The Strange Behavior of Asymptotic Series in Mathematics, Celestial Mechanics and Physical Geodesy

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In the beginning ... there was Poincaré
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1 Introduction

Yes, but not quite: Heinrich Bruns (1884) (*our* Bruns!) made the race, but Poincaré (1890, 1892–1899) went much farther and deeper. He proved that "most" series used in celestial mechanics were divergent but nevertheless perfectly useful. In fact, he recognized that *mathematical* convergence or divergence may be quite irrelevant for *numerical* convergence: the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots$$

is convergent but numerically practically useless because the convergence is so slow. On the other hand, the series

$$\frac{e^x}{x}\left(1 + \frac{1!}{x} + \frac{2!}{x^2} + \dots\right)$$

is divergent but numerically superbly useful as we shall see in sec. 2 of this paper.
Such divergent but numerically useful series have been called *asymptotic series* by Poincaré (1892–1899, beginning of vol. 2), a terminology generally accepted by numerical mathematicians (cf. Erdélyi 1956, Press et al 1992, p. 167).
What Poincaré (1890), foreshadowed by Bruns (1884), showed was that many (or even most) series in celestial mechanics were such asymptotic "practically convergent" series.
In geodesy, we have similar series of doubtful moral behavior: the spherical–harmonic series of the geopotential at the earth's surface and Molodensky's series for the solution of the geodetic boundary value problem.
Concerning the spherical–harmonic series, every scientist aspiring to fame in physical geodesy was bound to give a wrong proof of convergence or divergence. Hopfner (1933) proved convergence; his proof was wrong. Baeschlin (1948) proved divergence; his proof was wrong. Moritz (1961) proved that the question of convergence vs. divergence was meaningless; his proof was wrong. In fact, it was basically right: an arbitrarily small change of the attracting mass (the earth) by a sand grain can change convergence into divergence. I was wrong in taking for granted that, just as complex functions in the plane have a "circle of convergence", their three–dimensional analogues, spherical harmonic functions, had a "sphere of convergence". This mistake was pointed out by a beautiful counterexample by Krarup (1969, pp. 47–49). Whereas Moritz has shown the "direct" problem, that convergence can be readily changed into divergence (the "sandgrain effect"), Krarup proved the much more difficult "inverse problem", that divergence can be changed into convergence by *an arbitrarily small change* of the geopotential.

In analogy to the correspondent theorem for the complex functions in the plane, due to C. Runge (the "Runge" from the well–known "Runge–Kutta method"), he called the three–dimensional geodetic theorem, found by him independently, with his characteristic modesty, "Runge theorem". In my book (Moritz 1980) I called it at least "Runge–Krarup theorem" because of the enormous intellectual work Krarup put in. However, the Devil, irritated by human attempts to trespass mathematically into his empire, the earth's interior, was active also here: it turned out that this theorem was known already to G. Szegö around 1925(!), cf. (Frank–Mises 1930 pp. 760–762).
Under special assumptions, convergence can of course, be proved (Moritz 1980, p. 53), Balmino (1994) and Grafarend and Engels (1994).
For details, the reader may consult (Moritz 1980, sec. 6–8) or the slightly more humorous account (Moritz 1978).
So what? The question of the convergence or divergence of spherical harmonics at the earth's surface is perfectly meaningless. Practically, we anyway operate with finite bestfitting spherical harmonic polynomials, whether to degree and order 30, 360, 1000 or 3600 (or, if you can pay for it and are very patient, 3,600.000). These are exactly bestfitting polynomials in the Runge–Szegö–Krarup sense! The author gives his blessing and hopes (probably in vain) that the Devil will be impressed enough to step off this theater to look for more profitable problems ...
Concerning the convergence of Molodensky's series it is similar: all convergence proofs known to the author (including his own in Moritz 1980, sec. 47) are probably wrong. Don't waste your time, however, to look for errors: it is also an asymptotic series and you have the pleasant alternative: either the first 3, 4 or 5 terms are sufficient, or look for another job; the following terms are anyway pure noise because the effect of measuring errors rapidly increases and wipes out the gravitational "signal".

