; their sum defines the rate of volumetric dilatation $\boldsymbol{\Theta}$ by

$$\begin{aligned} 3\,\boldsymbol{\Theta} &= (\boldsymbol{\sigma}^{r}{}_{s}\, \mathbf{i}_{r}\, \mathbf{i}^{s} + \boldsymbol{\sigma}^{r}{}_{s}\, \mathbf{j}_{r}\, \mathbf{j}^{s} + \boldsymbol{\sigma}^{r}{}_{s}\, \mathbf{k}_{r}\, \mathbf{k}^{s}) \\ &= \boldsymbol{\sigma}^{r}{}_{s}\, \boldsymbol{\delta}^{s}{}_{r} = \boldsymbol{\sigma}^{r}{}_{r} = \mathbf{u}^{r}{}_{s}\, \boldsymbol{\delta}^{s}{}_{r} = \mathbf{u}^{r}{}_{r} \end{aligned} \tag{5}$$

where $\boldsymbol{\delta}^{s}{}_{r}$ is the Kronecker (or substitution) tensor in three dimensions, and $\boldsymbol{\sigma}^{r}{}_{r}$ is the trace of the symmetric tensor $\boldsymbol{\sigma}^{r}{}_{s}$.

Subtraction of the effect of isotropic expansion or contraction from the strain rate tensor $\boldsymbol{\sigma}^{r}{}_{s}$ leads to the following tensor of five independent terms that represents the rate of shearing, or pure change of shape

$$\begin{aligned} (\boldsymbol{\sigma}^{r}{}_{s} - \boldsymbol{\Theta}\, \boldsymbol{\delta}^{r}{}_{s}) = {} & (\boldsymbol{\sigma}^{u}{}_{t}\, \mathbf{i}_{u}\, \mathbf{i}^{t} - \boldsymbol{\Theta})\, \mathbf{i}^{r}\, \mathbf{i}_{s} + (\boldsymbol{\sigma}^{u}{}_{t}\, \mathbf{j}_{u}\, \mathbf{j}^{t} - \boldsymbol{\Theta})\, \mathbf{j}^{r}\, \mathbf{j}_{s} \\ & + (\boldsymbol{\sigma}^{u}{}_{t}\, \mathbf{k}_{u}\, \mathbf{k}^{t} - \boldsymbol{\Theta})\, \mathbf{k}^{r}\, \mathbf{k}_{s} \end{aligned} \tag{6}$$

where $\boldsymbol{\sigma}^{r}{}_{s}$ has been expanded in terms of the principal directions.

One way of representing this tensor is to introduce two scalar magnitudes, $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$, in addition to the three independent terms that define the three principal axes, *viz.*

$$\begin{aligned} \boldsymbol{\gamma}_1 &= \boldsymbol{\sigma}^{u}{}_{t}\, \mathbf{i}_{u}\, \mathbf{i}^{t} - \boldsymbol{\Theta} \\ \boldsymbol{\gamma}_2 &= -(\boldsymbol{\sigma}^{u}{}_{t}\, \mathbf{k}_{u}\, \mathbf{k}^{t} - \boldsymbol{\Theta}) \end{aligned} \tag{7}$$

where $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ correspond to the greatest and least rates of extensional strain in the principal directions, respectively, and are defined so that both $\boldsymbol{\gamma}_1 \geq 0$ and $\boldsymbol{\gamma}_2 \geq 0$.

In terms of these parameters, the eigenvalues are now

$$\begin{aligned} \boldsymbol{\sigma}^{r}{}_{s}\, \mathbf{i}_{r}\, \mathbf{i}^{s} &= \boldsymbol{\Theta} + \boldsymbol{\gamma}_1 \\ \boldsymbol{\sigma}^{r}{}_{s}\, \mathbf{j}_{r}\, \mathbf{j}^{s} &= \boldsymbol{\Theta} + (\boldsymbol{\gamma}_2 - \boldsymbol{\gamma}_1) \\ \boldsymbol{\sigma}^{r}{}_{s}\, \mathbf{k}_{r}\, \mathbf{k}^{s} &= \boldsymbol{\Theta} - \boldsymbol{\gamma}_2 \end{aligned} \tag{8}$$

and the deformation rate tensor can be expressed as

$$\mathbf{u}^{r}{}_{s} = \boldsymbol{\Theta}\, \boldsymbol{\delta}^{r}{}_{s} + \boldsymbol{\gamma}_1 (\mathbf{i}^{r}\, \mathbf{i}_{s} - \mathbf{j}^{r}\, \mathbf{j}_{s}) + \boldsymbol{\gamma}_2 (\mathbf{j}^{r}\, \mathbf{j}_{s} - \mathbf{k}^{r}\, \mathbf{k}_{s}) + \mathbf{a}^{rp}\, \boldsymbol{\varepsilon}_{pqs}\, \mathbf{T}^{q} \tag{9}$$

