Integral Equation Methods in Physical Geodesy

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1 Introduction

It is well-known since the days of G.G. Stokes (Stokes, 1849) that the main tasks of Geodesy, the determination of the geometry of the Earth's surface and its external gravity field, can be handled by solving geodetic boundary value problems. While Stokes's approach had been based on a reduction of observational data, related to the earth's surface, for gravitational effects induced by the topographical masses, M.S. Molodenskii provided a formulation in terms of an external boundary value problem associated to Laplace's differential equation with the topographical surface of the earth acting as boundary surface (Molodenskii et al., 1962). Further advances in the theory of the geodetic boundary value problem (GBVP) have been made in the past 30 years, especially by the work of H. Moritz, T. Krarup, P. Meissl, E. Grafarend and F. Sansò.

As a result, various formulations of the GBVP are discriminated today, depending on the type of boundary data given on the boundary surface and on the type and number of unknown functions to be solved for. A major criterion for the classification of the numerous types of the GBVP is the question whether the geometry of the boundary surface is known or to be determined from the boundary data itself as part of the GBVP. The concept and notion of "free" boundary value problems, involving a free boundary surface with unknown geometry, has first been introduced in Geodesy by E. Grafarend (Grafarend and Niemeier, 1971; Grafarend, 1972).

Since most of the original formulations of GBVPs are of non-linear type, the first step towards practically applicable solutions consists of a linearization of the primary, non-linear boundary conditions (observation equations) by introducing a reference ("normal") potential and – in the case of free BVPs – a reference surface ("telluroid") approximating the actual gravity potential and surface of the earth, respectively. In general, the linearized boundary conditions imply the derivative of the disturbing potential in a non-normal direction; thus the GBVPs at the level of the linearized problems are classified as fixed, oblique-derivative BVPs.

Further simplifications, e.g. the so-called spherical approximation and the planar approximation (Moritz, 1980, p. 349ff) are generally applied to the linear, oblique-derivative boundary operator in order to reduce the complexity of the GBVP. But still at this level of approximation the resulting BVPs cannot be solved in closed, analytical form due to the irregular boundary surface. Only at the level of the constant radius approximation, by replacing the topographic boundary surface by a sphere, closed solutions in the form of spherical integral formulae can be constructed by applying spherical harmonic expansions. For other geometrically simple substitutes of the boundary surface, e.g. a spheroid or ellipsoid of revolution, first-order solutions of the non-spherical GBVPs can be achieved by the procedure of ellipsoidal corrections (Heck, 1991, 1997; Seitz, 1997).

The more realistic case of an irregular, topographical boundary surface requires either direct discrete approaches such as finite element or finite difference methods (see the pioneering paper by Grafarend, 1975), or the integral equation approach, already applied by M.S. Molodenskii in combination with an analytical perturbation method. In the past decades the integral equation approach has been numerically adapted in the framework of the boundary Element Method (BEM); recent applications to the

GBVP proved the high flexibility and large potential of this promising approach (Klees, 1997; Lehmann, 1997).

The transformation of a BVP into an equivalent integral equation relies on the choice of a representation formula. For a BVP related to Laplace's differential equation admissible representation formulae are (generalized) Green's identities or the potentials of single or double layer mass distributions spread over the boundary surface. Taking advantage of the jump relations the representation formulae provide boundary integral equations which have to be solved for the unknown layer densitites or the potential on the boundary surface, Obviously any choice of representation formula yields a different boundary integral equation for one and the same boundary condition.

In the present paper several representation formulae are applied to the linearized, scalar free GBVP in spherical approximation ("Simple Molodensky Problem"). Section 2 gives a short review of the GBVP under consideration. Based on the representation of the disturbing potential by single and double layer potentials as well as by Brovar's generalized single layer and volume potentials, the transformation of the boundary condition is derived in section 3. For spherical boundary surfaces the solutions of the integral equations can be given in closed analytical form, which is the subject of section 4. Finally, section 5 summarizes some conclusions with respect to applications in Physical Geodesy.

2 The linearized, scalar free GBVP

In the formulation of the scalar free GBVP ("geodetic variant of Molodenskii's problem") it is presupposed that the "horizontal" coordinates of the point $P \in S$ situated on the closed boundary surface S – e.g. the geodetic coordinates with respect to an ellipsoid of revolution, fixed to the earth's rotating body – are known. As a consequence, this type of GBVP contains two unknown functions, identified by the ellipsoidal height H(P) of the boundary points and the gravity potential W(Q), fulfilling the extended Laplace equation

$$\text{Lap } W(Q) = 2T^2 \tag{2.1}$$

at any spatial point Q outside S; T denotes the angular velocity of the earth's rotation. Furthermore, the gravitational part V = W-Z (Z = ½ $T^2 p^2$ centrifugal potential) of the gravity potential is regular at infinity,

$$V = 0\left(r^{-1}\right) \quad , \quad r = \left|\vec{X}(Q)\right|. \tag{2.2}$$

