On the maintenance of a proper reference frame for VLBI and GPS global networks

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1. Introduction

The use of an appropriate terrestrial reference frame in order to describe the position of points on the earth, as well as its temporal variation, is a problem of both theoretical and practical importance. This is the zero order optimal design problem in the terminology of Grafarend (1974), extended from the space to the space-time domain.

A terrestrial reference frame consists of a particular point O, its origin and a triad of orthonormal vectors $\vec{\mathbf{e}}=[\vec{e}_1\ \vec{e}_2\ \vec{e}_3]$, and it is related to a quasi-intertial orthonormal frame $\vec{\mathbf{e}}^I=[\vec{e}_1^I\ \vec{e}_3^I\ \vec{e}_2^I]$ with origin at the geocenter C. A point P of the earth has position vector $\vec{x}=\overrightarrow{OP}=x^1\vec{e}_1+x^2\vec{e}_2+x^3\vec{e}_3=\vec{\mathbf{e}}\mathbf{x}$ and terrestrial coordinates $\mathbf{x}=[x^1 x^2 x^3]^T$; its inertial position vector is $\vec{x}_I\equiv\overrightarrow{CP}=x_I^1\vec{e}_1^I+x_I^2\vec{e}_2^I+x_I^3\vec{e}_3^I=\vec{\mathbf{e}}^I\mathbf{x}_I$ and it has inertial coordinates $\mathbf{x}_I=[x_I^1 x_I^2 x_I^2]^T$. To relate the two frames, we need the displacement vector $\vec{d}\equiv\overrightarrow{CO}=\vec{\mathbf{e}}^I\mathbf{d}_I=\vec{\mathbf{e}}\,\mathbf{d}$ and the components Q_i^k of the elements of the terrestrial triad with respect to the inertial ones, i.e. $\vec{e}_i=\sum_k\vec{e}_k^I Q_i^k$, or $\vec{\mathbf{e}}=\vec{\mathbf{e}}^I\mathbf{Q}$ in matrix form. The matrix $\mathbf{Q}$ is a proper orthogonal matrix ($\mathbf{Q}^{-1}=\mathbf{Q}^T$, $|\mathbf{Q}|=+1$) as a consequence of the orthonormality and common orientation (right-handed) of $\vec{\mathbf{e}}$ and $\vec{\mathbf{e}}^I$. The relation between the two frames is expressed by

$$\vec{x}_I=\vec{\mathbf{e}}^I\mathbf{x}_I\equiv\overrightarrow{CP}=\overrightarrow{CO}+\overrightarrow{OP}=\vec{d}+\vec{x}=\vec{\mathbf{e}}^I\mathbf{d}_I+\vec{\mathbf{e}}\mathbf{x}=\vec{\mathbf{e}}^I\mathbf{d}_I+\vec{\mathbf{e}}^I\mathbf{Q}\mathbf{x}\,, \tag{1}$$

or in component form

$$\mathbf{x}_I=\mathbf{d}_I+\mathbf{Q}\mathbf{x}\,,\qquad \mathbf{x}=\mathbf{Q}^T(\mathbf{x}_I-\mathbf{d}_I)=\mathbf{Q}^T\mathbf{x}_I-\mathbf{d}\,. \tag{2}$$

The motion of any earth point in inertial space described by the function $\mathbf{x}_I(t)$, where t denotes time, should be determined by observations. The corresponding motion $\mathbf{x}(t)$, with respect to a terrestrial frame, depends in addition on the more or less arbitrary choice of the terrestrial frame, i.e. of the functions $\mathbf{Q}(t)$ and $\mathbf{d}(t)$. If the earth was rigid (or in applications where rigidity is a valid approximation) there are choices of $\mathbf{Q}(t)$ and $\mathbf{d}(t)$, such that the terrestrial coordinates $\mathbf{x}=\mathbf{Q}^T(t)\mathbf{x}_I(t)-\mathbf{d}(t)$ are constant. For a deformable earth, where deformations are known to be small, it is reasonable to establish a terrestrial frame in a way that the temporal variations of $\mathbf{x}(t)=\mathbf{Q}^T(t)\mathbf{x}_I(t)-\mathbf{d}(t)$ appear to be "as small as possible", i.e., such that the largest part of the inertial motions $\mathbf{x}_I(t)$ is absorbed by the rotation and position of the terrestrial frame. The optimal choice of the terrestrial frame depends directly on the specific optimality criterion, which gives concrete mathematical meaning to the loose expression "as small as possible".

The solution of the problem requires, apart from the choice of the optimality criterion, the knowledge of the motion $\mathbf{x}_I(t)$ with respect to the inertial frame, of every point of the earth. Operationally, this is possible only for points on the surface of the earth, while the motion of internal points has to be deduced from theoretical arguments.