The change of shape of the medium in the vicinity of a point *P* can be measured by the rate of change in the angle between two distinct lines of material particles. Let $\mathbf{l}^{r}$, $\mathbf{m}^{r}$, $\mathbf{n}^{r}$, be an arbitrary right-handed set of orthogonal unit vectors, of which $\mathbf{l}^{r}$, $\mathbf{m}^{r}$, represent two such lines of particles. The rate of change of the right angle between $\mathbf{l}^{r}$ and $\mathbf{m}^{r}$ is given by the difference in the rates of rotation of the particles in these directions about the axis $\mathbf{n}^{r}$, *viz.*

$$\mathbf{u^r{}_s}\ \boldsymbol{\varepsilon}_{\mathbf{pqr}}\ \mathbf{n^q}\ (\ \mathbf{l^s\ l^p - m^s\ m^p}\ )\ =\ -\,\mathbf{u^r{}_s}\ (\ \mathbf{m^s\ l_r + l^s\ m_r}\ ) \tag{10}$$

In substituting for $\mathbf{u^r{}_s}$ from (9), the terms in $\boldsymbol{\Theta}$ and $\mathbf{T^q}$ will vanish, and the resultant expression will contain the scalar terms $\boldsymbol{\gamma}_{\mathbf{1}}$ and $\boldsymbol{\gamma}_{\mathbf{2}}$, and the three Euler angles that relate the ($\mathbf{l^r}$, $\mathbf{m^r}$, $\mathbf{n^r}$) triad to the principal directions ($\mathbf{i^r}$, $\mathbf{j^r}$, $\mathbf{k^r}$).

Because of the restriction of extensive geodetic measurements to the Earth's surface, the vertical gradient of the velocity vector is in general unobservable, *i.e.* if $\mathbf{n^r}$ is a unit vector in the vertical direction, the values of $\mathbf{u^r{}_s\ n^s}$ are usually unattainable. The "estimable quantities" are therefore reduced to six, and most of the discussion of the results of geodetic measurement of earth deformation is in terms of two-dimensional deformation, either in plane or spherical approximation.

Deformation in two dimensions

Four of the above six estimable quantities appear in the two-dimensional form of the expression for the velocity gradient of equation (9), *viz.*

$$\begin{aligned}\mathbf{u}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} &= \boldsymbol{\sigma}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} + \boldsymbol{\tau}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} \\ &= \boldsymbol{\Delta}\ \boldsymbol{\delta}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}} + \boldsymbol{\gamma}\ (\ \mathbf{j}^{\boldsymbol{\alpha}}\ \mathbf{j}_{\boldsymbol{\beta}} - \mathbf{k}^{\boldsymbol{\alpha}}\ \mathbf{k}_{\boldsymbol{\beta}}\ ) - \boldsymbol{\varepsilon}^{\boldsymbol{\alpha\eta}}\ \mathbf{a}_{\boldsymbol{\eta\beta}}\ \boldsymbol{\Omega}\end{aligned} \tag{11}$$

where the Greek subscripts & superscripts now denote two-dimensional surface vectors and tensors, and we have introduced $\boldsymbol{\Delta}$ as the rate of areal dilatation, $\boldsymbol{\Omega}$ as the scalar rate of mean rotation in the two-dimensional surface, and $\boldsymbol{\gamma}$ as the magnitude of the rate of shear strain (the *tensor shear*, in contrast to the *engineering shear*, 2 $\boldsymbol{\gamma}$). To these three invariants we can add one principal direction (either $\mathbf{j}^{\boldsymbol{\alpha}}$ or $\mathbf{k}^{\boldsymbol{\alpha}}$) to fully specify the rate of deformation. In the case of the intrinsic strain rate $\boldsymbol{\sigma}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}$ in two dimensions, the idealised test apparatus would comprise an equilateral triangle attached to the surface, an arrangement that is closely approximated by triangulation and trilateration networks.