The information for the determination of the unknown functions H(P) and W(Q) has to be extracted from two types of boundary data, presupposed to be given in continuous form over the whole surface S. In the framework of the scalar free GBVP it common to use the observable modulus э of the gravity vector and the geopotential number C with respect to a global fundamental point P_0 as boundary data. Assuming that the standard basic model of Physical Geodesy (Heck, 1997) holds, the relationship between the observables э(P), C(P) at P,S and the unknown functions W, H(P) is provided by the non-linear observation equations

$$\Gamma(P) = \left|\text{grad} W(P)\right| \tag{2.3a}$$

$$C(P) = W(P_0) - W(P). \tag{2.3b}$$

Linearization of these equations can be achieved by introducing a reference potential w, e.g. a Somigliana-Pizzetti normal gravity field, fulfilling the relationships

$$\begin{aligned} &\text{Lap } w(Q) = 2T^2 \\ &w\text{-}Z = 0(r^{-1}) \quad , \; r = \left|\vec{X}(Q)\right| \quad , \end{aligned} \tag{2.4}$$

if the centrifugal parts in W and w are identical. A reference surface $s \ni p$ suitable for linearization is constructed via Molodenskii's telluroid mapping (see Grafarend, 1978)

$$\varphi_g(p) = \varphi_g(P) \tag{2.5a}$$

$$\lambda(p) = \lambda(P) \tag{2.5b}$$

$$w(p) - w(p_0) = W(P) - W(P_0), \tag{2.5c}$$

where a one-to-one correspondence between the corresponding pairs of points p, P has been presupposed. The first and second mapping equation (2.5a,b) fix the telluroid point p,s on the ellipsoidal normal running through the surface point P,S; ν_g and 8 are the geodetic latitude and longitude, respectively, related to an ellipsoid of revolution with given size, form and orientation. The third equation (2.5c) provides the ellipsoidal height h(p)=h(ν_g,8) of the telluroid point p, which is numerically identical with the normal height of P.

Differencing the approximate quantities w, h from the original unknowns W, H yields the residual unknown ∗w (disturbing potential) and)h (height anomaly)

$$\delta w(Q) := W(Q) - w(Q) \tag{2.6a}$$

$$\Delta h \quad := H(P) - h(Q) \tag{2.6b}$$

where ∗w is assumed to be regular at infinity and harmonic in the space outside the telluroid s

$$\text{Lap}\ast w = 0 \tag{2.7}$$
$$\delta w = 0\left(r^{-1}\right) \quad , \quad r = \left|\vec{X}(Q)\right|.$$

After linearization of the boundary conditions (2.3a,b) with respect to the approximate information w, s and reducing for the unknown height anomaly)h the reduced linearized boundary condition

$$a \cdot \delta w + \langle \frac{\vec{\gamma}}{\gamma}, \text{grad}\,\delta w \rangle = \Delta\gamma + a \cdot \Delta w_o \tag{2.8}$$

$$a = -\frac{\langle \vec{\gamma}, \text{grad}\,\vec{\gamma} \cdot \vec{n}_e \rangle}{\gamma \cdot \langle \vec{\gamma}, \vec{n}_e \rangle} \tag{2.9}$$

is obtained, where $\vec{\gamma}$=grad w is the normal gravity vector with modulus $\gamma = \left|\vec{\gamma}\right|$, $\vec{n}_e$ is the unit vector in the direction of the external ellipsoidal normal, $\Delta\gamma := \Gamma(P) - \gamma(p)$ the scalar gravity anomaly and $\Delta w_o := W(P_o) - w(p_o)$ an unknown potential constant. For the derivation of (2.8), (2.9) and the representation of this boundary condition in various curvilinear coordinates see e.g. Heck (1991, 1997).

The directional derivative in (2.8) is related to the direction of the normal gravity vector $\vec{\gamma}(p)$ which deviates from the radial direction by no more than 12 arcmin. globally. By approximating the direction of $-\vec{\gamma}$ by the direction of the radius vector $\vec{x}$ the boundary condition (2.8) simplifies considerably, resulting in the boundary condition of the "simple" Molodenskii problem

$$\left(-\frac{2}{r} \cdot \delta w - \frac{\partial \delta w}{\partial r} \right)_s = \Delta\gamma - \frac{2}{r} \Delta w_o. \tag{2.10}$$

It should be noted that formally the same boundary condition in linear and spherical approximation is reproduced for the vectorial free GBVP. In the following, the unknown term proportional to Δw_o on the right hand side of (2.10) will be neglected, corresponding to a "proper" choice of the numerical value of the gravity potential at P_o.

3 Integral equations for the simple Molodensky problem

The transformation of partial differential equations, in particular Laplace's equation, into equivalent integral equations (considering the respective boundary conditions) can be achieved by applying either direct or indirect methods. The **direct method** is based on Green's identities: E.g. the standard BVPs of classical potential theory can be transformed by the aid of Green's 2nd or 3rd identities (Walter, 1971; Sigl, 1973), while the generalized Green's formula (Giraud, 1934) provides the transition for the oblique derivative BVP (Klees, 1992). A related procedure has been proposed by Molodensky (Molodenskii et al., 1962) and Moritz (Heiskanen and Moritz, 1967, p. 229) for the transformation of the simple Molodensky problem. A specific feature of the direct formulation is the fact that the potential function on the boundary surface can be solved for in a single step.

The **indirect methods** rely on the representation of the (harmonic) solution function by surface layer potentials, e.g. produced by single or double layer surface density functions defined over the boundary surface. Here the transformation into equivalent integral equations makes use of certain jump relations which occur when the computation point approaches the boundary surface in evaluating the surface layer potential or its derivatives. Indirect methods always provide two-step procedures: In the first step the integral equation for the unknown surface layer density, acting as an auxiliary unknown , is solved for; in the second step the representation formula has to be evaluated in order to calculate the potential function or its derivatives on or outside the boundary surface.