The general form of an optimality criterion is

$$\int_{t_0}^{t_F}\int_E F\left(x(t),\frac{d\mathbf{x}}{dt}(t)\right)d\mathbf{x}\,dt=\min\ , \tag{3}$$

where F is an appropriate known function and integration is carried out over the earth E and the time interval $[t_0, t_F]$ for which observational data are available.

A more modest problem is the maintenance of a reference frame for a set of discrete points P_i, $i=1,\ldots,n$, which are the positions of observation stations distributed all over the world and engaged in a collective analysis of the acquired data.

Formerly, the problem of frame definition was solved in a discrete way, corresponding to discrete data $\mathbf{x}_i(t_k)=\mathbf{x}(P_i,t_k)$, collected in repeated campaigns over short time intervals, which could efficiently considered as "instantaneous" data corresponding to a single epoch t_k. The most popular approach starts with a more or less arbitrary frame definition at the initial epoch t_0 and then fits the coordinates of each epoch t_k to those already for the previous epoch t_{k-1}, by applying the optimality criterion

$$\sum_{i=1}^{n}[\mathbf{x}_i(t_k)-\mathbf{x}_i(t_{k-1})]^T[\mathbf{x}_i(t_k)-\mathbf{x}_i(t_{k-1})]=\min\ . \tag{4}$$

Setting

$$\mathbf{x}(t_k)=\begin{bmatrix}\mathbf{x}_1(t_k)\\ \vdots\\ \mathbf{x}_n(t_k)\end{bmatrix}, \tag{5}$$

and introducing the notation $\mathbf{x}^0=\mathbf{x}(t_{k-1})$, $\mathbf{x}=\mathbf{x}(t_k)$, $\delta\mathbf{x}=\mathbf{x}-\mathbf{x}^0$, the optimality criterion $\delta\mathbf{x}^T\delta\mathbf{x}=\min$ can be incorporated in a linearized adjustment model $\mathbf{l}=\mathbf{A}\delta\mathbf{x}+\mathbf{v}$, where linearization is based on the approximate values $\mathbf{x}^0=\mathbf{x}(t_{k-1})$, by introducing a set of *inner constraints* $\mathbf{E}\delta\mathbf{x}=\mathbf{0}$, where the rows of $\mathbf{E}$ form a basis for the null space $N(\mathbf{A})=\{\mathbf{x}|\mathbf{A}\mathbf{x}=\mathbf{0}\}$ of the matrix $\mathbf{A}$.

Nowadays, observation stations are engaged in continuous data collection, operating as permanent stations within the framework of international organizations, such as the International GPS Service (IGS) and provide the means for the establishment of an official International Terrestrial Reference System (ITRS) with the care of the International Earth Rotation Service (IERS).

> *Remark*:
> The term reference frame used here corresponds to the term ITRS used by IERS, which preserves the term International Terrestrial Reference Frame (ITRF) to the set of stations engaged in the realization of the ITRS.

With the availability of continuously available data the stepwise approach of the past is no more desirable or practical from the implementation point of view. Instead, we propose to visualize the data (which are discrete but with a very high rate of repetition) as continuous and to seek a time-continuous solution to the reference frame choice problem. Such a solution may be eventually discretized or implemented in a discrete approximation. Furthermore, the optimality criterion to be introduced for the frame choice in the discrete point network, must coincide with (or at least attempt to imitate) the optimality criterion introduced for the whole earth on the basis of theoretical considerations.

We will distinguish between two types of networks, which we call for the sake of convenience VLBI- and GPS-type networks. In VLBI-type networks where observations are translation-invariant, the position of the geocenter C cannot be determined and the origin of the frame O must be also introduced. Thus we must determine both functions $\mathbf{Q}(t)$ and $\mathbf{d}(t)$. In GPS-type networks observations are linked to the geocenter through the use of satellite orbits, in which case $O\equiv C$ is already known and only $\mathbf{Q}(t)$ should be determined.

In Dermanis (1995) we introduced a methodology for the solution to the space-time datum problem, which considered also scale transformations. Here we will restrict ourselves to rigid transformations, since scale is provided for both VLBI- and GPS-type networks, within the framework of a non-relativistic approach, through the assumption that mean of the readings of a set of reference clocks does not accelerate or decelerate with respect to Newtonian time. Distance (and thus scale) is entering the problem only implicitly through the observation of time intervals. On the other hand we will generalize the approach by looking into alternative optimality criteria and also by introducing "masses" or "weights" m_i, for each station P_i, which may reflect either a measure of the quality of station data, or the degree of participation of the station to the optimality criterion, in relation to the part of earth masses closest to the particular point.

