Simplest solutions of Clairaut's equation and the Earth's density model

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Introduction

Starting from the first investigations of the Earth's density distribution some remarkable and simple density laws were constructed by Legendre, Laplace, G.Darvin, Roche, etc. These laws have a spherically symmetric density distribution with the volume density $\rho(\ell)$ that depends on the radial distance ℓ. At the Geodetic Week97 (Berlin,1977) the author had several interesting discussions with Prof. E. Grafarend about the exponential nature of the flattening distribution according to the Clairaut's equation. Maybe these meetings and discussions yielded now the presented consideration of the famous classic law of density in view of their mathematical descriptions. As a result, the latter is the main goal of this paper. On the other hand, we shall try to illustrate some our results by the numerical investigations on the ground of the fundamental constants of geodesy and astronomy together with global data of the seismic tomography of the Earth's interior.

1. Some basic relationships

In view of a mathematical formulation the traditional representation of the Earth's radial density can be treated as a function $\rho(\ell)$ (continuos or piecewise in form of shells) of one variable ℓ only, which is defined on the finite segment $(0 \le \ell \le R)$ only if we assume that the figure of the planet is spherical, (R is the mean Earth's radius (R = 6371 km)). It is well-known also (Moritz, 1990) that in this case the gravitational potential V is equal to the gravity potential W, since we use such simplest approximation of the ellipsoid by the sphere when the flattening $f = 0$.
First of all our initial (observed) information will be the Earth's mass M and the mean moment of inertia I. For latter use we shall write some well-known formulae within the sphere of the radius ℓ (the part of the Earth's mass which is restricted by this radius) for the *mass*

$$M(\ell) = 4\pi \int_0^{\ell} \rho(x) x^2 dx \quad , \tag{1}$$

where dx is the element of a line and the *mean density* $D(\ell)$:

$$D(\ell) = \frac{3}{4 \cdot \pi \cdot \ell^3} M(\ell) \quad . \tag{2}$$

The value $D(\ell)$ in the form of (2) leads to the following representations

$$g(\ell) = \frac{4 \cdot \pi \cdot G}{3} \ell \cdot D(\ell) \quad \Leftrightarrow \quad g(\ell) = \frac{GM}{\ell^2} \quad , \tag{3}$$

of the *gravity* $g(\ell)$ inside the Earth, where G=6.673·10^{-8} [cm^{3}s^{2}g^{-1}] is the gravitational constant. The *mean moment of inertia* is

$$I(\ell) = \frac{8\pi}{3}\int_0^{\ell} \rho(x)x^4 dx \quad . \tag{4}$$

We shall use also the seismic velocities V_p and V_s in the form of the function

$$\Phi = \Phi(\ell) = V(\ell)_P^2 - \frac{4}{3}V(\ell)_S^2 \quad , \tag{5}$$

by applying their grid values in accordance with (Dziewonski and Anderson, 1981), which practically represents the results of seismic tomography of the Earth interior.

2. The simplest solutions of Clairaut's equation

Now we recollect that the famous oldest hypothesis for the Earth's density distribution were proposed after solutions of Clairaut's equation for the flattening inside the Earth (see, for instance, Bullen, 1975; Moritz, 1990).
There exist (Bullen, 1975) three famous solutions of this equation for the density ρ. First one is *Legandre - Laplace law*

$$\rho(x) = \rho_0 \frac{\sin(\beta x)}{\beta x} = \rho_0 \frac{\exp(\sqrt{-1}\beta x) - \exp(-\sqrt{-1}\beta x)}{2\sqrt{-1}\beta x}, \qquad \beta\text{=const}, \tag{6}$$

where we apply the dimensionless „radius-vector"

$$x = \frac{\ell}{R} \quad , \tag{7}$$

regarding to R; ρ_0 = const and may be considered here as the density at the origin.
The second one is *Roche's law*

$$\rho(x) = \rho_0 (1 - Kx^2) = a + bx^2 \quad , \tag{8}$$

where

$$a = \rho_0 > 0 \quad \text{and} \quad b = \rho_0 K < 0 \quad . \tag{9}$$

Note now that Taylor series expansion of (6) (disregarding other higher powers of x) in view of mathematics leads to the similar expression:

