Geodetic Pseudodifferential Operators and the Meissl Scheme

Wolfgang Keller

Abstract
The concept of pseudodifferential operators (PDO) is introduced as a generalization of the usual concepts of differential and integral operators. Based on the PDO concept in *Euclidean* spaces the concept of a PDO on a manifold is developed. It is demonstrated that for PDOs on a manifold the main part of the operator coincides with the usual planar approximation of the operator.
The so-called *Meissl* scheme is identified as the direct consequence of the homomorphy of the algebra of PDOs and the algebra of their symbols.

1 Introduction

Let $f : \mathcal{R}^n \to \mathcal{R}$ be a so-called function of moderate growth. The function $\hat{f}$, defined by

$$\hat{f}(\omega) := (2\pi)^{-\frac{n}{2}} \int_{\mathcal{R}^n} f(\mathbf{x}) e^{-\imath\omega^\top \mathbf{x}} d\mathbf{x} = \mathcal{F}\{f\}(\omega) \tag{1}$$

is called the *Fourier* transform of the function f. The function $\hat{f}$ is again a function of moderate growth and the so called inverse *Fourier* transform can be applied to it:

$$\mathcal{F}^{-1}\{\hat{f}\}(\mathbf{x}) := (2\pi)^{-\frac{n}{2}} \int_{\mathcal{R}^n} \hat{f}(\omega) e^{\imath\omega^\top \mathbf{x}} d\omega \tag{2}$$

The *Fourier* transform enjoys several useful properties:

- $$\mathcal{F}^{-1}\{\mathcal{F}\{f\}\} = f \tag{3}$$

- $$\mathcal{F}\{f * g\} = (2\pi)^{-\frac{n}{2}} \mathcal{F}\{f\}\mathcal{F}\{g\} \quad \text{convolution theorem} \tag{4}$$

- $$\mathcal{F}\{D^\alpha f\} = \mathcal{F}\{\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\} = (-1)^{|\alpha|} \omega_1^{\alpha_1} \cdots \omega_n^{\alpha_n} \mathcal{F}\{f\} \tag{5}$$

  differentation theorem

The differentation theorem (5) of the *Fourier* transform is the starting point for the definition of the concept of pseudodifferential operators.
Let us consider the *Laplacian* in $\mathcal{R}^n$:

$$-\Delta u = -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_1^2}. \tag{6}$$

According to the differentiation theorem (5)

$$\mathcal{F}\{-\Delta u\} = \left(\sum_{i=1}^{n} \omega_i^2\right) \mathcal{F}\{u\} \tag{7}$$

holds. Applying the inverse *Fourier* transform to (7), one btains the following alternative representation of the *Laplacian*:

$$\begin{aligned} -\Delta u &= \mathcal{F}^{-1}\left\{\left(\sum_{i=1}^{n} \omega_i^2\right) \mathcal{F}\{u\}\right\} &(8)\\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathcal{R}^n} \left(\sum_{i=1}^{n} \omega_i^2\right) \hat{u}(\omega) e^{\imath\omega^\top \mathbf{x}} d\omega &(9)\end{aligned}$$

This is the representation of the *Laplacian*, which is a differential operator, in the form of an integral. Hence, the name *pseudodifferential operator* is motivated for the following type of operators.

Definition 1 *The mapping*

$$pu := \mathcal{F}^{-1}\{a(\mathbf{x}, \omega)\mathcal{F}\{u\}\} \tag{10}$$

is called pseudodifferential operator and the function a is called its symbol.

Note that the concept of a pseudodifferential operator is much more general than the usual concept of a differential operator: If a is a polynomial in ω then the pseudodifferential operator coincides with a classical differential operator. If a is a suitable transcendental function, the corresponding PDO is a certain combination of a differential and a singular integral operator.

The symbol a also determines the order of the PDO.

Definition 2 *The PDO p is called a PDO of order r if*

$$|D_x^\beta D_\omega^\alpha a(\mathbf{x}, \omega)| \leq C_{\alpha\beta}(1 + |\omega|)^{r-|\alpha|} \tag{11}$$

holds.

Example 1 *For the Laplacian $-\Delta$ the symbol is*

$$symb\{-\Delta\} = |\omega|^2 \tag{12}$$

Hence, it holds

$$|D_x^\beta D_\omega^\alpha |\omega|^2| \leq |D_\omega^\alpha |\omega|^2| \leq |D_\omega^\alpha (1 + |\omega|)^2| \leq C(1 + |\omega|)^{2-|\alpha|} \tag{13}$$

This means that the Laplacian is a PDO of order 2.