2. Transformation from a preliminary reference frame to an optimal one

The basic idea of our approach is to make use of the fact that the available observations can very well determine the shape of the network, but not the additional information of its orientation (and position) with respect to a reference frame, which is contained in a set of network coordinates $\mathbf{x}$. At any single epoch t there exist an infinite number of coordinates $\mathbf{x}(t)$ which give rise to the same shape for the network.
If $\mathbf{x}'(t)$ and $\mathbf{x}(t)$ are two coordinate sets which both correspond to the "observed" shape at epoch t, there exists a rigid transformation between the two, defined point-wise by

$$\mathbf{x}'_i(t) = \mathbf{R}(\boldsymbol{\theta}(t))\mathbf{x}_i(t) + \mathbf{b}(t), \tag{6}$$

where $\mathbf{R}$ is a proper orthogonal matrix. This means that if a preliminary solution $\mathbf{x}(t)$ is available, we can switch to an optimal solution $\mathbf{x}'(t)$, by applying an optimization principle and determining the optimal six functions $\boldsymbol{\theta}(t) = [\theta_1(t)\,\theta_2(t)\,\theta_3(t)]^T$ and $\mathbf{b}(t) = [b_1(t)\,b_2(t)\,b_3(t)]^T$ which transform to the coordinates $\mathbf{x}'(t)$ satisfying the optimality criterion. But such a solution is always available, because a reference frame must be introduced for the analysis of the data which lead to the coordinate estimates $\mathbf{x}(t)$ at every epoch t. The only requirement is that the function $\mathbf{x}(t)$ is a smooth one, i.e. continuous with continuous derivatives up to a certain order. The reference frame for $\mathbf{x}(t)$ may be introduced during the adjustment of the observations, by a set of minimal constraints, which define a frame without any influence on the shape of the network.
The original known rotation $\mathbf{Q}(t)$ and displacement $\mathbf{d}(t)$ in $\mathbf{x} = \mathbf{Q}^T\mathbf{x}_I - \mathbf{d}$ must be combined with the optimal relative rotation $\mathbf{R}(t)$ and relative displacement $\mathbf{b}(t)$ in $\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{b}$, in order to obtain the final ones $\mathbf{Q}'(t)$ and $\mathbf{d}'(t)$ in $\mathbf{x}' = \mathbf{Q}'^T\mathbf{x}_I - \mathbf{d}'$ by means of

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{b} = \mathbf{R}(\mathbf{Q}^T\mathbf{x}_I - \mathbf{d}) + \mathbf{b} = \mathbf{R}\mathbf{Q}^T\mathbf{x}_I - \mathbf{R}\mathbf{d} + \mathbf{b} = \mathbf{Q}'^T\mathbf{x}_I - \mathbf{d}' \quad \Rightarrow \quad \mathbf{Q}' = \mathbf{Q}\mathbf{R}^T \ \& \ \mathbf{d}' = \mathbf{R}\mathbf{d} - \mathbf{b}. \tag{7}$$

3. Optimal solutions of minimum energy and minimum length (geodesics)

A particular optimality criterion is based on the instantaneous (relative to the terrestrial frame) kinetic energy of the earth $T(t) = \frac{1}{2}\int_E |\mathbf{v}|^2\, dm$, where $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ is the velocity, which is integrated over a time interval $[t_0, t_F]$ and the resulting total energy is minimized

$$\int_{t_0}^{t_F} T_E(t)\, dt = \frac{1}{2}\int_{t_0}^{t_F}\int_E \mathbf{v}^T\mathbf{v}\, dm\, dt = \min. \tag{8}$$

The discrete analog for a network of points P_i, $i = 1, \ldots, n$, each of which has optimal coordinates $\mathbf{x}'_i$ and it is assigned a mass m_i, takes the form

$$\int_0^t T_{\mathrm{N}}(\mathrm{t})\,dt = \tfrac{1}{2}\int_0^t \left(\frac{d\mathbf{x}'}{dt}\right)^T \mathbf{M}\frac{d\mathbf{x}'}{dt}\,dt = \min\ , \tag{9}$$

where

$$T_N(t) = \frac{1}{2}\left(\frac{d\mathbf{x}'}{dt}\right)^T \mathbf{M}\frac{d\mathbf{x}'}{dt} = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\frac{d\mathbf{x}'_i}{dt}\right)^T \frac{d\mathbf{x}'_i}{dt} = \frac{1}{2}\sum_{i=1}^{n} m_i \mathbf{v}'^T_i \mathbf{v}'_i = \frac{1}{2}\sum_{i=1}^{n} m_i |\mathbf{v}'_i|^2\ , \qquad (M_{ik} = \delta_{ik} m_i), \tag{10}$$