Again denoting by $\mathbf{n^r}$ a unit vector in the vertical direction, the rate of areal dilatation $\boldsymbol{\Delta}$, a linear invariant of $\mathbf{u}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}$, can be related to the volumetric dilatation $\boldsymbol{\Theta}$ by the definition

$$\boldsymbol{\Delta}\ =\ \tfrac{1}{2}\,\mathbf{u}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}\ \boldsymbol{\delta}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}\ =\ \tfrac{1}{2}\,\mathbf{u^r{}_s}\ (\ \boldsymbol{\delta}^{\mathbf{s}}{}_{\mathbf{r}} - \mathbf{n^s\ n_r}\ )\ =\ \tfrac{1}{2}\,(\ 3\ \boldsymbol{\Theta} - \mathbf{u^r{}_s\ n^s\ n_r}\ ) \tag{12}$$

Thus the rate of areal dilatation differs from the rate of volumetric dilatation by the magnitude of the rate of vertical extensional strain. If it can be assumed that the volumetric strain $\boldsymbol{\Theta}$ is zero, *i.e.* that the medium is incompressible, then the rate of areal dilatation $\boldsymbol{\Delta}$ can be taken as a measure of the vertical extensional strain rate.

The scalar rotation rate $\boldsymbol{\Omega}$, a second linear invariant of $\mathbf{u}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}$, is related to the three-dimensional vector $\mathbf{T^q}$ through the definition

$$\boldsymbol{\Omega}\ =\ -\tfrac{1}{2}\,\mathbf{u}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}\ \mathbf{a}_{\boldsymbol{\alpha\eta}}\ \boldsymbol{\varepsilon}^{\boldsymbol{\eta\beta}}\ =\ -\tfrac{1}{2}\ \boldsymbol{\varepsilon}^{\mathbf{rst}}\ \mathbf{a_{su}\ u^u{}_t\ n_r}\ =\ \mathbf{T^r\ n_r} \tag{13}$$

The magnitude $\boldsymbol{\gamma}$ of the shear strain rate can be derived as the quadratic invariant

$$\boldsymbol{\gamma}^2\ =\ \tfrac{1}{4}\,\mathbf{u}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}\ \mathbf{u}^{\boldsymbol{\gamma}}{}_{\boldsymbol{\delta}}\ (\mathbf{a}_{\boldsymbol{\alpha\gamma}}\ \mathbf{a}^{\boldsymbol{\beta\delta}}\ -\ \boldsymbol{\varepsilon}_{\boldsymbol{\alpha\gamma}}\ \boldsymbol{\varepsilon}^{\boldsymbol{\beta\delta}}\ ) \tag{14}$$

The unit vectors $\mathbf{j}^{\alpha}$ and $\mathbf{k}^{\alpha}$ give the principal directions of the symmetric tensor $\sigma^{\alpha}{}_{\beta}$ corresponding to the directions of maximum and minimum extensional strain, respectively. The eigenvalues are thus

$$\sigma^{\alpha}{}_{\beta}\, j_{\alpha}\, j^{\beta} = u^{\alpha}{}_{\beta}\, j_{\alpha}\, j^{\beta} = \Delta + \gamma$$

$$\sigma^{\alpha}{}_{\beta}\, k_{\alpha}\, k^{\beta} = u^{\alpha}{}_{\beta}\, k_{\alpha}\, k^{\beta} = \Delta - \gamma \qquad (15)$$

By analogy with (10), the rate of shear with respect to an arbitrary pair of orthogonal unit vectors $\mathbf{l}^{\mathbf{r}}$, $\mathbf{m}^{\mathbf{r}}$, is

$$u^{\alpha}{}_{\beta}\, \varepsilon_{\alpha\delta}\, (l^{\beta}\, l^{\delta} - m^{\beta}\, m^{\delta}) = - u^{\alpha}{}_{\beta}\, (m^{\beta}\, l_{\alpha} + l^{\beta}\, m_{\alpha}) = 2\,\gamma \sin 2\,\varphi \qquad (16)$$

where φ is the angle between $\mathbf{l}^{\alpha}$ and the principal direction $\mathbf{i}^{\alpha}$ in the direction of maximum relative extension.

The advantage in calculating the shear strain rate is twofold:

- it can be derived from observations of changes in shape only, where no accurate length scale is available (as for repeated triangulations);
- it is the quantity that most accurately reflects the continuous accumulation of elastic strain energy, and thus presages brittle failure in elastic media.

Shear strain rate alone can be depicted as a line symbol of magnitude γ, with the direction (though not the sense) of either the maximum relative extension $\mathbf{j}^{\alpha}$, the maximum relative contraction $\mathbf{k}^{\alpha}$, or even of one or other of the directions of maximum shear which bisect the right angle between the directions $\mathbf{j}^{\alpha}$ and $\mathbf{k}^{\alpha}$. Alternatively, the magnitude γ of the shear strain rate can be plotted and contoured as a scalar variable without reference to the directions of the principal axes.