Generally speaking: PDOs of negative order are smoothing operators and PDOs of positive order are de-smoothing operators. In most cases a PDO cannot be given by only one symbol but by a sequence of symbols with decreasing order.

Definition 3 *(extended)*
A mapping

$$pu := \sum_{k=0}^{\infty} \mathcal{F}^{-1}\{a_k(\mathbf{x}, \omega)\mathcal{F}\{u\}\} \tag{14}$$

with

$$|D_x^\beta D_\omega^\alpha a_k(\mathbf{x}, \omega)| \leq C_{k,\alpha\beta}(1 + |\omega|)^{r+k-|\alpha|} \tag{15}$$

is called a PDO of order r.

The part

$$p_0 u := \mathcal{F}^{-1}\{a_0(\mathbf{x}, \omega)\mathcal{F}\{u\}\} \tag{16}$$

is called the main part of p.

The main part represents the essential properties of p. In most cases the behaviour of p can be deduced from the behaviour of p_0.

2 PDOs on a manifold

The core of the definition of a PDO on a manifold is the fact that for a local patch the manifold has approximatively the same properties as an *Euklidean* space. Hence, an operator p is called a PDO on a manifold, if for every local coordinate patch it has the form (14).
Let us consider the concept in more detail. The manifold is denoted by Γ. Let $U_i \subset \Gamma, i = 1, 2, \ldots$ be a sequence of open subsets of Γ with the property

$$\bigcup_i U_i = \Gamma \tag{17}$$

These open subsets are called charts of Γ. For each chart U_i a mapping $\Phi_i : U_i \to \mathcal{R}^n$ is defined. For each $P \in U_i \subset \Gamma$ the real numbers $\Phi(P)$ are called local coordinates of P.

Definition 4 *A mapping $p : C^\infty(\Gamma) \to C^\infty(\Gamma)$ is called a PDO on the manifold Γ, if for every local coordinate patch U_i, the mapping*

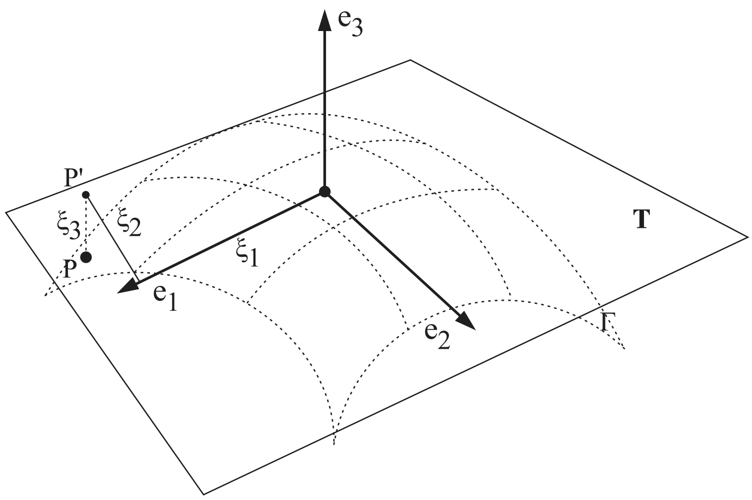
$$\Phi_i \circ p \circ \Phi_i^{-1} \tag{18}$$

is of the form (14).

Example 2 *Let Γ be a closed, orientable, smooth surface in $\mathcal{R}^3$. On Γ the following single-layer potential operator is defined:*

$$(pu)(\mathbf{x}) := \int_\Gamma \frac{u(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} \tag{19}$$

In the neighbourhood of an arbitrary $\mathbf{x}_0 \in \Gamma$ local coordinates are introduced in the following way:



First a tangential plane T is attached to Γ in $\mathbf{x}_0$. Secondly, T is equipped with a cartesian coordinate system, having its origin in $\mathbf{x}_0$.