is the kinetic energy of the network.
The minimization principle (9) is in fact equivalent to the minimization principle

$$\int_{t_0}^{t_F} \sqrt{\left(\frac{d\mathbf{x}'}{dt}\right)^T \mathbf{M}\frac{d\mathbf{x}'}{dt}}\,dt = \int_{t_0}^{t_F} \sqrt{d\mathbf{x}'^T \mathbf{M} d\mathbf{x}'} = \int_{t_0}^{t_F} d\mathrm{s} = s|_{t_0}^{t_F} = \min \tag{11}$$

which is leading to a solution $\mathbf{x}'(t)$ which is a geodesic curve (curve of minimum length $s|_{t_0}^{t_F}$) in the network coordinate space X where any set of network coordinates $\mathbf{x}$ belong, $\mathbf{x}\in X$. Distance is measured by an element of length ds defined by $ds^2 = d\mathbf{x}'^T \mathbf{M} d\mathbf{x}' = \sum_{i=1}^{n} m_i d\mathbf{x}'^T_i d\mathbf{x}'_i$. This means that the "distance" between two network coordinate sets $\mathbf{x}'$ and $\mathbf{x}''$ is measured by

$$\rho(\mathbf{x}',\mathbf{x}'') = \|\mathbf{x}' - \mathbf{x}''\| = \sqrt{\sum_{i=1}^{n} m_i (\mathbf{x}'_i - \mathbf{x}''_i)^T (\mathbf{x}'_i - \mathbf{x}''_i)} = \sqrt{\sum_{i=1}^{n} m_i d_i^2}\ , \tag{12}$$

where $d_i = \|\mathbf{x}'_i - \mathbf{x}''_i\|$ is the usual euclidean distance between the two positions of point P_i . The minimization problem (10) is a standard problem of the calculus of variations. Its solution $\mathbf{x}'(t)$, described by means of the curvilinear coordinates

$$\mathbf{u}(s) = \begin{bmatrix} \boldsymbol{\theta}(s) \\ \mathbf{b}(s) \\ t(s) \end{bmatrix} \tag{13}$$

expressed as functions of arc length s , satisfies the Euler-Lagrange differential equations

$$\frac{\partial L}{\partial \boldsymbol{\theta}} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}}\right) = \mathbf{0}\ , \qquad \frac{\partial L}{\partial \mathbf{b}} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\mathbf{b}}}\right) = \mathbf{0}\ , \qquad \frac{\partial L}{\partial t} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right) = 0\ , \qquad \left(\dot{\boldsymbol{\theta}} = \frac{d\boldsymbol{\theta}}{ds},\ \dot{\mathbf{b}} = \frac{d\mathbf{b}}{ds},\ \dot{t} = \frac{dt}{ds}\right). \tag{14}$$

The derivation of the explicit form of the Euler-Lagrange equations has been carried out in Dermanis (1995), for the special case $\mathbf{M} = \mathbf{I}$, but they can be easily generalized to the present case of varying point masses m_i . Additionally the geodesic differential equations, corresponding to the optimality criterion (11) have been derived, yielding (as expected) identical results.

The resulting equations are

$$[(\mathbf{R}^T \boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt})\times]\left(\mathbf{h}_x + \mathbf{C}_x \mathbf{R}^T \boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt}\right) + \left[\frac{d\mathbf{h}_x}{dt} + \frac{d\mathbf{C}_x}{dt}\mathbf{R}^T \boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt} + \mathbf{C}_x \mathbf{R}^T \left(\frac{d\boldsymbol{\Omega}}{dt}\frac{d\boldsymbol{\theta}}{dt} + \boldsymbol{\Omega}\frac{d^2\boldsymbol{\theta}}{dt^2}\right)\right] - \frac{\ddot{s}}{\dot{s}}\left(\mathbf{h}_x + \mathbf{C}_x \mathbf{R}^T \boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt}\right) = \mathbf{0} \tag{15}$$

$$\frac{d^2\mathbf{b}}{dt^2}-\frac{\ddot{s}}{\dot{s}}\frac{d\mathbf{b}}{dt}=\mathbf{0} \tag{16}$$