Originally, the integral equation method has been used in potential theory in order to prove the existence of solutions of various boundary value problems, this concept being strongly related to Fredholm's alternative (Martensen and Ritter, 1997). In the past two decades the integral equation approach has become the basis for numerical solutions, too, in the framework of the boundary element method (Hackbusch, 1989). Substantial numerical savings can be expected in many practical applications by reducing the dimension of the problem from 3 (dimension of the "spatial" Laplace operator) to 2 (dimension of the boundary surface on which the density function is defined).

Obviously, the transformation of a BVP into an equivalent integral equation is not unique, since any representation formula produces another type of integral equation for one and the same BVP. Since the analytical and numerical behaviour of these integral equations may be quite different, it is necessary to select, for a given BVP, those representations which possess optimal properties in this respect. In the following, several representation formulae related to the indirect approach will be applied to the simple Molodensky problem; for the special case of a spherical boundary surface the solution of the respective integral equation can be explicitly described.

3.1 Representation by a single layer potential

Since the potential of a single layer mass distribution on a closed surface, e.g. on the telluroid s, is harmonic in the external space and regular at infinity, a single layer potential can be used for representing the disturbing potential δw

$$\delta w(\vec{X}) = \frac{1}{4\pi}\int_s \frac{\mu(\vec{y})}{|\vec{X}-\vec{y}|}\cdot ds(\vec{y}) \tag{3.1}$$

$\vec{X}$ denotes the position vector of the point of evaluation in space, $\vec{y}$ of the variable point of integration on the boundary surface s where the density function takes the value $\mu(\vec{y})$. The single layer potential (3.1) is continuous throughout $\mathbb{R}^3$, but in general not continuously differentiable with respect to each side of s. Considering the limiting relations of the normal derivative when the point $\vec{X}$ in space tends to the surface point $\vec{x}$, situated on the same surface normal (Martensen and Ritter, 1997), the gradient of the disturbing potential at the positive side of the surface s is given by the expression

$$\begin{aligned}(\mathrm{grad}\,\delta w(\vec{x}))_+ &= -\frac{1}{2}\mu(\vec{x})\cdot\vec{n}_x - \\ &- \frac{1}{4\pi}\cdot \text{p.v.}\int_s \frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^3}\cdot\mu(\vec{y})\cdot ds(\vec{y})\ ,\end{aligned} \tag{3.2}$$

p.v. denoting Cauchy's principal value. Concerning the function spaces it should be presupposed that the surface is Hölder-continuously differentiable, $s \in C^{1+\alpha}, \alpha > 0$, and $\mu \in L^2(s)$ is quadratically integrable on the surface s.

Inserting the representation formula (3.1) and its gradient (3.2) into the reduced boundary condition (2.10) of the "simple" Molodenskii problem produces the following integral equation of second kind for the unknown auxiliary density function μ:

$$\frac{1}{2}\mu(\vec{x})\cdot\cos(\vec{n}_x,\vec{x}) + \frac{1}{4\pi}\,\text{p.v.}\int_s \frac{|\vec{x}|^2 - |\vec{y}|^2 - 3|\vec{x}-\vec{y}|^2}{2|\vec{x}|\cdot|\vec{x}-y|^3}\cdot\mu(\vec{y}).ds(\vec{y}) = \Delta\gamma(\vec{x}) \tag{3.3}$$

where $(\vec{n}_x,\vec{x})$ is the angle between the external surface normal and the position vector, which is roughly identical with the inclination angle β of the terrain. The integral equation (3.3) involves a pseudo-differential operator of order r=0 and contains a strongly singular integral kernel

$$k(\vec{x},\vec{y}-\vec{x}) := \frac{|\vec{x}|^2 - |\vec{y}|^2 - 3|\vec{x}-\vec{y}|^2}{2|\vec{x}|\cdot|\vec{x}-\vec{y}|^3} \tag{3.4}$$

in conventional notation $(|\vec{x}| = r, |\vec{y}| = r', |\vec{x}-\vec{y}| = l)$

$$k(r,r',l) = \frac{r^2 - r'^2}{2r\cdot l^3} - \frac{3}{2r\cdot l} \tag{3.5}$$

The integral equation (3.3) was the starting point in M.S. Molodenskii's series expansion for the analytical solution of the GBVP (Moritz, 1980, p. 354 ff).

3.2 Representation by a double layer potential

Since the potential of a surface dipole distribution on the closed surface s is harmonicc outside the surface and regular at infinity, the double layer potential involving the density function ν can be used for representing the disturbing potential δw

$$\delta w(\vec{X}) = \frac{1}{4\pi}\int_s \frac{\partial}{\partial n_y}\frac{1}{|\vec{X}-\vec{y}|}\cdot \nu(\vec{y})\cdot ds(\vec{y}). \tag{3.6}$$

It is well-known (Martensen and Ritter, 1997) that the double layer potential is discontinuous when the point $\vec{X}$ in space tends to the surface point $\vec{x}$, fulfilling the limiting relations for the potential and its gradient

$$(\delta w(\vec{x}))_+ = \frac{1}{2}\nu(\vec{x}) + \frac{1}{4\pi}\int_s < \vec{n}_y, \frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^3} > \nu(\vec{y})\cdot ds(\vec{y}) \tag{3.7}$$

$$\begin{aligned}(\mathrm{grad}\,\delta w(\vec{x}))_+ &= \frac{1}{2}\mathrm{Grad}\,\nu(\vec{x}) + \\ &+ \frac{1}{4\pi}\mathrm{p.f.}\int_s\left[\frac{\vec{n}_y}{|\vec{x}-\vec{y}|^3} - 3\frac{<\vec{n}_y, \vec{x}-\vec{y}>\cdot(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^5}\right]\cdot\nu(\vec{y})\cdot ds(\vec{y})\end{aligned} \tag{3.8}$$

The integral (3.7) exists as an improper integral if it is assumed that s is piecewise Hölder-continuously differentiable, $s0C^{1+\alpha}$, $\alpha>0$ and ν is continuous, $\nu\in C^o$. In contrast, the integral in (3.8) has to be understood in the sense of Hadamard's part fini integral (Hackbusch, 1989, p. 284), presupposing $s\in C^{1+\alpha}$, $\nu\in C^{1+\alpha}(s)$, $\alpha>0$. Grad $\nu(\vec{x})$ denotes the surface gradient of the density function ν at $\vec{x}$.