Heterogeneous strain in two dimensions: bending

If the rate of strain is constant over some region, the strain is said to be *homogeneous*, and the gradient of the deformation rate tensor is then zero

$$u^{\alpha}{}_{\beta\gamma} = 0 \qquad (17)$$

On the other hand, if the rate of strain varies within the region, the strain is *heterogeneous*.

If the observed strain is heterogeneous, this will be made obvious in plotting different values of the rate of dilatation, or different values of the magnitude of the shear strain rate and its associated direction across the region. The question arises, however, as to whether one or more functions of $u^{\alpha}{}_{\beta\gamma}$ might be used to display the character of the heterogeneous strain.

From an inspection of equation (11), it is apparent that the gradients of the two linear invariants, Δ and Ω, will yield vectors that point towards regions of greater or lesser rates of dilatation or rotation. However, as even the gradient of the intrinsic strain rate tensor $\sigma^{\alpha}{}_{\beta\gamma}$ has six independent parameters, there is a large number of derived functions available.

One characteristic of homogeneous strain is that any line of material particles that was originally straight remains straight after straining. Under heterogeneous strain, such a line of particles would in general become curved. This suggests that the *rate of bending*, or of change of curvature, is a quantity that not only reflects an observed characteristic of many geological structures, but also could be used to describe one aspect of heterogeneous strain.

Introducing an arbitrary pair of orthogonal unit vectors, $\mathbf{l}^{\alpha}$ and $\mathbf{m}^{\alpha}$, we may express the velocity of a line of particles in the direction of $\mathbf{l}^{\alpha}$ resolved into components along the line and normal to it

$$\mathbf{u}^{\alpha}{}_{\beta}\,\mathbf{l}^{\beta} = \mathbf{e}\,\mathbf{l}^{\alpha} + \mathbf{r}\,\mathbf{m}^{\alpha} \tag{18}$$

where $\mathbf{e}$ is the rate of extensional strain in the direction $\mathbf{l}^{\alpha}$, and $\mathbf{r}$ is the rate of rotation of the line of particles in the same direction. Hence

$$\mathbf{r}(\mathbf{l}^{\alpha}) = \mathbf{u}^{\alpha}{}_{\beta}\,\mathbf{l}^{\beta}\,\mathbf{m}_{\alpha} \tag{19}$$

We define the rate of bending $\boldsymbol{\rho}(\mathbf{l}^{\alpha})$ of the line of particles in the direction $\mathbf{l}^{\alpha}$ as the gradient of the rate of rotation in that same direction, *viz.*

$$\boldsymbol{\rho}(\mathbf{l}^{\alpha}) = \mathbf{r}_{\gamma}(\mathbf{l}^{\alpha})\,\mathbf{l}^{\gamma} = \mathbf{u}^{\alpha}{}_{\beta\gamma}\,\mathbf{l}^{\beta}\,\mathbf{m}_{\alpha}\,\mathbf{l}^{\gamma} \tag{20}$$

If we have the values of $\mathbf{u}^{\alpha}{}_{\beta\gamma}$, then we can derive the directions of extreme or zero bending at a point. Since (20) is a cubic function of direction, there will be either one or three axes of maximum bending, and one or three axes of zero bending, which is a little more complicated than the intuitive concept of the bending of a linear structure, such as a beam.

There are in all three distinct contractions of $\mathbf{u}^{\alpha}{}_{\beta\gamma}$ with the orthogonal unit vectors $\mathbf{l}^{\alpha}$ and $\mathbf{m}^{\alpha}$ of the form given in (20), as well as analogous forms in three dimensions (Reilly, 1986), but the bending rate in two dimensions is probably the most accessible.

Tilting and warping of surfaces

The vertical component of the velocity of a material point *P* is denoted by

$$\mathbf{h}_{s} = \mathbf{u}^{r}{}_{s}\,\mathbf{n}_{r} \tag{21}$$

where $\mathbf{n}_{r}$ is a unit vector in the direction of the vertical. The *tilt rate vector* in three dimensions is defined as the gradient of the vertical velocity

$$\mathbf{h}_{s} = \mathbf{u}^{r}{}_{s}\,\mathbf{n}_{r} = \boldsymbol{\sigma}^{r}{}_{s}\,\mathbf{n}_{r} + \mathbf{n}^{p}\,\boldsymbol{\varepsilon}_{pqs}\,\mathbf{T}^{q} \tag{22}$$

The tilt rate in the direction of an arbitrary horizontal unit vector $\mathbf{l}^{r}$ is

$$\mathbf{h}_{s}\,\mathbf{l}^{s} = \boldsymbol{\sigma}^{r}{}_{s}\,\mathbf{n}_{r}\,\mathbf{l}^{s} + \mathbf{n}^{p}\,\mathbf{l}^{s}\,\boldsymbol{\varepsilon}_{pqs}\,\mathbf{T}^{q} \tag{23}$$

showing that the tilt rate is a combination of the intrinsic shear strain rate in the vertical plane containing $\mathbf{l}^{r}$, and the extrinsic mean rotation rate about a horizontal axis normal to $\mathbf{l}^{r}$. These components cannot be separated on the basis of horizontal tilt measurements alone, as where the tilt rate is found by such geodetic measurements as repeated spirit levelling, or repeated GPS observations.