2 A Computer Study of a Mathematical Asymptotic Series

A simple but typical example is the well–known *exponential integral* defined by

$$Ei(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt \qquad (x > 0) \quad . \tag{1}$$

This is a standard mathematical function contained e.g., in the programming language MATHEMATICA; it can be called there by the name $ExpIntegralEi[x]$, cf. Fig. 1.
The standard way of computing it if MATHEMATICA is not available, is the source code in C as given by (Press et al, 1992, sec. 6.3, p. 225), function "ei(x)". Since this book is the standard work for modern mathematical computation, it is available not only in C, but also in FORTRAN, PASCAL and even BASIC. Therefore, the interested reader is simply referred to this book.
It turns out that for small x one uses an ordinary convergent power series:

$$Ei(x) = \gamma + \ln x + \frac{x}{1 \cdot 1!} + \frac{x^2}{2 \cdot 2!} + \dots \tag{2}$$

where $\gamma = 0.5772\dots$ is Euler's constant. For $x > 16.62\dots$, however, the convergence of this series becomes too slow, and one uses the asymptotic series

$$Ei(x) \doteq \frac{e^x}{x}\left(1 + \frac{1!}{x} + \frac{2!}{x^2} + \dots\right) \quad . \tag{3}$$

To compare the "true" (or rather highly accurate) function $ExpIntegralEi[x]$ in MATHEMATICA with our asymptotic series (3), one could program this asymptotic series in MATHEMATICA. Since a beautiful and fast source code for (3) is already contained in the C Program "$ei(x)$"

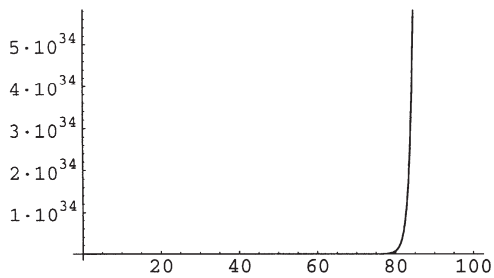
by Press et al., as mentioned above, we can use it too and convert it to MATHEMATICA by the auxiliary tool MATHLINK (Wolfram 1996, sec. 2.12). This is extremely simple: we add the header file "mathlink.h" in our source code "eiconv.c" (Fig. 2), and compile it (this is a bit technical) to get the exe–file "eiconv" to be installed in the MATHEMATICA program of Fig. 1 as shown there (all relevant programs may be obtained from moritz@phgg.tu-graz.ac.at).
The result is the deviation

$$err[n, x] = ExpIntegralEi[x] - expIntAsy[n, x] \tag{4}$$

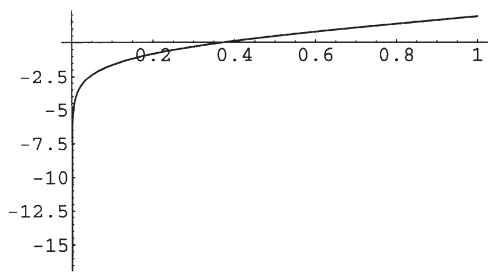
of our home–made function (3) truncated after the n–th term. It is thus a function of the usual variable x and the truncation value n. The deviation is with respect to the standard MATHEMATICA function $ExpIntegralEi[x]$.

Mathematica Program ExpIntConvergence.nb

```
In[1]:= Install["eiconv"]    (* C exe-file from Mathlink *)

Out[1]= LinkObject [eiconv , 2, 2]

In[2]:= LinkPatterns[%]

Out[2]= {expIntAsy [n_Integer , x_Real ]}

In[3]:= ? expIntAsy

        expIntAsy [n,x] approximates the Mathematica function ExpIntegralEi [x] by the sum
          of the first n terms of an asymptotic series .

In[4]:= (* Mathematica function *)

In[5]:= ? ExpIntegralEi

        ExpIntegralEi [z] gives the exponential integral function Ei (z).

In[6]:= Plot[ExpIntegralEi[x], {x, 0, 100}]
```

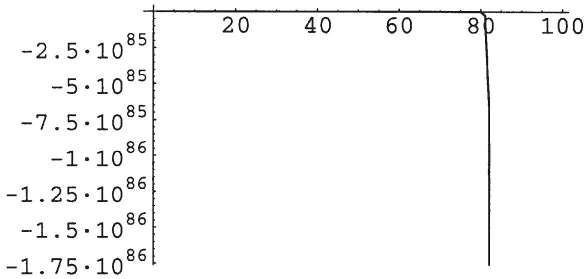



```
Out[6]= - Graphics -

In[7]:= Plot[ExpIntegralEi[x], {x, 0, 1}]
```

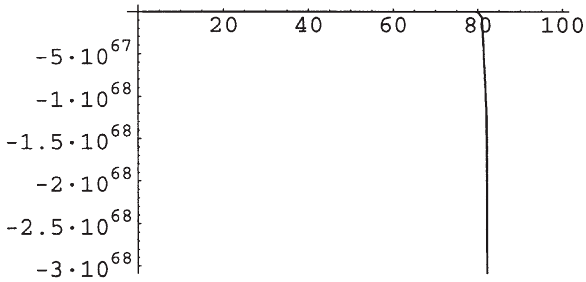



```
Out[7]= - Graphics -

In[8]:= err[n_, x_] := ExpIntegralEi[x] - expIntAsy[n, x]
```