Let $P \in \Gamma$ be and $P' \in T$ its orthogonal projection onto the tangential plane. Let ξ_1, ξ_2 be the Cartesian coordinates of P' and ξ_3 the distance between P and P'. Then the local coordinates of $P \in \Gamma$ are defined by

$$\Phi(P) = (\xi_1, \xi_2, \xi_3) = \xi \tag{20}$$

Consequently, we have

$$(\Phi \circ p \circ \phi^{-1})u(\xi) = \int_{\mathcal{R}^3} \frac{u(\Phi^{-1}(\eta))}{|\Phi^{-1}(\xi) - \Phi^{-1}(\eta)|} |det(\Phi^{-1})'| d\eta \tag{21}$$

For Φ^{-1} the following Taylor expansion is valid

$$\begin{aligned} \Phi^{-1}(\eta) &= \Phi^{-1}(0) + (\Phi^{-1})'(0) && (22) \\ &= 0 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \eta && (23) \\ &= (\eta_1, \eta_2, 0) && (24) \end{aligned}$$

Hence,

$$\begin{aligned} (\Phi \circ p \circ \Phi^{-1})u &\approx \int_{\mathcal{R}^2} \frac{u(\eta_1, \eta_2)}{\sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}} d\eta && (25) \\ &= \frac{1}{|\xi|} * u && (26) \\ &= \frac{1}{2\pi} \mathcal{F}^{-1}\{\frac{1}{|\omega|} \mathcal{F}\{u\}\} && (27) \end{aligned}$$

This means that p is a PDO with the main part

$$p_0 u := \int_{\mathcal{R}^2} \frac{u(\eta)}{|\xi - \eta|} dy = \frac{1}{2\pi} \mathcal{F}^{-1}\{\frac{1}{|\omega|} \mathcal{F}\{u\}\} \tag{28}$$

3 Planar approximation

One typical technique in Physical Geodesy is the local approximation of globally defined integral operators. For this purpose the mean sphere S of the Earth is approximated by a tangential plane T. Consequently, the integral operator p defined on the sphere S has to be approximated by an integral operator p_0 on the tangential plane T. Usually, this is done by the following technique:

- In the point x_0 a *Cartesian* coordinate system is attached to the tangentialplane T, so that its ξ_3- axis coincides with the outer normal vector n of the sphere S in x_0
- A one-to-one relationship between S and T is established by orthogonal projection.
- The (ξ_1, ξ_2) coordinates of the projection are used as local coordinates on S.

It is easy to see that the mapping Φ^{-1} is given by

$$\Phi^{-1}(\xi_1, \xi_2) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \sqrt{R^2 - \xi_1^2 - \xi_2^2} - R \end{pmatrix} \tag{29}$$

Let

$$(pu)(P) := \int_S K(\psi) u(Q) dS(Q) \tag{30}$$

be an invariant operator on S with ψ as the spherical distance between P and Q. Let the projections of the points P and Q be denoted by P' and Q', and let ξ and η denote their coordinates.
Obviously,

$$\psi = 2\arcsin(\frac{l}{2R}) \ , \quad l = \sqrt{|P' - Q'|^2 + (\xi_3 - \eta_3)^2} \tag{31}$$

holds and the representation of the invariant operator p in local coordinates is

$$pu = \int_{\mathcal{R}^2} K(2\arcsin(\sqrt{\frac{|P' - Q'|^2}{4R^2} + \frac{(\xi_3 - \eta_3)^2}{4R^2}}))u(Q')|det(\Phi^{-1})'|dQ' \tag{32}$$

Since $h := \frac{\xi_3 - \eta_3}{2R}$ is a small quantity, a *Taylor* expansion of K at the place $h = 0$ can be made. The replacement of K by the first term of its expansion is called *planar approximation* p_0 of p :

$$\begin{aligned}(p_0 u)(\xi) &= \int_{\mathcal{R}^2} K(2\arcsin(\frac{|\xi - \eta|}{2R})u(\eta)d\eta & (33)\\ &= K(2\arcsin(\frac{\bullet}{2R})) * u & (34)\\ &= \mathcal{F}^{-1}\{\hat{K} \cdot \hat{u}\} & (35)\end{aligned}$$

Now, the similarities between the main part of a PDO on a manifold and the planar approximation are obvious: The relation (29) defines the local coordinates, the representation $\Phi \circ p \circ \Phi^{-1}$ is given by (32) and the first term of the *Taylor* expansion gives the main part (33) of the corresponding PDO on the sphere S.
Usually, the planar approximation is understood intuitively. Its identification with the main part of the corresponding PDO gives an additional justification for this approximation: it already represents all essential properties of the original operator.

4 Meissl's Scheme

One of the most exiting things about PDOs is the homorphity of the algebra of PDOs with the algebra of its symbols. In detail this homomorphity is expressed by the following two relations

Theorem 1

$$\begin{aligned}symb(p + q) &= symb(p) + symb(q) & (36)\\ symb(p \circ q) &= symb(p) \cdot symb(q) & (37)\end{aligned}$$

In a maner of speaking, this means that one could work with the symbols instead of the operators themselves. Since the symbols are real function and the operators are mostly singular integral operators the handling of the former is much easier than the handling of the latter.