It is usually more convenient to define the tilt rate as a two-dimensional vector $\mathbf{h}_{\alpha}$ in the horizontal plane: this accounts for the remaining two of the six ”estimable quantities” generally attainable by geodetic measurements on the Earth’s surface. An accessible measure of the *intrinsic* strain rate is then given by its gradient $\mathbf{h}_{\alpha\beta}$ (*cf*. Hein & Kistermann 1981). If $\mathbf{l}^{\alpha}$ is a unit surface vector coincident with the space vector $\mathbf{l}^{r}$ of (23) above, the *rate of change of surface curvature* in the direction of $\mathbf{l}^{\alpha}$ is

$$\mathbf{h}_{\alpha\beta}\,\mathbf{l}^{\alpha}\,\mathbf{l}^{\beta} \;=\; \mathbf{u}^{r}{}_{st}\,\mathbf{n}_{r}\,\mathbf{l}^{s}\,\mathbf{l}^{t} \tag{24}$$

By comparison with equation (20), the expression on the right-hand-side of (24) is seen to be equivalent to the *rate of bending* in the vertical plane of the line of particles in the direction $\mathbf{l}^{r}$.

The tensor $\mathbf{h}_{\alpha\beta}$ is symmetric; its decomposition is analogous to that for two-dimensional strain in (11), *viz.*

$$\mathbf{h}_{\alpha\beta} \;=\; \mathbf{H}\,\mathbf{a}_{\alpha\beta} + \mathbf{D}\,(\,\mathbf{j}_{\alpha}\,\mathbf{j}_{\beta} - \mathbf{k}_{\alpha}\,\mathbf{k}_{\beta}\,) \tag{25}$$

where the *rate of change of mean curvature* of the surface is

$$\mathbf{H} \;=\; \tfrac{1}{2}\,\mathbf{h}_{\alpha\beta}\,\mathbf{a}^{\alpha\beta} \tag{26}$$

and the *rate of change of torsion* **D** of the surface is found from

$$\mathbf{D}^{2} \;=\; \tfrac{1}{4}\,\mathbf{h}_{\alpha\beta}\,\mathbf{h}_{\gamma\delta}\,(\mathbf{a}^{\alpha\gamma}\,\mathbf{a}^{\beta\delta} \;-\; \boldsymbol{\varepsilon}^{\alpha\gamma}\,\boldsymbol{\varepsilon}^{\beta\delta}\,) \tag{27}$$

The maximum and minimum values of the rate of change of curvature (or of bending in the vertical plane) are given by the eigenvalues of the symmetric tensor $\mathbf{h}_{\alpha\beta}$, *viz.*

$$\begin{aligned} \mathbf{h}_{\alpha\beta}\,\mathbf{j}^{\alpha}\,\mathbf{j}^{\beta} &= \mathbf{H} + \mathbf{D} \\ \mathbf{h}_{\alpha\beta}\,\mathbf{k}^{\alpha}\,\mathbf{k}^{\beta} &= \mathbf{H} - \mathbf{D} \end{aligned} \tag{28}$$

Determination of strain from geodetic observations

Following the general principle introduced by Bibby (1973, 1976, 1981), the displacements of geodetic bench-marks can be modelled so as to permit a unified solution of geodetic measurements made at different epochs. The results comprise

- a set of coordinates for each bench-mark at some reference epoch;
- a set of parameters defining the velocity field, either continuous or discrete in time and space.

Amongst the continuum models used or proposed for interpolating velocities and rates of deformation, Grafarend (1986) has noted that the geodetic network is an actualisation of a finite element model, and coupled this with a local spline interpolation. Spline interpolation is also the basis of an application by Haines & Holt (1993) of a finite element model to the spherical surface of the Earth. Two further models — polynomial expansion and least-squares collocation — will be briefly mentioned here.

