Intrinsic Parameters and Satellite Orbital Elements

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ABSTRACT

The relations between the curvature and torsion of the satellite orbit with the orbital elements and the equipotential surface counterparts are revisited, using some angular quantities which define the geometry of the orbit and its relation to the equipotential surface and the line of force, i.e. the *slope* of the orbit (ζ), the *zenith distance* of the orbit (Z), the *separation* (θ) of the orbital plane from the equipotential surface and the *separation* (β) of the orbital plane from the Frenet osculating plane.

1. INTRODUCTION

The motion of a satellite along its orbit, a curve S in space, is governed by the well known differential equation of celestial mechanics $\ddot{\vec{x}} = \vec{g}(\vec{x})$, where $\ddot{\vec{x}}$ is the acceleration vector of the satellite and $\vec{g}$ the gravity vector, at the position $\vec{x}$, ignoring additional small forces due to drag, solar pressure and luni-solar attraction. Differences in acceleration at two indeed neighbouring points on S, e.g. $\vec{x}$ and $\vec{x}+\mathrm{d}\vec{x}$/dS, establish the linear relation of differential changes of acceleration with relevant changes of satellite position, namely

$$\frac{\mathrm{d}\ddot{\vec{x}}}{\mathrm{dS}} = \frac{\mathrm{d}\vec{g}(\vec{x})}{\mathrm{dS}} = w(\vec{x})\,\frac{\mathrm{d}\vec{x}}{\mathrm{dS}} \tag{1.1}$$

where $w(\vec{x})$ is a linear operator (homography) synthesising all the mechanical properties of the gravity field. In terms of matrix notation, $w(\vec{x})$ is represented by the gravity gradient tensor (or the Bruns tensor), **W**, in the equivalent linear transformation,

$$\frac{\mathrm{d}\ddot{\mathbf{x}}}{\mathrm{dS}} = \frac{\mathrm{d}\mathbf{g}}{\mathrm{dS}} = \mathbf{W}\,\frac{\mathrm{d}\mathbf{x}}{\mathrm{dS}} \tag{1.2}$$

where $\ddot{\mathbf{x}}$ is the acceleration components, **g** the geocentric components of the gravity vector and **x** the geocentric co-ordinates of the satellite which refer to the geocentric reference frame $\mathbf{e}_X$ represented by a triad of mutually orthogonal unit vectors ($\mathbf{e}_X$: $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$), where $\vec{e}_x$ is directed to the vernal equinox and $\vec{e}_z$ to the pole. Gravity gradient tensor **W** is a basic topic of study in satellite gradiometry *(Rummel 1986)*. It describes fully (see, e.g., *Marussi 1985*) the intrinsic geometry of the gravity field *(Grafarend 1974-)* since it contains the curvatures and torsions of the equipotential surface as well as the curvatures of the line of force, at the satellite point. On the other hand, the differential changes of satellite acceleration, $\mathrm{d}\ddot{\mathbf{x}}$/dS, can be expressed in terms of curvature and torsion of the satellite orbit. The same holds for the differential change of position d**x**/dS, since it can be shown the relation between the variation of relevant Kepler elements with the curvature and torsion of the satellite orbit. This interrelation between the intrinsic properties of the gravity field with those of the satellite orbit has not been studied extensively in the geodetic literature. Some indeed isolated examples can only be mentioned treating the satellite orbit in terms of its intrinsic properties *(Hotine 1969)* and in relation with the intrinsic properties of the gravity field *(Marussi 1962)*. In this paper the relations between the curvature and torsion of the satellite orbit and the equipotential surface counterparts are revisited, using some angular quantities which define the geometry of the orbit and its relation with the equipotential surface and the line of force, i.e. the slope of the orbit (ζ), the zenith distance of the orbit (Z), the separation (θ) of the orbital plane from the equipotential surface and the separation (β) of the orbital plane from the Frenet osculating plane.