Inserting (3.7) and (3.8) into the reduced boundary condition (2.10) of the simple Molodenskii poblem yields the following integro-differential equation for the unknown auxiliary density function ν:

$$\begin{aligned}&-\frac{1}{2}< \mathrm{Grad}\,\nu(\vec{x}), \frac{\vec{x}}{|\vec{x}|} > -\frac{\nu(\vec{x})}{|\vec{x}|} + \\ &+\frac{1}{4\pi}\mathrm{p.f.}\int_s\Big[-<\vec{n}_y,\vec{y}>\cdot\Big[3\left(|\vec{x}|^2-|\vec{y}|^2\right)-|\vec{x}-\vec{y}|^2\Big]+ \\ &+3<\vec{n}_y,\vec{x}>\cdot\Big[|\vec{x}|^2-|\vec{y}|^2-|\vec{x}-\vec{y}|^2\Big]\Big]\cdot\frac{\nu(\vec{y})\cdot ds(\vec{y})}{2|\vec{x}|\cdot|\vec{x}-\vec{y}|^5} = \Delta\gamma(\vec{x})\end{aligned} \tag{3.9}$$

This integro-differential equation involves a pseudo-differential operator of order r = 1 and contains a hypersingular integral kernel

$$\begin{aligned}k(\vec{x},\vec{y}-\vec{x}) := &\Big[-<\vec{n}_y,\vec{y}>\cdot\Big[3\left(|\vec{x}|^2-|\vec{y}|^2\right)-|\vec{x}-\vec{y}|^2\Big]+ \\ &+3<\vec{n}_y,\vec{x}>\cdot\Big[|\vec{x}|^2-|\vec{y}|^2-|\vec{x}-\vec{y}|^2\Big]\Big]\cdot\frac{1}{2|\vec{x}|\cdot|\vec{x}-\vec{y}|^5}\ ,\end{aligned} \tag{3.10}$$

in conventional notation

$$k(r, r', l) = \frac{3(r^2 - r'^2)(r\cos\varepsilon - r'\cos\beta')}{2r \cdot l^5} + \\ + \frac{r'\cos\beta' - 3r\cos\varepsilon}{2r \cdot l^3} \tag{3.11}$$

where $\beta' = \supseteq(\vec{n}_y, \vec{x})$
the angle between the surface normal at $\vec{y}$ and the radius vector of the evaluation point $\vec{x}$.

3.3 Representation by Brovar's generalized single layer potential

Attempting to obtain simpler expressions for the solution of Molodenskii's problem, Brovar (1963, 1964) introduced two alternative representations of harmonic functions, regular at infinity, by generalized surface layer potentials. The first representation formula generalizes the single layer potential, extending the inverse distance kernel to the Stokes-Pizzetti kernel:

$$\delta w(\vec{X}) = \frac{1}{4\pi} \int_s E_1(\vec{X}, \vec{y}) \cdot \lambda(\vec{y}) \cdot ds(\vec{y}) \tag{3.12}$$

$$E_1(\vec{X}, \vec{y}) := \frac{2}{|\vec{X} - \vec{y}|} - 3\frac{|\vec{X} - \vec{y}|}{|\vec{X}|^2} - \frac{5}{|\vec{X}|^3} < \vec{X}, \vec{y} > - \\ - \frac{3}{|\vec{X}|^2} \cdot < \vec{X}, \vec{y} > \cdot \ln \frac{|\vec{X}|^2 - < \vec{X}, \vec{y} > + |\vec{X}| \cdot |\vec{X} - \vec{y}|}{2|\vec{X}|^2} \tag{3.13}$$

Despite of the extension of the kernel by a logarithmically singular term the integral (3.12) still exists as an improper integral if $s \in C^1$ (piecewise) and $\lambda \in L^2(s)$. Like in section 3.1 the generalized single layer potential (3.12) is continuous throughout ^{3}; its gradient is discontinuous, fulfilling the limiting relation

$$(\operatorname{grad} \delta w(\vec{x}))_+ = -\lambda(\vec{x}) \cdot \vec{n}_x + \\ + \frac{1}{4\pi} \text{p.v.} \int_s \operatorname{grad}_x E_1(\vec{x}, \vec{y}) \cdot \lambda(\vec{y}) \cdot ds(\vec{y}) \tag{3.14}$$

where the integral is understood in the sense of Cauchy's principal value and $s \in C^{1+\alpha}$, $\lambda \in C^{\alpha}(s)$, $\alpha > 0$.