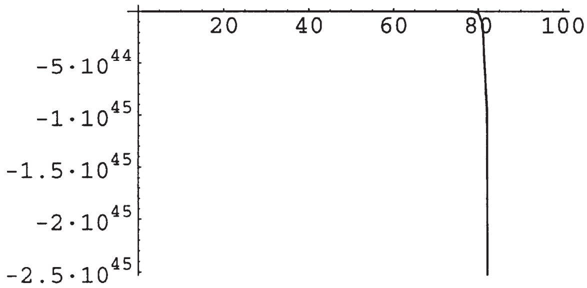
In[9]:= `ListPlot[Table[err[n, 3.], {n, 2, 100}], PlotJoined -> True];`



In[10]:= `ListPlot[Table[err[n, 5.], {n, 2, 100}], PlotJoined -> True];`



In[11]:= `ListPlot[Table[err[n, 10.], {n, 2, 100}], PlotJoined -> True];`



In[12]:= `relativeErr[n_, x_] := err[n, x] / ExpIntegralEi[x]`

In[13]:= `{relativeErr[5, 20.], relativeErr[10, 20.], relativeErr[15, 20.]}`

Out[13]= $\{0.0000174303,\ 5.49235\times10^{-7},\ 9.37426\times10^{-8}\}$

In[14]:= `{relativeErr[17, 20.], relativeErr[18, 20.], relativeErr[19, 20.] }`

Out[14]= $\{3.78099\times10^{-8},\ 1.4681\times10^{-8},\ -7.29146\times10^{-9}\}$

In[15]:= `Uninstall["eiconv"]`

Out[15]= eiconv

```
// Source Code eiconv.c
// Study of the convergence of the asymptotic series
// for the exponential integral Ei(x) -

#include <math.h>
#include "mathlink.h"

double eias(int n, double x)  // in C : Function eias(n,x)
{                             // in MATHEMATICA : ExpIntAsy[n,x]
   int k;                     // Asymptotic Series to power n
   double prev, sum, term;

   if (x <= 0) return 0;
   else
   {
     sum=0.0;
     term=1.0;
     for(k=1;k<=n;k++)
     {
       prev=term;
       term *= k/x;
       sum += term;
     }
     return exp(x)*(1.0+sum)/x;
   }
}
int main(argc, argv)
   int argc; char* argv[];
{
   return MLMain(argc, argv);
}
```

Figure 2: Source code

To repeat, $err[n, x]$ is the deviation of the (divergent!) asymptotic series (3) to order n from the "true" function $Ei(x)$.
Of course, I expected such a behavior, but nevertheless the extremeness of the result was shocking, and I could hardly believe my eyes. Even with three terms of the asymptotic series ($n = 3$) one gets an excellent approximation and with $n = 5, 10, 15, 17, 18, 19$, we get phantastic accuracies on the order of 10^{-18}. And these we get with a few terms of a divergent series!

3 Celestial Mechanics and Geodesy

It is not surprising that Poincaré, when he first recognized these facts and applied them to the series of celestial mechanics, was overwhelmed with the joy of discovery. His work, and still less that by Bruns, was hardly understood for more than half a century. Only with the advent of fast computers was one able to render visible the phantastic pictures of modern "general nonlinear dynamics", now popularly called "chaos theory", which Poincaré had in the back of his mind and, as he said, to his regret was unable to draw.
Another fact common to Poincaré's series and spherical–harmonic series was that regular (stable) and "chaotic" (unstable) trajectories, convergent and divergent series are arbitrarily close to each other. In geodesy this is the *Runge–Szegö–Krarup (RSK) theorem* mentioned at the very beginning of this paper. (Other people could be included in this list, cf. Moritz 1980, p. 74, but questions of priority are usually rather questionable ...) In chaotic dynamics it is the *KAM*

(Kolmogorov–Arnold–Moser) theorem.
The RSK theorem, in a rather simplified form (Moritz 1980, p. 67) may be stated:

> Let K be a compact set and Γ and Ω open sets in R^3, such that their boundaries are homeomorphic to a sphere and such that $K \subset \Gamma$ and $\Gamma \subset \Omega$. If the function ϕ is harmonic in Γ and if $\epsilon > 0$ is arbitrarily small, then there exists a function ψ, harmonic in Ω, such that
>
> $$|\phi - \psi| < \epsilon \tag{5}$$
>
> uniformly on K.