2. FRAMES AND TRANSFORMATIONS

Traditionally, the geometry of the satellite orbit S is respectively associated with the geocentric and the perigee-related reference frames $\mathbf{e}_X$ and $\mathbf{e}_p$, the second represented here by the triad of mutually orthogonal unit vectors ($\mathbf{e}_p$: $\vec{e}_p$, $\vec{e}$, $\vec{e}_n$), where $\vec{e}_p$ is directed to the perigee and $\vec{e}_n$ is normal to the orbital plane. The rotational transformation of these frames are given by

$$\mathbf{e}_X = \mathbf{R}_3(-\Omega)\, \mathbf{R}_1(-i)\, \mathbf{R}_3(-\omega)\, \mathbf{e}_p \tag{2.1}$$

where ω the argument of perigee, i the inclination of the orbit and Ω the longitude of the ascending node, three quantities which define the space orientation of the orbit, with respect to the geocentric frame $\mathbf{e}_X$. One more triad of mutually orthogonal unit vectors, is also used, as a reference frame in satellite geodesy, namely the moving orbital triad ($\mathbf{e}$: $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$), where $\vec{e}_1$ is collinear with the radial vector from the geo-centre to the satellite, $\vec{e}_2$ is directed along the satellite orbit and $\vec{e}_3(=\vec{e}_n)$ is normal to the orbital plane. This moving frame is related with $\mathbf{e}_p$ via true anomaly f

$$f = \cos^{-1}\,(\vec{e}_p \cdot \vec{e}_1)\,, \tag{2.2}$$

by the transformation

$$\mathbf{e} = \mathbf{R}_3(f)\, \mathbf{e}_p\ . \tag{2.3}$$

True anomaly f and the radial distance r, of the satellite from the geo-centre, define as polar co-ordinates, the position of the satellite with respect to the perigee-related frame $\mathbf{e}_p$. Considering the unit tangent vector of the orbit $\vec{t}$, we can define the "slope" of the orbit ζ, as the angle from the radial vector $\vec{e}_1$

$$\zeta = \cos^{-1}\,(\vec{e}_1 \cdot \vec{t}\,)\,. \tag{2.4}$$

If β is the small angle separating the orbital plane from the osculating plane of the orbit, in terms of the Frenet triad, ($\mathbf{e}_F$: $\vec{t}$, $\vec{n}$, $\vec{b}$), where $\vec{t}$ the unit tangent vector, $\vec{n}$ the unit normal and $\vec{b}$ the unit binormal, the relation between the $\mathbf{e}$ triad and the Frenet triad is given by the transformation

$$\mathbf{e}_F = \mathbf{R}_1(\beta)\, \mathbf{R}_3(\zeta)\, \mathbf{e} \tag{2.5}$$

which combined with (2.1) and (2.3) gives the relation between the Frenet and the geocentric triads

$$\boxed{\mathbf{e}_F = \mathbf{R}_1(\beta)\, \mathbf{R}_3(q)\, \mathbf{R}_1(i)\, \mathbf{R}_3(\Omega)\, \mathbf{e}_X} \tag{2.6}$$

where q,

$$q = \omega + f + \zeta \tag{2.7}$$

is the orientation of the orbit-tangent with respect to the equatorial plane. At each satellite point, the relevant equipotential surface (W=const.) intersects the orbital plane by an angle θ. The intersection of the equipotential surface with the orbital plane defines the satellite trajectory on the equipotential surface, associated with the surface unit tangent vector $\vec{t}^*$. The angle ε, between the tangent to the orbit and its counterpart on the equipotential surface is, thus

$$\varepsilon = \cos^{-1} (\vec{t} \cdot \vec{t}^*) \tag{2.8}$$

where ε, when added to q, gives the orientation of the tangent of the satellite trajectory on the equipotential surface, with respect to the equatorial plane. The unit vector $\vec{N}$, normal to the equipotential surface along the vertical at the satellite point, is obviously orthogonal with $\vec{t}^*$, both belonging to a "natural" triad of mutually orthogonal unit vectors ($\mathbf{e}_N$: $\vec{t}^*$, $\vec{T}$, $\vec{N}$), where $\vec{t}^*$ defines the direction of the orbit on the equipotential surface, $\vec{T}$ the perpendicular direction, both vectors $\vec{t}^*$ and $\vec{T}$, on the horizontal plane, and $\vec{N}$ the opposite direction of the gravity vector $\vec{g}$,