Inserting the representation formula (3.12) and its gradient (3.14) into the reduced boundary condition (2.10) of the "simple" Molodenskii problem yields the following integral equation of second kind for the unknown auxiliary density function λ:

$$\lambda(\vec{x}) \cdot \cos(\vec{n}_x, x) + \frac{1}{4\pi} \text{p.v.} \int_s \left(\frac{|\vec{x}|^2 - |\vec{y}|^2}{|\vec{x}| \cdot |\vec{x} - \vec{y}|^3} - 3\frac{< \vec{x}, \vec{y} >}{|\vec{x}|^4} \right) \cdot \lambda(\vec{y}) \cdot ds(\vec{y}) = \Delta\gamma(\vec{x}). \tag{3.15}$$

This integral equation again involves a pseudo-differential operator of order r=0 and contains a strongly singular integral kernel

$$k(\vec{x},\vec{y}-\vec{x}) := \frac{|\vec{x}|^2 - |\vec{y}|^2}{|\vec{x}|\cdot|\vec{x}-\vec{y}|^3} - 3\frac{<\vec{x},\vec{y}>}{|\vec{x}|^4} \quad , \tag{3.16}$$

in conventional notation

$$k(r,r',l) = \frac{r^2 - r'^2}{r \cdot l^3} - \frac{3r' \cos\psi}{r^3} \tag{3.17}$$

where $\psi = \angle(\vec{x},\vec{y})$ denotes the angle between the position vectors $\vec{x}$ (fixed point of evaluation) and $\vec{y}$ (variable point of integration). By comparing (3.17) with (3.5) it becomes obvious that the weakly singular term proportional to l^{-1} has disappeared; the additional term in (3.17) is essentially a spherical harmonic term of first degree.

3.4 Representation by Brovar's generalized "volume" potential

A second alternative surface layer representation of the disturbing potential, given by Brovar (1963, 1964) contains a kernel with an even weaker degree of singularity:

$$\delta w(\vec{X}) = \frac{1}{4\pi}\int_s E_2(\vec{X},\vec{y})\cdot\chi(\vec{y})\cdot ds(\vec{y}) \tag{3.18}$$

$$E_2(\vec{X},\vec{y}) := \frac{|\vec{X}-\vec{y}|}{|\vec{X}|^2} - \frac{<\vec{X},\vec{y}>}{|\vec{X}|^3}\left(1 + \ln\frac{|\vec{X}|^2 - <\vec{X},\vec{y}> + |\vec{X}|\cdot|\vec{X}-\vec{y}|}{2|\vec{X}|^2}\right) \tag{3.19}$$

The spatial function (3.18) is harmonic in $\mathbb{R}^3$ besides on s, and regular at infinity. Due to the logarithmic (weak) singularity of $E_2(\vec{X},\vec{Y})$ the surface layer potential (3.18) is continuously differentiable in $\mathbb{R}^3$; since this property holds generally for volume potentials, the notion "generalized volume potential" has been chosen by Brovar. The gradient of this potential representation at the point $\vec{x}$ on the surface is given by the improper integral

$$\begin{aligned} \operatorname{grad}\delta w(\vec{x}) = \frac{1}{4\pi}\int_s \Bigg[& \frac{2|\vec{x}-\vec{y}|}{|\vec{x}|^3} - \frac{1}{|\vec{x}|\cdot|\vec{x}-\vec{y}|} + 3\frac{<\vec{x},\vec{y}>}{|\vec{x}|^4} + \\ & + \frac{2<\vec{x},\vec{y}>}{|\vec{x}|^4}\cdot\ln\frac{|\vec{x}|^2 - <\vec{x},\vec{y}> + |\vec{x}|\cdot|\vec{x},\vec{y}|}{2|\vec{x}|^2}\Bigg]\cdot\chi(\vec{y})\cdot ds(\vec{y}) \end{aligned} \tag{3.20}$$

Due to the continuity of $\operatorname{grad}\delta w(\vec{X})$, $\vec{X} \in \mathbb{R}^3$ there is no residual term outside the integral (3.20). For this reason an integral equation of first kind for the unknown density function χ is produced when the representation formula (3.18) and its gradient (3.20) are inserted into the reduced boundary condition (2.10) of the "simple" Molodenskii problem:

$$\frac{1}{4\pi}\int_s\left(\frac{1}{|\vec{x}|\cdot|\vec{x}-\vec{y}|}-\frac{<\vec{x},\vec{y}>}{|\vec{x}|^4}\right)\cdot\chi(\vec{y})\cdot ds(\vec{y})=\Delta\gamma(\vec{x}) \tag{3.21}$$

This integral equation involves a pseudo-differential operator of order r = -1 and contains a weakly singular integral kernel

$$k(\vec{x},\vec{y}-\vec{x}):=\frac{1}{|\vec{x}|\cdot|\vec{x}-\vec{y}|}-\frac{<\vec{x},\vec{y}>}{|\vec{x}|^4} \tag{3.22}$$

in conventional notation

$$k(r,r',l)=\frac{1}{r\cdot l}-\frac{3r'\cdot\cos\psi}{r^3} \tag{3.23}$$

Again the second term in (3.23) is essentially a spherical harmonic of first degree.

4 The special case of a spherical boundary surface

It is well-known that the formulae of Physical Geodesy become rather simple as soon as the boundary surface is a sphere. If any relationship in spherical approximation is applied, the respective problem in addition becomes a "normal" problem in the sense of potential theory, since the radial derivative is automatically a normal derivative on the spherical surface. Spherical BVPs play a dominant role in Physical Geodesy since on a global scale the earth can be approximated rather well by a sphere, the approximation error having the order of 0.3%. For this reason reduction methods, aiming at the creation of a "spherical" situation, have become very familiar; instead of calculating those reductions from prior information more rigorous approaches can be constructed on the basis of iterative schemes. These reduction procedures form the background of e.g. the so-called "ellipsoidal corrections" (Heck, 1997; Seitz, 1997).