Example 3 *Let p be a PDO with the symbol $a(\omega)$*

$$pu = \mathcal{F}^{-1}\{a(\omega)\mathcal{F}\{u\}\} \tag{38}$$

and I the identity operator which also can be written as

$$Iu = \mathcal{F}^{-1}\{1 \cdot \mathcal{F}\{u\}\} \tag{39}$$

The determination of the inverse p^{-1} of p means that the following PDO-equation has to be solved:

$$p \circ p^{-1} = I \tag{40}$$

The corresponding symbol equation is

$$a(\omega) \cdot symb(p^{-1}) = 1 \tag{41}$$

which can be solved for $symb(p^{-1})$ and giving the following representation of the inverse operator

$$p^{-1}u = \mathcal{F}^{-1}\{\frac{1}{a(\omega)}\mathcal{F}\{u\}\} \tag{42}$$

The homomorphy means that a concatenation of several operators can be described by the multiplication of their symbols. For operators with geodetic relevance this relationship was already found earlier and independently of the context of PDO . It is called *Meissl Scheme* after its discoverer *P. Meissl*.

5 Construction of the Meissl scheme from the PDOs

The operators which are involved in the *Meissl* scheme are

- the upward continuation operator,
- the normal derivative operator,
- the gravity anomaly operator and the
- Stokes operator

For each of them the main part and its symbol has to be found. The upward continuation operator on the spehere is given by *Poisson*'s integral

$$Uu := u(r, \vartheta, \lambda) = \frac{R^2 - r^2}{4\pi} \int_\sigma \frac{u(\vartheta', \lambda')}{(R^2 - 2Rr\cos\psi + r^2)^{\frac{3}{2}}} \sin\vartheta' d\sigma(\vartheta', \lambda') \tag{43}$$

Its planar approximation, according to section 3, is the PDO

$$U_0 u = u(\mathbf{x}, h) = \frac{1}{2\pi} \int_{\mathcal{R}^2} \frac{u(\mathbf{x}')}{(|\mathbf{x} - \mathbf{x}'|^2 + h^2)^{\frac{3}{2}}} d\mathbf{x}' \tag{44}$$

having the symbol $e^{-h\omega}$.
The normal derivative operator is derived from *Greens* representation theorem

Theorem 2 *(Greens representation theorem)*
Let u be a harmonic function and $\mathbf{n}$ be the normal vector of S. For every $\mathbf{x}$ in S holds

$$u(\mathbf{x}) = -\frac{1}{2\pi} \int_S \left(\frac{1}{|\mathbf{x} - \mathbf{y}|} \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial}{\partial \mathbf{n}} \frac{1}{|\mathbf{x} - \mathbf{y}|} \right) d\sigma(\mathbf{y}) \tag{45}$$

Denoting the single layer potential by s and the double layer potential by d

$$su = \frac{1}{2\pi} \int_S \frac{1}{|\mathbf{x} - \mathbf{y}|} u d\sigma \tag{46}$$

$$du = \frac{1}{2\pi} \int_S \frac{\partial}{\partial \mathbf{n}} \frac{1}{|\mathbf{x} - \mathbf{y}|} u d\sigma \tag{47}$$

the equation (45) can be rewritten as

$$Iu = -s(\frac{\partial u}{\partial \mathbf{n}}) + du \tag{48}$$

which can be solved for the normal derivative

$$nu := \frac{\partial}{\partial \mathbf{n}} u = -s^{-1}(I - d)u \tag{49}$$

The planar approximations of s and d are

$$s_0 u = \int_{\mathcal{R}^2} \frac{u}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} \tag{50}$$

$$d_0 u = 0, \tag{51}$$

which leads to

$$n_0 u = -s_0^{-1} u. \tag{52}$$

Since the symbol of s_0 equals

$$symb(s_0) = 4\pi \frac{1}{|\omega|} \tag{53}$$

the main part of the normal derivative operator is given by

$$n_0 u = \frac{1}{4\pi} \mathcal{F}^{-1}\{|\omega| \mathcal{F}\{u\}\} \tag{54}$$

and its symbol is

$$symb(n_0) = \frac{1}{4\pi} |\omega| \tag{55}$$

In spherical approximation the gravity anomaly operator g is given by

$$gu := -\frac{\partial}{\partial \mathbf{n}} u - \frac{2}{R} u = -(n + \frac{2}{R} I)u \tag{56}$$