Polynomial approximation

The velocity vector at a point P can be expressed as a Taylor's series expansion about a suitable origin P_0

$$\mathbf{u}^{\alpha} \;=\; \mathbf{b}^{\alpha} + \mathbf{b}^{\alpha}{}_{\beta}\,y^{\beta} + \mathbf{b}^{\alpha}{}_{\beta\gamma}\,y^{\beta}\,y^{\gamma}/2! + \mathbf{b}^{\alpha}{}_{\beta\gamma\delta}\,y^{\beta}\,y^{\gamma}\,y^{\delta}/3! + \ldots \tag{29}$$

where

$\mathbf{y}^{\beta}$ is a position vector, with gradient $\mathbf{y}^{\beta}{}_{\gamma} \;=\; \boldsymbol{\delta}^{\beta}{}_{\gamma}$ (in a Euclidean space), and

$\mathbf{b}^{\alpha}$, $\mathbf{b}^{\alpha}{}_{\beta}$, $\mathbf{b}^{\alpha}{}_{\beta\gamma}$, $\mathbf{b}^{\alpha}{}_{\beta\gamma\delta}$, …, are constant coefficients to be determined.

The deformation rate tensor is found by covariant differentiation of (29) as

$$\mathbf{u}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\eta}} = \mathbf{b}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\eta}} + \mathbf{b}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta\eta}}\, \mathbf{y}^{\boldsymbol{\beta}} + \mathbf{b}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta\gamma\eta}}\, \mathbf{y}^{\boldsymbol{\beta}}\, \mathbf{y}^{\boldsymbol{\gamma}} /2! + \ldots \tag{30}$$

With adequate error determination, the series expansion can be truncated to exclude insignificant terms. Low-order expansions are of value in smoothing the results from noisy data. Application to extensive regions (such as the order of 10^5 km^2 in Reilly, 1990) can be criticised as forcing a pattern on complex data, a universal hazard of polynomial approximation methods.

Least-squares collocation

The interpolation of a vector field of displacements or velocities of material points would seem to be an ideal subject for least-squares collocation. Deakin *et al.* (1994) have applied the method to interpolating the displacements of the three-dimensional coordinates of points of a geodetic network in Victoria, Australia. In this they used a triplet of covariance functions, one for each coordinate direction. In a study of the prediction of horizontal strain in Japan, El-Fiky && Kato (1999) assumed the covariance between point displacements to be ”homogeneous and isotropic”, but used a separate covariance function for each of the E-W and N-S components of the observed displacement vectors.

In neither of these examples is it clear that the difference between the covariance function for different components is of any real significance in the interpolation process. Moreover, the assignation of different covariance functions to different components of the displacement field amounts to defining a covariance function for the *vector field* that is anisotropic with respect to the azimuth of the parallel components of the two vectors, with axes of anisotropy coinciding with arbitrarily chosen coordinate directions. In short, there seems to be no good reason to go beyond a simple function that is isotropic both with respect to the relative orientation of pairs of points, and also with respect to the orientation of any arbitrary pair of parallel vector components, and where the correlation between orthogonal vector components is zero.

Let a material particle P have a position denoted by the vector $\mathbf{x}^{\mathbf{i}}$, and to be moving with a velocity $\mathbf{u}^{\mathbf{i}}(P) = \mathbf{d}\,\mathbf{x}^{\mathbf{i}}(P) / \mathbf{d}\,\mathbf{t}.$ Let a similar particle Q have a position $\mathbf{x}^{\mathbf{i}}(Q)$ and velocity $\mathbf{u}^{\mathbf{i}}(Q)$. Assuming a Euclidean space, we denote the vector PQ by

$$\mathbf{p}^{\mathbf{i}} = \mathbf{r}\,\mathbf{m}^{\mathbf{i}} = \mathbf{x}^{\mathbf{i}}(Q) - \mathbf{x}^{\mathbf{i}}(P) \tag{31}$$

where $\mathbf{r}$ is the length of the vector $\mathbf{p}^{\mathbf{i}}$,

$\mathbf{m}^{\mathbf{i}}$ is the unit vector in the direction of $\mathbf{p}^{\mathbf{i}}$, such that $\mathbf{m}^{\mathbf{i}}\,\mathbf{m}_{\mathbf{i}} = 1$.