In the following the integral equations derived in section 3 will be specified for a sphere of radius R acting as boundary surface s with surface element $ds=R^2 Ad\Phi$. It is shown that the solutions of the integral equations for the various representations can easily be expressed in the frequency domain; in space domain the respective relationships are represented by spherical integrals.

4.1 Representation by a single layer potential

From equation (3.1) follows the representation of the disturbing potential by the potential of a single layer spread over the sphere with radius R

$$\delta w(\vec{X})=\frac{R^2}{4\pi}\int_\sigma\frac{1}{l}\cdot\mu(\vec{y})\cdot d\sigma(\vec{y}) \quad . \tag{4.1}$$

The Euclidean distance l between the points $\vec{X}$ in space and $\vec{y}$ on the sphere can be expressed by the angle P between the position vectors $\vec{X}$ and $\vec{y}$

$$l=\sqrt{r^2+R^2-2rR\cos\psi} \quad ; \tag{4.2}$$

for a computation point on the sphere $\left(\vec{X} \to \vec{x}, r = R\right)$ this relationship is simply

$$l_o = 2 \cdot R \cdot \sin\frac{\psi}{2} \quad , \tag{4.3}$$

hence

$$\delta w(\vec{x}) = \frac{R}{4\pi}\int_\sigma \frac{\mu(\vec{y})}{2 \cdot \sin\frac{\psi}{2}} \cdot d\sigma(\vec{y}) \quad . \tag{4.4}$$

In a similiar way the integral equation (3.3) reduces to

$$\frac{1}{2}\mu(\vec{x}) + \frac{1}{4\pi}\int_\sigma \left(\frac{-3}{4\sin\frac{\psi}{2}}\right) \cdot \mu(\vec{y}) \cdot d\sigma(\vec{y}) = \Delta\gamma(\vec{x}) \quad . \tag{4.5}$$

Obviously the strongly singular integral kernel in (3.5) has now been transformed into a weakly singular kernel; conversely expressed this means that the strongly singular kernel in (3.5) is produced by the topography and ellipticity of the boundary surface.

Expanding the disturbing potential outside the boundary sphere into solid spherical harmonics

$$\delta w(\vec{X}) = \sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1} \cdot \delta w_n(\vec{x}) \qquad , \quad \vec{X} = \frac{r}{R} \cdot \vec{x} \tag{4.6}$$

and the functions $\Delta\gamma(\vec{x})$ and $\mu(\vec{x})$ in surface spherical harmonics

$$\Delta\gamma(\vec{x}) = \sum_{n=0}^{\infty}\Delta\gamma_n(\vec{x}) \quad , \quad \mu(\vec{x}) = \sum_{n=0}^{\infty}\mu_n(\vec{x}) \quad , \tag{4.7}$$

and inserting these series in (4.4) and (4.5) yields the following frequency-domain relations

$$\mu_n(\vec{x}) = \frac{2n+1}{n-1} \cdot \Delta\gamma_n(\vec{x}) \quad , \quad n \neq 1 \tag{4.8}$$

$$\delta w_n(\vec{x}) = \frac{R}{2n+1} \cdot \mu_n(\vec{x}) \quad . \tag{4.9}$$

These spectral relationships show that the single layer density μ as a function on the spherical boundary is about as rough as the gravity anomaly data; on the other hand the disturbing potential δw on the sphere is smoother than the density function since the high degree, short wavelength constituents are damped by the factor 1/(2n+1). Combining formulae (4.8) and (4.9) results in the well-known spectral Stokes formula (Heiskanen and Moritz, 1967)

$$\delta w_n(\vec{x}) = \frac{R}{n-1} \cdot \Delta\gamma_n(\vec{x}) \quad , \qquad n \neq 1 \tag{4.10}$$

It should be noted that the first degree (n=1) terms are forbidden in (4.8) and (4.10), expressing the fact that $\mu_1(\vec{x})$ and $\delta w_1(\vec{x})$ cannot be determined from gravity anomaly data. On the other hand it must be postulated that the boundary data $\Delta\gamma$ fulfill the consistency condition

$$\Delta\gamma_1(\vec{x}) := \frac{3}{4\pi}\int_\sigma \Delta\gamma(\vec{y}) \cdot \cos\psi \cdot d\sigma(\vec{y}) = 0 \quad . \tag{4.11}$$

Using the spherical harmonic expansion of the function 1/l the spectral relationship (4.8) can easily be transformed into the space domain, resulting in the spherical integral

$$\mu(\vec{x}) = 2\Delta\gamma(\vec{x}) + \frac{3}{4\pi}\int_\sigma \Delta\gamma(\vec{y}) \cdot (S(\psi) - 1) \cdot d\sigma(\vec{y}) + \mu_1(\vec{x}) \tag{4.12}$$

where $S(\psi)$ denotes Stokes's function. By the combination of formulae (4.12) and (4.1), respectively (4.4) the solution of the simple Molodenskii problem in constant radius approximation is provided in two steps $(\Delta\gamma \to \mu \to \delta w)$, while a one-step procedure is based on a direct application of Stokes's integral formula equivalent to (4.10)

$$\delta w(\vec{x}) = \frac{R}{4\pi}\int_\sigma \Delta\gamma(\vec{y}) \cdot (S(\psi) - 1) \cdot d\sigma(\vec{y}) + \delta w_1(\vec{x}) \quad . \tag{4.13}$$