$$\rho(x) = \rho_0 \left(1 - \frac{\beta^2}{6}x^2\right) \quad . \tag{10}$$

The third one is *G. Darwin law*

$$\rho(x) = C \cdot x^{-n} \quad , \tag{11}$$

where C is a constant. His solution involves an „assumption of the form for the law of the internal density of the planet and subsequent determination of the law of compressibility„ (Darwin, 1884). Clearly, the expression (11) represents the density with a singularity at the origin. G. Darwin noted already that case $n = 0$ for the model (11) corresponds to the case of homogeneous density; for $n = 3$ the Earth's mass M will become infinite; for $n > 3$ the mass M must be assumed to be negative. As a result, we get the inequality $0 < n < 3$ which agrees with the determination $n = 1.011$ (Bullen, 1975). Thus the expression (11) represents a power function.

3. Williamson-Adams equation

The density ρ may fulfil the so-called Williamson-Adams equation for each shell of the stratified Earth under the following assumptions: the Earth is globally in hydrostatic equilibrium; chemical composition and phase transformation are homogeneous in every shell; the temperature is adiabatic in each shell. Thus, if we have the observable seismic velocity (5), in view of the gravitational (3) and hydrostatic relationships

$$grad p(\ell) = \rho(\ell) \cdot grad V(\ell) \qquad \Rightarrow \qquad \frac{dp(\ell)}{d\ell} = -\rho(\ell) \cdot g(\ell) \quad , \tag{12}$$

finally the Williamson-Adams equation can be written as

$$\frac{d \ln \rho(\ell)}{d\ell} = -\frac{g(\ell)}{\Phi(\ell)} \quad , \tag{13}$$

where p is the pressure inside the Earth. Thus (13) is a formula to derive the radial density distribution from the seismic velocity data, fulfilled under the assumptions listed above.

In order to use (13) we must first try to solve this equation and to express the observed seismic data by a suitable function of depth, separating the Earth into convenient shells. Traditionally we shall assume that the separation into shells has to be choice at those spheres, where discontinuities in the parameter Φ or in its derivative can be observed.

It is evident that the formal solution of (13) may be obtained after the integration of Williamson-Adams equation. The result is

$$\rho(\ell) = \rho_0 \exp\left(-\int_0^{\ell} \frac{g(x)}{\Phi(x)} dx \right) \quad , \tag{14}$$

and we get the functional dependence for radial density as an exponential function. The right hand side of the expression (14) is unknown. For this reason, we shall apply instead of (14) the simplest approximating function

$$\rho(\ell) = \rho_0 \exp\left(-\gamma^2 x^2\right) \quad , \qquad \gamma = \text{const} \quad , \tag{15}$$

where the power 2 is the lowest power for which we may get a non-zero value Φ at the origin. Taylor expansion of (15) leads again to the Roche's model

$$\rho(x) = \rho_0 \left(1 - \gamma^2 x^2\right) = a + bx^2 \quad , \tag{16}$$

if we disregard other higher powers of x.

4. Poisson's equation

The density ρ must fulfil the Poisson's equation for the gravity potential $W = V$ of the Earth. Using the spherical coordinates after simple manipulations we get for a radial layered Earth, that is for $\rho = \rho(\ell)$, in spherical approximation

$$-\Delta V = 4\pi G\rho = \frac{dg}{d\ell} + \frac{2g}{\ell} = M[g] \quad . \tag{17}$$

The operator

$$M[\ ] = \frac{d}{d\ell} + \frac{2}{\ell} \quad , \tag{18}$$

is well-known in geodesy as Molodensky operator (see, for instance, Neyman, 1979) and it was introduced first for the basic boundary problem of geodesy in the next form

$$M[T] = \frac{dT}{dr} + \frac{2T}{r} = -\Delta g \quad , \tag{19}$$

where T is the anomalous potential, Δg is the gravity anomaly, r is the radius-vector of an *external* point (the parameter ℓ represents the radius-vector of an *internal* point).
The expression (19) is used for the determination of T on the ground of known gravity anomalies. In the expression (17) we have as unknown values both the density and gravity inside the Earth. Nevertheless, if the gravity g is known we get a simple rule for the computation of radial density profile in accordance with Poisson's equation. If gravity is known approximately, we get one of the most important additional information for a stable creation of the density models. So, one of our next steps will connected with the gravity distribution inside the Earth.