$$\vec{N} = -\frac{1}{g}\vec{g} \tag{2.9}$$

where g the intensity of gravity at the satellite orbit. The zenith distance Z, of the orbit, is defined by

$$Z = \cos^{-1} (\vec{N} \cdot \vec{t}) \tag{2.10}$$

and due to (2.8) and (2.9), it is

$$Z = 90° - \varepsilon \tag{2.11}$$

$$\vec{g} \cdot \vec{t} = -\, g \cos Z = -\, g \sin\varepsilon \tag{2.12}$$

The definitions of the Frenet and the natural frames give their rotational transformation, as function of the angles β, ε, θ,

$$\boxed{\mathbf{e}_N = \mathbf{R}_1(\theta)\, \mathbf{R}_3(\varepsilon)\, \mathbf{R}_1(-\beta)\, \mathbf{e}_F} \tag{2.13}$$

from which, combining with (2.6), we obtain

$$\boxed{\mathbf{e}_N = \mathbf{R}_1(\theta)\, \mathbf{R}_3(\varepsilon + q)\, \mathbf{R}_1(i)\, \mathbf{R}_3(\Omega)\, \mathbf{e}_X} \quad . \tag{2.14}$$

For the model spherical field and for a polar circular orbit (eccentricity zero, inclination i = 90°), the above angular quantities ζ, θ, β, ε reduce to

$$\zeta = \theta = 90° \tag{2.15}$$
$$\beta = \varepsilon = 0° \tag{2.16}$$

and consequently, the frames $\mathbf{e}_F$ and $\mathbf{e}_N$ coincide with $\mathbf{e}$,

$$\begin{aligned} \vec{e}_1 &= -\vec{n} = \vec{N} \\ \vec{e}_2 &= \vec{t} = \vec{t}^* \\ \vec{e}_3 &= \vec{b} = \vec{T} \end{aligned} \tag{2.17}$$

In such approximation, $\vec{e}_1$ is in the vertical direction, $\vec{e}_2$ is the tangent and $\vec{e}_3$ normal to the orbital plane. The slope ζ is thus, the zenith distance Z of the orbit. It is clear that the slope of the orbit ζ, the angular separations β and θ of the orbital plane from the Frenet osculating plane and from the equipotential surface respectively as well as the angular separation ε of the orbit-tangent from its counterpart on the equipotential surface, reflect the contribution of the anomalous field of forces which affect the satellite orbit.

3. ACCELERATION

The differential change of $\mathbf{e}_F$ along the orbit S, is given by the Frenet-Serret relation

$$\frac{d\mathbf{e}_F}{dS} = \mathbf{F}\mathbf{e}_F \tag{3.1}$$

$$\mathbf{F} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \tag{3.2}$$

where κ, τ are respectively the curvature and the torsion of the orbit. The time derivative of $\vec{x}$ is the velocity vector $\dot{\vec{x}}$ along the unit tangent vector $\vec{t}$, of the orbit

$$\dot{\vec{x}} = \mathrm{v}\ \vec{t}\ . \tag{3.3}$$

The acceleration vector is the time derivative of (3.3)

$$\ddot{\vec{x}} = \dot{\mathrm{v}}\vec{t} + \mathrm{v}\ \dot{\vec{t}} \tag{3.4}$$

and since $\dot{\vec{t}} = \mathrm{v}\ \frac{d\vec{t}}{dS}$, equation (3.4) with the help of (3.1), (3.2) is written

$$\ddot{\vec{x}} = \dot{\mathrm{v}}\vec{t} + \mathrm{v}^2\ \kappa\ \vec{n} \tag{3.5}$$