As an example of a covariance tensor $\mathbf{C}^{\mathbf{jk}}$ between the vectors $\mathbf{u}^{\mathbf{j}}(P)$ and $\mathbf{u}^{\mathbf{k}}(Q)$ based on a simple Gaussian function, that is simultaneously homogeneous, and isotropic in both the senses discussed above, we can write

$$\mathbf{C}^{\mathbf{jk}}\{\mathbf{u}^{\mathbf{j}}(P), \mathbf{u}^{\mathbf{k}}(Q)\} = \mathbf{C_0}\,\mathbf{a}^{\mathbf{jk}} \exp(-\mathbf{r}^2 / 2\mathbf{d}^2) \tag{32}$$

where $\mathbf{C_0}$ is a constant of dimension $(velocity)^2$,

$\mathbf{d}$ is a constant of dimension *length*,

$\mathbf{a}^{\mathbf{jk}}$ is the metric tensor in three dimensions.

If $\mathbf{f}_{\mathbf{j}}$, $\mathbf{g}_{\mathbf{j}}$ are two arbitrary unit vectors, then the covariance between two velocity components is the scalar quantity

$$\mathbf{C}^{\mathbf{jk}}\{\mathbf{u}^{\mathbf{j}}(P)\,\mathbf{f}_{\mathbf{j}},\mathbf{u}^{\mathbf{k}}(Q)\,\mathbf{g}_{\mathbf{k}}\}\,\mathbf{f}_{\mathbf{j}}\,\mathbf{g}_{\mathbf{k}}$$

$$= \mathbf{C_0}\,\mathbf{a}^{\mathbf{jk}}\,\mathbf{f}_{\mathbf{j}}\,\mathbf{g}_{\mathbf{k}}\,\exp(-\mathbf{r}^2/2\mathbf{d}^2)$$

$$= \mathbf{C_0}\cos\boldsymbol{\theta}\,\exp(-\mathbf{r}^2/2\mathbf{d}^2) \tag{33}$$

where $\boldsymbol{\theta}$ is the angle between $\mathbf{f}_{\mathbf{j}}$ and $\mathbf{g}_{\mathbf{j}}$.

The argument can be extended to calculate the covariance between the velocity vector $\mathbf{u}^{\mathbf{k}}(Q)$ at Q and the deformation rate tensor $\mathbf{u}^{\mathbf{j}}_{\mathbf{l}}(P)$ at P by taking the covariant derivative of (32) with respect to $\mathbf{x}^{\mathbf{j}}(P)$

$$\mathbf{C}^{\mathbf{jk}}_{\mathbf{l}}\{\mathbf{u}^{\mathbf{j}}_{\mathbf{l}}(P),\mathbf{u}^{\mathbf{k}}(Q)\} = \mathbf{C_0}\,\mathbf{a}^{\mathbf{jk}}\,\mathbf{m}_{\mathbf{l}}\,(\mathbf{r}/\mathbf{d}^2)\exp(-\mathbf{r}^2/2\mathbf{d}^2) \tag{34}$$

If $\mathbf{f}_{\mathbf{j}}$, $\mathbf{g}_{\mathbf{j}}$, $\mathbf{h}_{\mathbf{j}}$ are three arbitrary unit vectors, then the scalar covariance between arbitrary components of the velocity and of the deformation rate tensor

$$\mathbf{C}^{\mathbf{jk}}_{\mathbf{l}}\{\mathbf{u}^{\mathbf{j}}_{\mathbf{l}}(P)\,\mathbf{f}_{\mathbf{j}},\mathbf{u}^{\mathbf{k}}(Q)\,\mathbf{g}_{\mathbf{k}}\,\mathbf{h}^{\mathbf{l}}\}\,\mathbf{f}_{\mathbf{j}}\,\mathbf{g}_{\mathbf{k}}\,\mathbf{h}^{\mathbf{l}}$$

$$= \mathbf{C_0}\,\mathbf{a}^{\mathbf{jk}}\,\mathbf{f}_{\mathbf{j}}\,\mathbf{g}_{\mathbf{k}}\,\mathbf{h}^{\mathbf{l}}\,\mathbf{m}_{\mathbf{l}}\,(\mathbf{r}/\mathbf{d}^2)\exp(-\mathbf{r}^2/2\mathbf{d}^2)$$

$$= \mathbf{C_0}\cos\boldsymbol{\theta}\cos\boldsymbol{\psi}\,(\mathbf{r}/\mathbf{d}^2)\exp(-\mathbf{r}^2/2\mathbf{d}^2) \tag{35}$$

where $\boldsymbol{\psi}$ is the angle between $\mathbf{h}^{\mathbf{l}}$ and the direction $\mathbf{m}_{\mathbf{l}}$ of the line PQ. This provides a basis for interpolating the deformation rate tensor $\mathbf{u}^{\mathbf{j}}_{\mathbf{l}}$ at P from observed velocities at a series of N points (Q_1, $Q_2 \ldots Q_N$).