5. Some remarks on the regular Darwin's law

If we want to avoid a singularity at the origin in (11), this function may be transform to the expression

$$\rho(x) = C \cdot x^{-f(x)} = C \cdot \exp(-f(x)\ln x) \quad , \tag{20}$$

where $f(x)$ is any suitable function. Such a function can represent a regular form of Darwin's law without a singularity at the origin (Marchenko and Lelgemann, 1997). The expression (20) may be considered as an exponential function.
Taking into account the relationships (14), (15) we may try to insert into (20) another function $f(x) = F(x)/\ln(x)$ (in particular, $F(x) = \gamma^2 x^2$) that leads on the whole again to

$$\rho(x) = C \cdot \exp(-F(x)) \quad , \tag{21}$$

the solution (14) of Williamson-Adams equation and to the considered case (15) in particular.
Note that the direct integration of (20) is impossible for mass (1), for moment of inertia (4), etc. The expression (21) in the form of (15):

$$\rho(\ell) = \rho_0 \exp\left(-\gamma^2 x^2\right) \quad , \tag{22}$$

admits according to (1) and (4) the next remarkable expressions for the mass

$$M(\ell)=\frac{4\pi\rho_0 R^3}{\gamma^2}\left[\frac{\sqrt{\pi}\cdot\operatorname{erf}(\gamma\cdot x)}{4\gamma}-\frac{x}{2\rho_0}\rho(\ell)\right] \quad , \tag{23}$$

and for the mean moment of inertia

$$I(\ell)=\frac{8\pi\rho_0 R^5}{3\gamma^4}\left[\frac{3\sqrt{\pi}\cdot\operatorname{erf}(\sqrt{\gamma}\cdot x)}{8\gamma}-\frac{x}{4\rho_0}\rho(\ell)\cdot\left(2\gamma^2x^2+3\right)\right]=\frac{R^2}{\gamma^2}\left[M(\ell)-\frac{4\pi\ell^3}{3}\rho(\ell)\right] \quad , \tag{24}$$

where erf(z) is the integral of the Gaussian distribution from 0 to z or the probability integral with the density distribution according to (22).
Thus we come to a remarkable result: one of solutions of Williamson-Adams equation in the regular Darwin's form is nothing else but the famous Gaussian distribution, which may be approximated by the Roche's model, represented the possible solution of the Clairaut's equation.
In spite of the difference between considered above various expressions for density, we come to their exponential nature on the whole. Roche's model we may treat now as a truncated Taylor series of them.

6. Saigey's theorem and the Roche's model

According to the so-called Saigey theorem the gravity $g(\ell)$ has a maximum inside the Earth. We shall use the Roche's model as a basic tool for next study. So that, it is necessary to find such a point(s), where the radial derivative $\frac{dg(\ell)}{d\ell}$ is equal to zero. As a result, for the stationary point(s) we get the well-known expression

$$\frac{dg(\ell)}{d\ell}=\frac{4\pi G}{3}\left(D(\ell)+\ell\frac{dD(\ell)}{d\ell}\right)=4\pi G\left(\rho(\ell)-\frac{2}{3}D(\ell)\right)=0 \quad \Rightarrow \quad \rho(\ell)=\frac{2}{3}D(\ell) \quad . \tag{25}$$

Now applying the Roche's model (8) or (16) to (25) we get immediately

$$D(\ell)=a+\frac{3\cdot b}{5}\left(\frac{\ell}{R}\right)^2=a+\frac{3\cdot b}{5}x^2 \quad , \tag{26}$$

and the solution of (25) for the parameter x

$$x=\frac{\ell}{R}=\frac{\sqrt{5}\cdot\sqrt{a}}{3\cdot\sqrt{-b}} \quad . \tag{27}$$

Note that this root of (25) corresponds to (9) and $a>0$. In this case the sign of b must be negative: $b<0$. Moreover applying such dimensionless $x\in[0,1]$ and (27) the following inequality

$$\frac{a}{-b}\leq\frac{9}{5} \quad , \tag{28}$$

may be found for the coefficients of the Roche's model. Note only that the sign of the second radial derivative follows from the coefficient *b*. For this reason $\frac{d^2 g(\ell)}{d\ell^2} < 0$ in the point (27) and our function $g(\ell)$ has a maximum only at this point.