The KAM theorem requires considerable knowledge from number theory. A relatively accessible presentation can be found in (Schuster 1988, p. 191). The criterion of stability or instability of trajectories is

$$\left|\frac{\omega_1}{\omega_2} - \frac{m}{s}\right| > \frac{k(\epsilon)}{s^{2.5}} \qquad (k(\epsilon \to 0) \to 0) \quad . \tag{6}$$

What does this mean? The state space of a one–dimensional oscillation (frequency ω_1) is a circle, and the state space of two regular oscillators is the topological product circle $\times$ circle, which is a torus. If the motion is perturbed, some tori will be conserved (stable motion), but some tori will break up (chaotic motion). This depends on "how irrational" the ratio ω_1/ω_2 is, how well or how poorly ω_2/ω_1 can be approximated by a rational number (m and s are integers). This is a difficult number–theoretic problem solved by eq. (6). The interested reader can work himself through the enormous literature; for the present purpose it is better to insert a picture (Fig. 3)

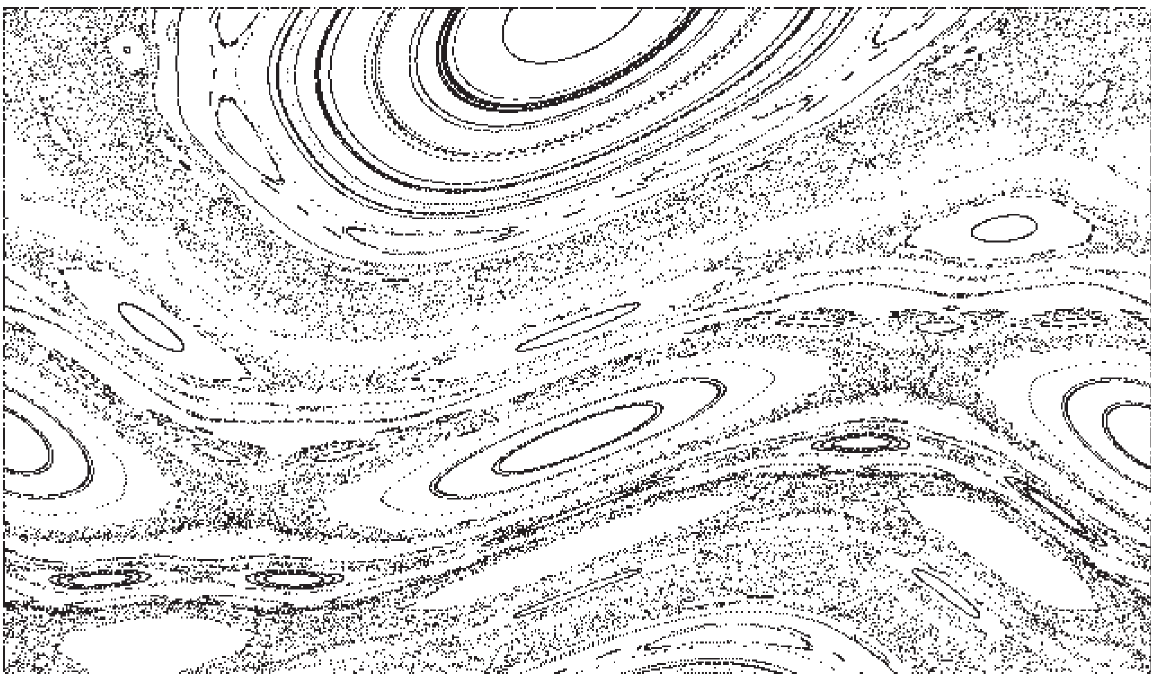


Figure 3: Poincaré

which shows the "Poincaré section" of a certain set of three–dimensional trajectories: every point corresponds to a trajectory, every "elliptic island" to a torus that has been preserved.

There are many books on chaos by computers, but to me the book (Herrmann 1994) is still unsurpassed. (Fig. 3 was computed by the author using MATHEMATICA and MATHLINK to C, but has been inspired by Herrmann's algorithms.)
From the very beginning, the problem of the present paper has fascinated me (Moritz 1969). My guess that there are relations between astronomic and geodetic series, both being asymptotic series, later proved to be correct. The work of the mathematician Heinrich Bruns both in astronomy and geodesy was striking. My early guess on Runge–type behavior of spherical harmonics (Moritz 1961) was intuitively correct and mathematically wrong, but it inspired pioneering work by Krarup (1969). When I studied the difficult but fascinating book (Sternberg 1969), I was struck by the use of "hard inverse problems" of non–linear functional analysis in the KAM problem, especially by a method of Nash which the famous mathematician Lars Hörmander later (1976) used in the first partially successful mathematical attack of existence and uniqueness of Molodensky's problem! In fact, Hörmander wsa the second eminent mathematician in this century who did significant work in geodesy. The first was Poincaré: Throughout his life (1854–1912), Henri Poincaré never ceased to work on problems of astronomy and geodesy. (In his last years, he was even the French Chief Delegate to the International Geodetic Association.)
I still I firmly believe that there is a deep mathematical relationship between nonlinear dynamic systems ("chaos theory") and geodetic problems which I, however, was never able to penetrate really deeply. We would badly need a new Poincaré. What about the young geniuses of our 60 years young Professor Erik Grafarend?

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