where $\dot{\mathrm{v}}$ is the tangential acceleration and $\mathrm{v}^2\kappa$ the normal, or centripetal, acceleration. Differentiating (3.5), along the orbit S, we obtain, with the help of (3.1), (3.2),

$$\frac{d\ddot{\vec{x}}}{dS} = \frac{d\dot{\mathrm{v}}}{dS}\vec{t} + (\ 3\ \dot{\mathrm{v}}\kappa + \mathrm{v}^2\ \frac{d\kappa}{dS})\ \vec{n} - \mathrm{v}^2\ \kappa\ \tau\ \vec{b} \tag{3.6}$$

which in matrix form is written

$$\frac{d\ddot{\vec{x}}}{dS} = \frac{d\ddot{\mathbf{x}}_F^T}{dS}\ \mathbf{e}_F \tag{3.7}$$

where

$$\frac{d\ddot{\mathbf{x}}_F}{dS} = \begin{bmatrix} 0 \\ 3\dot{\mathrm{v}}\kappa \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{v}^2 & 0 \\ 0 & 0 & -\mathrm{v}^2\kappa \end{bmatrix} \begin{bmatrix} \frac{d\dot{\mathrm{v}}}{dS} \\ \frac{d\kappa}{dS} \\ \tau \end{bmatrix} \tag{3.8}$$

Combining with (1.2) and (2.6) we obtain

$$\frac{d\ddot{\mathbf{x}}}{dS} = \mathbf{R}_3(-\Omega)\ \mathbf{R}_1(-i)\ \mathbf{R}_3(-q)\ \mathbf{R}_1(-\beta)\ \frac{d\ddot{\mathbf{x}}_F}{dS} \tag{3.9}$$

from which, with the approximations i=90°, β=0°, ζ=90°, it is

$$
\begin{bmatrix} \frac{d\ddot{x}}{dS} \\ \frac{d\ddot{y}}{dS} \\ \frac{d\ddot{z}}{dS} \end{bmatrix} = 3\,\dot{v}\,\kappa \begin{bmatrix} \cos\Omega\cos(\omega+f) \\ \sin\Omega\cos(\omega+f) \\ \sin(\omega+f) \end{bmatrix} + \tag{3.10}
$$

$$
+ \begin{bmatrix} \cos\Omega\sin(\omega+f) & v^2\cos\Omega\cos(\omega+f) & -v^2\kappa\sin(\omega+f) \\ \sin\Omega\sin(\omega+f) & v^2\sin\Omega\cos(\omega+f) & v^2\kappa\cos(\omega+f) \\ -\cos(\omega+f) & v^2\sin(\omega+f) & 0 \end{bmatrix} \begin{bmatrix} \frac{d\dot{v}}{dS} \\ \frac{d\kappa}{dS} \\ \tau \end{bmatrix}
$$

4. INTRINSIC PROPERTIES OF THE ORBIT

Differentiating (2.6) and due to (2.1), recalling the relevant transformations, we obtain

$$
\boxed{\begin{aligned} \mathbf{F} = {} & \mathbf{P}_1 \frac{d\beta}{dS} \\ & + \mathbf{R}_1(\beta)\mathbf{P}_3\mathbf{R}_1(-\beta)\ \frac{dq}{dS} \\ & + \mathbf{R}_1(\beta)\mathbf{R}_3(q)\mathbf{P}_1\mathbf{R}_3(-q)\mathbf{R}_1(-\beta)\ \frac{di}{dS} \\ & + \mathbf{R}_1(\beta)\mathbf{R}_3(q)\mathbf{R}_1(i)\mathbf{P}_3\mathbf{R}_1(-i)\mathbf{R}_3(-q)\mathbf{R}_1(-\beta)\ \frac{d\Omega}{dS} \end{aligned}} \tag{4.1}
$$

where

$$
\mathbf{P}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix};\ \mathbf{P}_3 = \begin{bmatrix} 0 & 1,0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & \end{bmatrix}. \tag{4.2}
$$