7. Piecewise Roche's model

If a suitable stratification of the Earth leads to its division into *m* shells, first we shall represent the density distribution by own Roche's model within every shell separately

$$\rho_i(x) = a_i + b_i x^2 \quad , \qquad i = 1,2,\ldots m \,. \tag{29}$$

Inserting (29) into the expressions (1), (2), and (4) we get finally the recurrence formulae for the mass, the mean density and mean moment of inertia, respectively:

$$M_{1,m}(\ell) = M_{1,m-1}(\ell_{m-1}) + [M_m(\ell) - M_m(\ell_{m-1})] \quad , \qquad (\ell_{m-1} \le \ell \le R) \,, \tag{30}$$

$$D_{1,m}(\ell) = \left(\frac{\ell_{m-1}}{\ell}\right)^3 D_{1,m-1}(\ell_{m-1}) + \left[D_m(\ell) - \left(\frac{\ell_{m-1}}{\ell}\right)^3 D_m(\ell_{m-1})\right] \,, \tag{31}$$

$$I_{1,m}(\ell) = I_{1,m-1}(\ell_{m-1}) + [I_m(\ell) - I_m(\ell_{m-1})] \quad , \tag{32}$$

where for the piecewise Roche's model

$$M_i(\ell) = \frac{4\pi}{3}\ell^3\left[a_i + \frac{3}{5}b_i x^2\right] \quad , \qquad M_{1,1}(\ell) = M_1(\ell) \,, \tag{33}$$

$$D_i(\ell) = \left[a_i + \frac{3}{5}b_i x^2\right] \quad , \qquad D_{1,1}(\ell) = D_1(\ell) \,, \tag{34}$$

$$I_i(\ell) = \frac{8\pi}{3}\ell^5\left[\frac{a_i}{5} + \frac{b_i}{7}x^2\right] \quad , \qquad I_{1,1}(\ell) = I_1(\ell) \,, \tag{35}$$

starting from the first shell $(0 \le \ell \le \ell_1)$.In these formulae ℓ_j (j =1,2,…m-1) are the fixed radius-vectors, where jumps of radial density are presupposed. The recurrence formulae for gravity is based on the expressions (3) and (31):

$$g_{1,m}(\ell) = \frac{4 \cdot \pi \cdot G}{3}\ell \cdot D_{1,m}(\ell) \quad , \qquad (\ell_{m-1} \le \ell \le R) \,, \tag{36}$$

again starting from the first shell.
For the recurrence formulae of the seismic parameter Φ and it jumps first we shall find

$$\frac{d \ln \rho_i(\ell)}{d\ell} = \frac{2 b_i \ell}{R^2 \rho_i(\ell)} \quad . \tag{37}$$

Further by applying the Williamson-Adams equation (13) for the piecewise model (29) in view of (38) after some manipulations we get

$$\Phi_{1,m}(\ell) = -\frac{2\cdot\pi\cdot G\cdot R^2}{3\cdot b_m}\rho_m(\ell)\cdot D_{1,m}(\ell) \quad , \qquad (\ell_{m-1} \le \ell \le R) \ , \tag{38}$$

$$\Phi_i(\ell) = -\frac{2\cdot\pi\cdot G\cdot R^2}{3\cdot b_i}\rho_i(\ell)\cdot D_i(\ell) \quad , \qquad \Phi_{1,1}(\ell) = \Phi_1(\ell)\ , \quad (0 \le \ell \le \ell_1)\ . \tag{39}$$

By the definition (5) the parameter Φ must be always positive and we shall consider the ratio

$$\frac{\Phi_{1,i}(\ell_{j-1})}{\Phi_{1,i-1}(\ell_{j-1})} = \frac{b_i}{b_{i-1}}\frac{\rho_i(\ell_{j-1})}{\rho_{i-1}(\ell_{j-1})} > 0 \quad , \tag{40}$$

which must be positive for each boundary of two shells. From this inequality together with (9), (28) (for one shell) we come to a remarkable results: *all coefficients a_i will be positive and all coefficients b_i will be negative for the piecewise Roche's model of density.*
Finally we may compute the seismic jump of Φ at the j - boundary

$$\Delta\Phi = \Delta\Phi_{i,i+1} = \Phi_i(\ell_j) - \Phi_{i+1}(\ell_j) = -\frac{2\cdot\pi\cdot G\cdot R^2}{3}D_{1,i}(\ell_j)\left[\frac{a_i}{b_i} - \frac{a_{i+1}}{b_{i+1}}\right] \ . \tag{41}$$

This formula may use as the additional condition between the coefficients of every shell, because the left hand side of (41) is known from seismic data.