Given that the line PQ between two points on the Earth's surface will generally be nearly horizontal, then for vertical derivatives of $\mathbf{u}^{\mathbf{j}}$, the angle $\boldsymbol{\psi}$ will be close to a right angle, and the scalar covariance in this case will tend to zero. This is just another way of stating that the vertical gradient of the velocity field is not determinable from observations of the velocity vector distributed over the surface, and that only six of the nine components of the deformation rate tensor can in general be found from such observations.

Conclusions

Repeated geodetic observations yield estimates of the particle velocity field of the deforming Earth, sampled at the network of observing points. Evaluation of such observations in terms of the spatial gradient of a continuous velocity field leads to the determination of such invariants as the rates of dilatation, rotation, and shear. These are the "estimable quantities" that best characterise the state of strain in the Earth, and for which purpose are more suited than the velocity field itself. The methods of the tensor calculus are particularly apt for the clear and unambiguous derivation of such invariants, as for many other manipulations of vector fields, and thus are very much in the spirit of the rigorous approach to geodetic problems that has been demonstrated by Erik Grafarend and his co-workers over many years.

I am grateful to Dr Hugh Bibby for his comments on the draft of this manuscript.

References

Bibby, H. M., 1973: The reduction of geodetic survey data for the detection of earth deformation. *Geophysics Division Report No. 84*, Department of Scientific & Industrial Research, Wellington.

Bibby, H. M., 1976: Crustal strain across the Marlborough Faults, New Zealand. *New Zealand Journal of Geology & Geophysics 19:* 407-425.

Bibby, H. M., 1981: Geodetically determined strain across the southern end of the Tonga-Kermadec-Hikurangi subduction zone. *Geophysical Journal of the Royal Astronomical Society 66:* 513-533.

Deakin, R. E., Collier, P. A., Leahy, F. J., 1994: Transformation of coordinates using least squares collocation. *The Australian Surveyor, March 1994:* 6-20.

El-Fiky, G. S., Kato, T., 1999: Continuous distribution of horizontal strain in the Tohoku district, Japan, predicted by least-squares collocation. *Journal of Geodynamics 27:* 213-236.

Grafarend, E., Schaffrin, B., 1974: Unbiased free net adjustment. *Survey Review 22*: 200-218.

Grafarend, E., Schaffrin, B., 1976: Equivalence of estimable quantities and invariants in geodetic networks. *Zeitschrift für Vermessungswesen 101:* 485-491.

Grafarend, E. W., 1986: Three-dimensional deformation analysis; global vector spherical harmonic and local finite element representation. *Tectonophysics 130:* 337-359.

Haines, A. J., Holt, W. E., 1993: A procedure for obtaining the complete horizontal motions within zones of distributed deformation from the inversion of strain rate data. *Journal of Geophysical Research 98:* 12,057-12,082.

Hein, G. W., Kistermann, R., 1981: Mathematical foundation of non-tectonic effects in geodetic recent crustal movement models. *Tectonophysics 71:* 315-334.

Hotine, Martin, 1969: *Mathematical Geodesy*. U. S. Department of Commerce, Washington, D.C.

Lyell, Charles, 1872: *Principles of Geology*. 11th Ed., John Murray, London.

McKay, A., 1890: On the earthquakes of September 1888, in the Amuri and Marlborough districts of the South Island. *New Zealand Geological Survey report of geological explorations 20:* 1-16.

Playfair, John, 1802: *Illustrations of the Huttonian Theory of the Earth*. Dover, New York (reprint of 1964).

Reilly, W. I., 1986: Heterogeneous strain in earth deformation. *Geophysics Division Report No. 210*, Department of Scientific & Industrial Research, Wellington.

Reilly, W. I., 1990: Horizontal crustal deformation on the Hikurangi Margin. *New Zealand Journal of Geology & Geophysics 33:* 393-400.

Terada, T., Miyabe, N., 1929: Deformation of the Earth Crust in Kwansai Districts and its relation to the Orographic Feature. *Bulletin Earthquake Research Institute 7:* 223-239.

Wellman, H. W., 1955: New Zealand Quaternary tectonics. *Geologische Rundschau 43:* 248-257.

Xu, P., Grafarend, E., 1995: A multi-objective second-order optimal design for deforming networks. *Geophysical Journal International 120:* 577-589.

Xu, P., Grafarend, E., 1996a: Probability distribution of eigenspectra and eigendirections of a two-dimensional, symmetric rank two random tensor. *Journal of Geodesy 70:* 419-430.

Xu, P., Grafarend, E., 1996b: Statistics and geometry of the eigenspectra of thee-dimensional second-rank symmetric random tensors. *Geophysical Journal International 127:* 744-756.