Equation (4.1) gives the curvature and torsion of the orbit, in terms of differential change of parameters ω, i, Ω, f, ζ, β along the orbit,

$$
\begin{bmatrix} \kappa \\ \tau - \frac{d\beta}{dS} \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta\sin q & \cos\beta\cos i - \sin\beta\cos q\sin i \\ 0 & \cos q & \sin q\sin i \\ \sin\beta & -\cos\beta\sin q & \sin\beta\cos i + \cos\beta\cos q\sin i \end{bmatrix} \begin{bmatrix} \frac{dq}{dS} \\ \frac{di}{dS} \\ \frac{d\Omega}{dS} \end{bmatrix} \tag{4.3}
$$

and the inverse relations, for i ≠ 0°, are

$$\begin{bmatrix} \frac{dq}{dS} \\ \frac{di}{dS} \\ \frac{d\Omega}{dS} \end{bmatrix} = \begin{bmatrix} \cos\beta + \sin\beta\cos q\cot i & -\sin q\cot i & \sin\beta - \cos\beta\cos q\cot i \\ \sin\beta\sin q & \cos q & -\cos\beta\sin q \\ -\sin\beta\frac{\cos q}{\sin i} & \frac{\sin q}{\sin i} & \cos\beta\frac{\cos q}{\sin i} \end{bmatrix} \begin{bmatrix} \kappa \\ \tau - \frac{d\beta}{dS} \\ 0 \end{bmatrix} \tag{4.4}$$

5. INTRINSIC PROPERTIES OF THE GRAVITY FIELD

The differential change of $\mathbf{e}_N$ along the orbit-trace S* on the equipotential surface, is given by

$$\frac{d\mathbf{e}_N}{dS^*} = \mathbf{K}^* \, \mathbf{e}_N \tag{5.1}$$

with

$$\mathbf{K}^* = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \tag{5.2}$$

where κ_n, κ_g the normal and the geodetic curvatures respectively and τ_g the geodetic torsion of the equipotential surface, namely the second derivatives of the geopotential W,

$$\begin{aligned} \kappa_n &= \frac{1}{g}\frac{\partial^2 W}{\partial \tau^{*2}} = \frac{1}{g} W_{t^*t^*} \\ \kappa_g &= \frac{1}{g}\frac{\partial^2 W}{\partial T^2} = \frac{1}{g} W_{TT} \\ \tau_g &= \frac{1}{g}\frac{\partial^2 W}{\partial \tau^* \partial T} = \frac{1}{g} W_{t^*T} \end{aligned} \tag{5.3}$$

The curvatures and torsion of the equipotential surface, in (5.3), the components of the curvature of the line of force χ, tangent to the vertical direction $\vec{N}$, given by

$$\begin{aligned} \chi_{t^*} &= \frac{1}{g}\frac{\partial^2 W}{\partial \tau^* \partial T} = \frac{1}{g}\frac{\partial g}{\partial \tau^*} = \frac{1}{g} W_{t^*N}, \\ \chi_T &= \frac{1}{g}\frac{\partial^2 W}{\partial T \partial N} = \frac{1}{g}\frac{\partial g}{\partial T} = \frac{1}{g} W_{TN} \end{aligned} \tag{5.4}$$

and the gradient along the vertical

$$\frac{\partial g}{\partial N} = \frac{\partial^2 W}{\partial N^2} = W_{NN} \; . \tag{5.5}$$

form the gravity gradient tensor $\mathbf{W}_N$ with respect to the to the $\mathbf{e}_N$ triad,

$$\mathbf{W}_N = g\begin{bmatrix} \kappa_n & \tau_g & \chi_{t*} \\ \tau_g & \kappa_g & \chi_T \\ \chi_{t*} & \chi_T & \frac{1}{g}\frac{\partial g}{\partial N} \end{bmatrix} = \begin{bmatrix} W_{t*t*} & W_{t*T} & W_{t*N} \\ W_{t*T} & W_{TT} & W_{TN} \\ W_{t*N} & W_{TN} & W_{NN} \end{bmatrix} \tag{5.6}$$