8. First iteration for piecewise density distribution

Now we recollect (see, for instance, Moritz, 1990) that „any global density law must satisfy three basic conditions:

1. It must provide the correct total mass or, equivalently, the mean density;
2. It must give the value for the mean moment of inertia;
3. It must reproduce the density at the base of continental layers, which may be taken as about 3.2 to 3.3 g/cm^3, e.g. the conventional density just below Mohorovichich discontinuity much used in isostasy ρ_1 =3.27 g/cm^3 „.

These three conditions may lead to the construction of the continuos radial density distribution. First two conditions can apply for the determination of the continuos Roche's model. In this case we get a remarkable expression for the coefficient b of such a model

$$\left.\begin{aligned} b &= \frac{5}{3}[D - \rho_0] \\ a &= \rho_0 \end{aligned}\right\} \quad . \tag{42}$$

Nevertheless, we may add according to (39) the additional condition for density at the origin, which will depend on the observe value of Φ:

$$\Phi(0) = -\frac{2\cdot\pi\cdot G\cdot R^2}{3\cdot b_1}a_1^2 \quad . \tag{43}$$

and use then forth conditions for determinations of the coefficients of two (m=2) models (29). We presuppose also that the first model will describe the density on the interval [0, 3480km] and the second model is valid for the interval [3480km, 6371km]. Now according to three condition listed above we get

$$a_1\left(\frac{\ell_1}{R}\right)^3 + a_2\left[1-\left(\frac{\ell_1}{R}\right)^3\right] + \frac{3b_1}{5}\left(\frac{\ell_1}{R}\right)^5 + \frac{3b_2}{5}\left[1-\left(\frac{\ell_1}{R}\right)^5\right] = D \quad , \tag{44}$$

$$\frac{2}{D}\left\{\frac{a_1}{5}\left(\frac{\ell_1}{R}\right)^5 + \frac{a_2}{5}\left[1-\left(\frac{\ell_1}{R}\right)^5\right] + \frac{b_1}{7}\left(\frac{\ell_1}{R}\right)^7 + \frac{b_2}{7}\left[1-\left(\frac{\ell_1}{R}\right)^7\right]\right\} = I_d \quad , \tag{45}$$

$$a_2 + b_2 = \rho_s \quad , \tag{46}$$

where $I_d = I/MR^2$ is the dimensionless Earth's moment of inertia, ρ_s is the surface density, $D=D(R)$ is the Earth's mean density. Because the equation (43) is non-linear, on the first step we shall add the following linear equation

$$a_1 = \rho_0 \quad , \tag{47}$$

and will solve this system (44)-(47) with respect to the density (47) at the origin. On the second step the non-linear equation (43) may be solved numerically in a traditional way. After iterations, we can get these four coefficients and compute now the basic jump of the Earth's density. To our own surprise such solution of the equations (43)-(46) together with the seismic data alone provided finally (in this step) the density jump at the core/mantle boundary $\Delta\rho = 4.454$ g/cm^3, the density at the centre mass of the Earth ρ =12.953 g/cm^3, and the remarkable restoring of the main behaviour of the gravity distribution according to PREM model.

For this reason after the creation of these two models we may continue such approach for the further division of the Earth and determination of the set of the models (29) which should be agreed with the whole initial information about the seismic data. Thus on the first step we may get a preliminary solution for every shell separately by the „golden section" technique (in view of the necessity of the Earth's stratification and solution of the non-linear equation (43)). The second step consists of the readjustment of these independent pieces of density to the piecewise density distribution which agrees with the set of the seismic data and other additional information about fundamental constants.