Obviously, its main part is

$$g_0 u = -n_0 u = s_0^{-1} u \tag{57}$$

The *Stokes* operator is given by

$$Stu := \frac{1}{4\pi\gamma R} \int_S S(\psi) u dS \tag{58}$$

with $S(\psi)$ being the *Stokes* function and γ being the normal gravity. The main part of St equals the planar approximation

$$St_0 u = \frac{1}{2\pi\gamma} \int_{\mathcal{R}^2} \frac{1}{|\mathbf{x} - \mathbf{y}|} u(\mathbf{y}) d\mathbf{y} = -\frac{1}{2\pi\gamma} s_0 u \tag{59}$$

having the symbol

$$symb(St_0) = \frac{1}{2\pi\gamma} \frac{1}{|\omega|} \tag{60}$$

The following table summarizes the results

Name	main part	symbol
upward continuation U	$U_0 u = \frac{1}{2\pi} \int_{\mathcal{R}^2} \frac{u(\mathbf{x}')}{(\lvert\mathbf{x}-\mathbf{x}'\rvert^2+h^2)^{\frac{3}{2}}} d\mathbf{x}'$	$e^{-h\lvert\omega\rvert}$
normal derivation n	$n_0 u = \frac{1}{4\pi} \mathcal{F}^{-1}\{\lvert\omega\rvert \mathcal{F}\{u\}\}$	$\frac{1}{4\pi}\lvert\omega\rvert$
gravity anomaly g	$g_0 u = -n_0 u$	$-\frac{1}{4\pi}\lvert\omega\rvert$
Stokes St	$St_0 u = \frac{1}{2\pi\gamma} \int_{\mathcal{R}^2} \frac{1}{\lvert\mathbf{x}-\mathbf{y}\rvert} u(\mathbf{y}) d\mathbf{y}$	$\frac{1}{2\pi\gamma} \frac{1}{\lvert\omega\rvert}$

With the help of these four operators different geodetic quantities as

- disturbing potential T
- geoid undulations N
- gravity anomalies Δg
- vertical gravity gradients Γ

can be connected at ground level as well as at a certain height H. The following picture shows the commutative diagram of the previously mentioned quantities.

$$
\begin{tikzcd}
\text{h = H} & T_H \arrow[r, "\frac{1}{\gamma}"] & N_H \arrow[r, "s_0^{-1}"] & \Delta g_H \arrow[r, "n_0"] & \Gamma_H \\
\text{h = 0} & T_0 \arrow[r, "\frac{1}{\gamma}"] \arrow[u, "U_0"] & N_0 \arrow[r, "s_0^{-1}"] \arrow[u, "U_0"] & \Delta g_0 \arrow[r, "n_0"] \arrow[u, "U_0"] & \Gamma_0 \arrow[u, "U_0"]
\end{tikzcd}
$$

If this relationship is transformed into the frequency domain a relationship between the spectra of the used quantities is obtained.

$$
\begin{tikzcd}
\text{h = H} & T_H \arrow[r, "\frac{1}{\gamma}"] & N_H \arrow[r, "4\pi\gamma|\omega|"] & \Delta g_H \arrow[r, "|\omega|"] & \Gamma_H \\
\text{h = 0} & T_0 \arrow[u, "e^{-H|\omega|}"] \arrow[r, "\frac{1}{\gamma}"] & N_0 \arrow[u, "e^{-H|\omega|}"] \arrow[r, "4\pi\gamma|\omega|"] & \Delta g_0 \arrow[u, "e^{-H|\omega|}"] \arrow[r, "|\omega|"] & \Gamma_0 \arrow[u, "e^{-H|\omega|}"]
\end{tikzcd}
$$

This commutative diagram of the spectra is frequently called *Meissl* scheme.

6 Summary

The concept of a PDO is a usefull notion since it comprises both differential and integral operators under one term. The techniques, which were discussed here do not necessarily rely on PDOs, but the usage of the concept of PDOs simplifies the work much in the same way as matrix notation simplifies arithmetic calculations.
The use of singular integral operators in Physical Geodesy dates back to [4] and [2],[3] . In this papers the name PDO is never mentioned but the typical techniques are already used.
The introduction of PDOs into Geodesy was done by the famous article [6] and it is nowadays frequently used for the treatment of geodetic boundary value problems [5] and in connection with wavelets on the sphere [1].

References

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