with the condition

$$\kappa_n + \kappa_g = \frac{1}{g}(2\omega^2 - \frac{\partial g}{\partial N}) \tag{5.7}$$

where ω the Earth rotation. Equation (5.1) is written as

$$\frac{d\mathbf{e}_N}{dS}\frac{dS}{dS^*} = \mathbf{K}^* \, \mathbf{e}_N \tag{5.8}$$

which, with the help of (2.8) and (2.11), it is

$$\frac{d\mathbf{e}_N}{dS} = \sin Z \, \mathbf{K}^* \, \mathbf{e}_N = \mathbf{K} \, \mathbf{e}_N \tag{5.9}$$

where obviously

$$\mathbf{K} = \sin Z \, \mathbf{K}^* \, . \tag{5.10}$$

6. ORBIT AND GRAVITY FIELD CURVATURES AND TORSIONS

Differentiating (2.6) and due to (3.1), (5.1), with the relevant transformations, we obtain the relation between **F** and **K** matrices of orbit and equipotential surface curvatures and torsions.

$$\begin{aligned} \mathbf{F} = {} & \mathbf{R}_1(\beta)\mathbf{R}_3(-\varepsilon)\mathbf{R}_1(-\theta)\mathbf{K}\mathbf{R}_1(\theta)\mathbf{R}_3(\varepsilon)\mathbf{R}_1(-\beta) \\ & - \mathbf{R}_1(\beta)\mathbf{R}_3(-\varepsilon)\mathbf{P}_1\mathbf{R}_3(\varepsilon)\mathbf{R}_1(-\beta) \; \frac{d\theta}{dS} \\ & - \mathbf{R}_1(\beta)\mathbf{P}_3\mathbf{R}_1(-\beta) \; \frac{d\varepsilon}{dS} \\ & + \mathbf{P}_1 \; \frac{d\beta}{dS} \end{aligned} \tag{6.1}$$

from which we obtain

$$\begin{bmatrix} \kappa \\ \tau - \frac{d\beta}{dS} \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\beta\sin Z & \sin\beta\sin^2 Z & -\sin\beta\cos Z \\ 0 & \sin Z\cos Z & \sin Z \\ \sin\beta\sin Z & -\cos\beta\sin^2 Z & \cos\beta\cos Z \end{bmatrix} \begin{bmatrix} \kappa_g\cos\theta - \kappa_n\sin\theta \\ \kappa_g\sin\theta + \kappa_n\cos\theta \\ \tau_g\sin Z - \frac{d\theta}{dS} \end{bmatrix} + \begin{bmatrix} \cos\beta \\ 0 \\ \sin\beta \end{bmatrix} \frac{dZ}{dS} \tag{6.2}$$

and due to (4.3), for $i \neq 0°$, we obtain

$$
\begin{bmatrix} \dfrac{dq}{dS} \\[2ex] \dfrac{di}{dS} \\[2ex] \dfrac{d\Omega}{dS} \end{bmatrix} = \begin{bmatrix} \sin Z & \sin(Z-q)\cot i \sin Z & -\cos(Z-q)\cot i \\[2ex] 0 & \cos(Z-q)\sin Z & \sin(Z-q) \\[2ex] 0 & \dfrac{-\sin(Z-q)}{\sin i}\sin Z & \dfrac{\cos(Z-q)}{\sin i} \end{bmatrix} \begin{bmatrix} \kappa_g \cos\theta - \kappa_n \sin\theta \\[2ex] \kappa_g \sin\theta + \kappa_n \cos\theta \\[2ex] \tau_g \sin Z - \dfrac{d\theta}{dS} \end{bmatrix} + \begin{bmatrix} \dfrac{dZ}{dS} \\[2ex] 0 \\[2ex] 0 \end{bmatrix} \qquad (6.3)
$$

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