Regarding the discontinuities in the seismic velocities as sampled for PREM, we are led to the following separation into shells (Table 1) as a particular case. Based on this separation a mathematical description of the Earth's density based on the piecewise Roche's model was derived and presented in Table1. This model (see, Figure 1) can be used further for an improvement as a starting model using another – exponential solution of Williamson-Adams equation. Figure 1 reflects its good agreement with the PREM-density model, with the exception of the crust shells: we try to create on the final step a „geodetic version" of the Earth density profile with surface density ρ_s=2.67 g/cm^3.

Table 1. Piecewise Roche's density model (m=7)

Shell	a_i	b_i	ℓ_j, km	Density jump
1	13.061	-8.891	1221.5	0.558
2	12.483	-8.343	3480.0	4.392
3	6.370	-2.574	5701.0	0.314
4	6.058	-2.577	5971.0	0.228
5	5.784	-2.524	6151.0	0.080
6	6.057	-2.903	6346.6	0.476
7	6.622	-3.952		

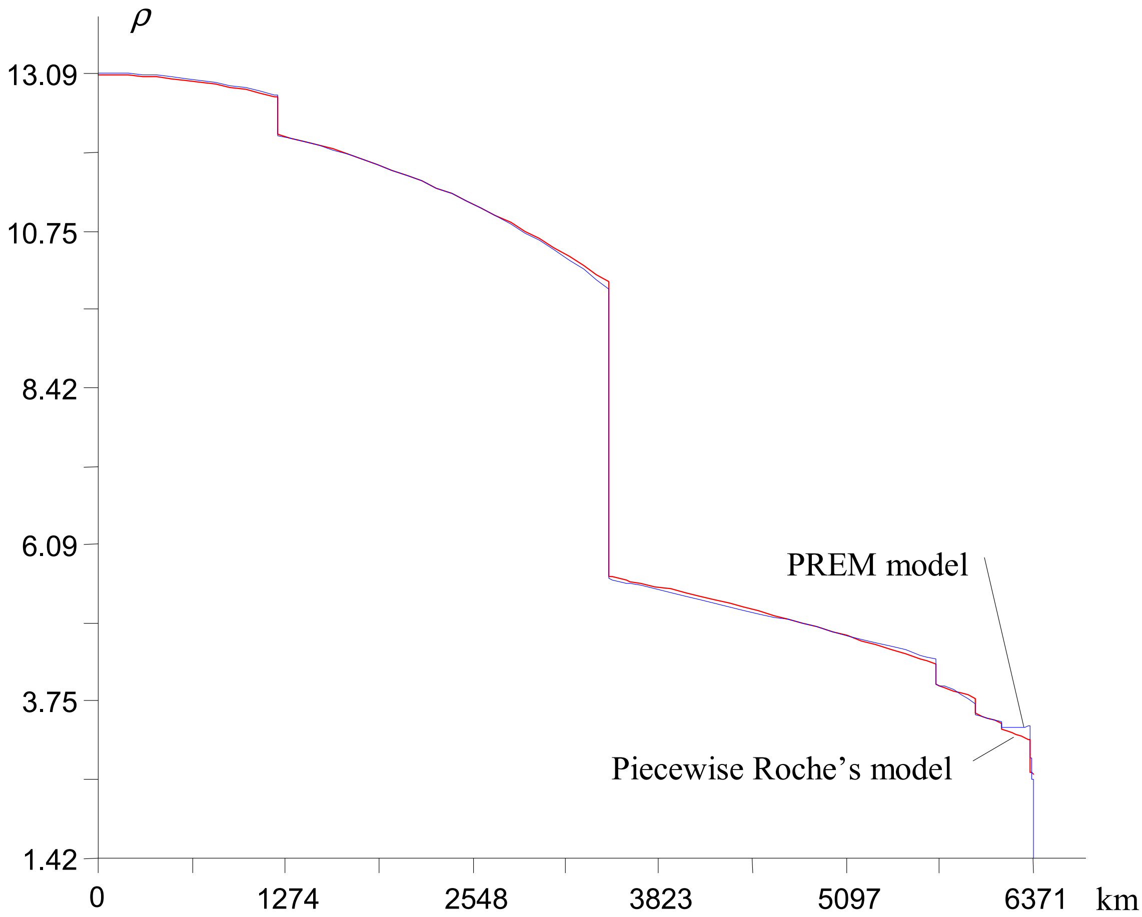

Figure 1. Comparison of the PREM-density ρ [g/cm^3], with the constructed density model.

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