4.2 Representation by a double layer potential

Considering the fact that the normal derivative

$$\begin{aligned} \frac{\partial}{\partial n_y}\frac{1}{|\vec{X} - \vec{y}|} &= \lim_{r' \to R}\frac{\partial}{\partial r'}\left(r^2 + r'^2 - 2rr'\cos\psi\right)^{-1/2} \\ &= -\frac{1}{2Rl} + \frac{r^2 - R^2}{2Rl^3} \end{aligned} \tag{4.14}$$

contains a part which acts as a spherical Dirac pulse for $r \to R$, the representation formula (3.6) can be specified for a computation point situated on the spherical boundary

$$\delta w(\vec{x}) = \frac{1}{2}\nu(\vec{x}) - \frac{1}{4\pi}\int_\sigma \frac{\nu(\vec{y})}{4 \cdot \sin\frac{\psi}{2}} \cdot d\sigma(\vec{y}) \quad . \tag{4.15}$$

In a similar way the integral equation (3.9) reduces to

$$-\frac{\nu(\vec{x})}{R} + \frac{1}{4\pi}\int_\sigma \frac{3\nu(\vec{y})}{8R \cdot \sin\frac{\psi}{2}} \cdot d\sigma(\vec{y}) - \frac{1}{4\pi}\,\mathrm{p.f.}\int_\sigma \frac{\nu(\vec{y})}{8R \cdot \sin^3\frac{\psi}{2}} d\sigma(\vec{y}) = \Delta\gamma(\vec{x}) \quad . \tag{4.16}$$

The hypersingular part fini integral can be regularized by shifting the constant value $\nu(\vec{x})$ under the integral. This procedure results in the integral equation for the unknown double layer density ν:

$$\frac{3}{4\pi}\int_\sigma \frac{\nu(\vec{y})-\nu(\vec{x})}{8R\cdot\sin\frac{\psi}{2}}\cdot d\sigma(\vec{y})-\frac{1}{4\pi}\,\mathrm{p.v.}\!\int_\sigma \frac{\nu(\vec{y})-\nu(\vec{x})}{8R\cdot\sin^3\frac{\psi}{2}}\cdot d\sigma(\vec{y})=\Delta\gamma(\vec{x}) \quad . \tag{4.17}$$

Obviously the part fini hypersingular integral degenerates into a simple Cauchy principal value integral containing a strongly singular kernel. Furthermore it can be recognized that in the spherical case the differential part of the integro-differential equation (3.9) disappears.

By the aid of the expansions (4.6) and (4.7) the following spectral domain relationships are obtained

$$\nu_n(\vec{x})=R\cdot\frac{2n+1}{n(n-1)}\cdot\Delta\gamma_n(\vec{x}) \quad , \quad n\neq 1 \tag{4.18}$$

$$\delta w_n(\vec{x})=\frac{n}{2n+1}\cdot\nu_n(\vec{x})=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)\cdot\nu_n(\vec{x}) \quad , \tag{4.19}$$

proving that the double layer density ν as a function on the spherical boundary is smootehr than the gravity anomaly data; on the other hand the density function ν has the same degree of smoothness as the disturbing potential δw on the sphere. Again, the first degree terms $\nu_1(\vec{x})$ and $\delta w_1(\vec{x})$ cannot be determined from the gravity anomaly data, and the consistency condition (4.11) must be fulfilled. By combining equations (4.18) and (4.19) again the spectral Stokes's formula (4.10) is reproduced.

4.3 Representation by Brovar's generalized single layer potential

Brovar's first representation formula (3.12) can be easily specified for a computation point situated on the spherical boundary

$$\delta w(\vec{x})=\frac{R}{4\pi}\int_\sigma \lambda(\vec{y})\cdot(S(\psi)-1)\cdot d\sigma(\vec{y})+\delta w_1(\vec{x}) \tag{4.20}$$

where the kernel function is now Stokes's function; the strongly singular integral kernel $E_1(\vec{x},\vec{y})$ (3.13) has degenerated into a weakly singular one. The last term in (4.20) reflects the fact that the first degree terms of $\delta w(\vec{x})$ are indefinite.

In a similar way the integral equation (3.15) reduces to

$$\lambda(\vec{x})-\frac{3}{4\pi}\int_\sigma \cos\psi\cdot\lambda(\vec{y})\cdot d\sigma(\vec{y})=\Delta\gamma(\vec{x}) \quad . \tag{4.21}$$

The second term on the right hand side corresponds to the first degree harmonic term $\lambda_1(\vec{x})$ in the expansion of $\lambda(\vec{x})$. On the other hand, due to (4.11) the first-degree term in $\Delta\gamma$ is forced to be zero, thus it follows that $\lambda_1(\vec{x})=0$, too. As a consequence, equation (4.21) reduces to the "integral" equation

$$\lambda(\vec{x})=\Delta\gamma(\vec{x}) \quad , \tag{4.22}$$

i.e. the density function λ is identical with the boundary data $\Delta\gamma$.

A transformation of (4.20) and (4.22) into the spectral domain yields

$$\lambda_n(\vec{x}) = \Delta\gamma_n(\vec{x}) \tag{4.23}$$

$$\delta w_n(\vec{x}) = \frac{R}{n-1} \cdot \lambda_n(\vec{x}) \quad ,n \neq 1 \tag{4.24}$$

the combination of both resulting again in Stokes's formulae (4.10) and (4.13) in the spectral and in the space domain, respectively.

4.4 Representation by Brovar's generalized "volume" potential

Brovar's second representation formula (3.18) can be specified for a computation point situated on the spherical boundary

$$\begin{aligned} \delta w(\vec{x}) = \frac{R}{4\pi} \int_\sigma & \left[-2\sin\frac{\psi}{2} - \cos\psi \cdot \left(1 + \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2} \right) \right) \right] \cdot \\ & \cdot \chi(\vec{y}) \cdot d\sigma(\vec{y}) + \delta w_1(\vec{x}) \quad . \end{aligned} \tag{4.25}$$

Again the first degree term $\delta w_1(\vec{x})$ is indefinite since the first degree term

$$\int_\sigma \cos\psi \cdot \chi(\vec{y}) \cdot d\sigma(\vec{y})$$

is subtracted on the right hand side of (4.25).

In a similar way the integral equation (3.21) reduces to

$$\frac{1}{4\pi} \int_\sigma \left(\frac{1}{2\sin\frac{\psi}{2}} - \cos\psi \right) \cdot \chi(\vec{y}) \cdot d\sigma(\vec{y}) = \Delta\gamma(\vec{x}) \quad . \tag{4.26}$$

Due to the fact that the boundary data $\Delta\gamma$ have to fulfill the consistency condition (4.11) the first degree term in the auxiliary density function χ vanishes too, i.e. $\chi_1(\vec{x}) = 0$, as the analysis of (4.26) proves. Consequently (4.26) reduces to the simple integral equation of first kind

$$\frac{1}{4\pi} \int_\sigma \frac{\chi(\vec{y})}{2\sin\frac{\psi}{2}} \cdot d\sigma(\vec{y}) = \Delta\gamma(\vec{x}) \tag{4.27}$$

A transformation of (4.25) and (4.27) into the spectral domain yields

$$\chi_n(\vec{x}) = (2n+1) \cdot \Delta\gamma_n(\vec{x}) \tag{4.28}$$

$$\delta w_n(\vec{x}) = \frac{R}{(2n+1)(n-1)} \cdot \chi_n(\vec{x}) \quad , \quad n \neq 1 \tag{4.29}$$

It can be recognized from (4.28) that the density function χ as a function on the boundary is rougher than the boundary data $\Delta\gamma$ since the short wavelength components in $\Delta\gamma$ are amplified by the factor (2n+1). This behaviour could be expected from (3.21), because the inverse of the operator $K:\chi\rightarrow\Delta\gamma$, being a pseudo-differential operator of order r = -1, naturally has a de-smoothing property and is unstable. On the other hand, the operator $I:\chi\rightarrow\delta w$ is strongly smoothing. As a consequence, a two-step approach for the solution of the GBVP, which is based on Brovar's second representation formula, will be senseless for numerical reasons, since the procedure used in the first step will not be stable. This behaviour is also visible when (4.28) is transformed into the space domain

$$\chi(\vec{x}) = \frac{1}{4\pi}\,\text{p.v.}\int_{\sigma}\left(\Delta\gamma(\vec{y}) - \Delta\gamma(\vec{x})\right)\left(\frac{1}{2\sin^3\frac{\psi}{2}} - 9\cos\psi\right)d\sigma(\vec{y}) + \chi_1(\vec{x}) \tag{4.30}$$

where the hypersingular integral has been regularized, leaving an integral in the sense of Cauchy's principal value.

5 Closing remarks

The preceding derivations have shown that there exist numerous alternative and competitive representations of the disturbing potential, providing as many integral equations for the solution of one and the same formulation of the GBVP. The two-step approach described above arrives at the solution after having solved the integral equation for the auxiliary density function which is inserted into the representation formula. For an arbitrary density function κ, $\kappa \in \{\mu, \nu, \lambda, \chi\}$ this is indicated by the sequence of mappings

$$\Delta\gamma \rightarrow \kappa \rightarrow \delta w \ .$$

In numerical solutions of the GBVP via the integral equation method (BEM) the properties of the respective operators play a dominant role (Klees, 1992, 1997; Lehmann, 1997). For numerical reasons it it advantageous to apply only non-desmoothing operators in this process. The variants described in sections 3.1, 3.2, 3.3 and 4.1, 4.2, 4.3 respectively are characterized by a sequence of two transformations, one of which retaining the same degree of roughness and the other one being of smooting type. An exception is provided in sections 3.4 and 4.4 where by the use of Brovar's second alternative of representation a desmoothing mapping $\Delta\gamma\rightarrow\chi$ has been applied which has to be counterbalanced in the second step $\chi\rightarrow\delta w$ by a much stronger smoothing. Since the degree of smooting of the composed mapping $\Delta\gamma\rightarrow\delta w$ is fixed, a smoothing gain in one step will be lost in the other step of the indirect BEM approach. For the same reason the use of surface layer representations involving higher order derivatives of the inverse distance

$$\frac{\partial^k}{\partial n^k}\left(\frac{1}{l}\right) \quad , \quad k \geq 2$$

cannot be recommended, in general.

Finally it should be noted that the integral equation method is applicable to the linearized GBVP in the strict sense, too, without presupposing spherical and planar approximations. The integral equation method in its modern numerical version, the Boundary Element Method, is capable of taking care of very irregular boundary surfaces, making it a most excellent and efficient tool for solving the GBVP. The considerable numerical expenditure can be managed today by the use of modern supercomputers (vector and parallel computers), as the results by Klees (1992, 1997) and Lehmann (1997) have confirmed.

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