$$\left(\frac{d\mathbf{x}}{dt}\right)^T\frac{d^2\mathbf{x}}{dt^2}+\mathbf{h}_x^T\mathbf{R}^T\left(\frac{d\mathbf{\Omega}}{dt}\frac{d\boldsymbol{\theta}}{dt}+\mathbf{\Omega}\frac{d^2\boldsymbol{\theta}}{dt^2}\right)-\frac{1}{2}\left(\frac{d\boldsymbol{\theta}}{dt}\right)^T\mathbf{\Omega}^T\mathbf{R}\frac{d\mathbf{C}_x}{dt}\mathbf{R}^T\mathbf{\Omega}\frac{d\boldsymbol{\theta}}{dt}-\frac{\ddot{s}}{\dot{s}}\left[\left(\frac{d\mathbf{x}}{dt}\right)^T\frac{d\mathbf{x}}{dt}+\mathbf{h}_x^T\mathbf{R}^T\mathbf{\Omega}\frac{d\boldsymbol{\theta}}{dt}\right]=0 \tag{17}$$

where $\mathbf{C}_x$ is the moment of inertia matrix of the network with respect to the initial reference frame

$$\mathbf{C}_x=-\sum_{i=1}^{n}m_i[\mathbf{x}_i\times]^2=\sum_{i=1}^{n}m_i[(\mathbf{x}_i^T\mathbf{x}_i)-\mathbf{x}_i\mathbf{x}_i^T]=\mathbf{x}^T\mathbf{M}\mathbf{x}-\sum_{i=1}^{n}m_i\mathbf{x}_i\mathbf{x}_i^T\ , \tag{18}$$

$\mathbf{h}_x$ is its relative angular momentum vector of the network with respect to the initial reference frame

$$\mathbf{h}_x=\sum_{i=1}^{n}m_i[\mathbf{x}_i\times]\dot{\mathbf{x}}_i \tag{19}$$

and $\mathbf{\Omega}=\mathbf{\Omega}(\boldsymbol{\theta})$ is a matrix defined by

$$[\boldsymbol{\omega}_k\times]\equiv\frac{\partial\mathbf{R}}{\partial\theta_k}\mathbf{R}^T\ ,\qquad \mathbf{\Omega}=[\boldsymbol{\omega}_1\boldsymbol{\omega}_2\boldsymbol{\omega}_3]\ . \tag{20}$$

Remark: We make repeated use of the notation

$$\mathbf{a}=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\qquad\Rightarrow\qquad[\mathbf{a}\times]\equiv\begin{bmatrix}0&-a_3&a_2\\a_3&0&-a_1\\-a_2&a_1&0\end{bmatrix} \tag{21}$$

and of the properties ($\mathbf{Q}^{-1}=\mathbf{Q}^T$)

$$[\mathbf{a}\times]\mathbf{b}=-[\mathbf{b}\times]\mathbf{a}\ ,\qquad[(\mathbf{Q}\mathbf{a})\times]=\mathbf{Q}[\mathbf{a}\times]\mathbf{Q}^T\ ,\qquad[\mathbf{a}\times][\mathbf{b}\times]=\mathbf{b}\mathbf{a}^T-(\mathbf{a}^T\mathbf{b})\mathbf{I}\ . \tag{22}$$

We have also assumed that reference frame $\mathbf{x}(t)$ has been chosen in a way that $\bar{\mathbf{x}}=\mathbf{0}$, where $\bar{\mathbf{x}}$ are the coordinates of the center of mass of the network, defined by

$$\bar{\mathbf{x}}=\frac{1}{m}\sum_{i=1}^{n}m_i\mathbf{x}_i\ ,\qquad m=\sum_{i=1}^{n}m_i\ . \tag{23}$$

(It is always possible to switch from any reference frame $\mathbf{x}_0(t)$ to a "centered" one $\mathbf{x}(t)$ with $\bar{\mathbf{x}}=\mathbf{0}$, using $\mathbf{x}_i(t)=\mathbf{x}_{0i}(t)-\bar{\mathbf{x}}_0$).

Of the seven equations (15), (16), (17), the last one can be solved for the factor $\frac{\ddot{s}}{\dot{s}}$ (s been length and dots denoting differentiation with respect to time) which replaced in the first two, will yield a system of six non-linear differential equations for the six unknown functions $\boldsymbol{\theta}(t)=[\theta_1(t)\,\theta_2(t)\,\theta_3(t)]^T$ and $\mathbf{b}(t)=[b_1(t)\,b_2(t)\,b_3(t)]^T$. The resulting equations are very complicated and they can be solved only by numerical methods. Furthermore any solution yields a frame definition where the "curve" $\mathbf{x}'(t)$ is the closest between its end points $\mathbf{x}'(t_0)$ and $\mathbf{x}'(t_F)$, but not necessarily the shortest possible. To arrive at a truly optimal solution, which by the way is not unique, we must select the optimal among all initial (or boundary) values which are necessary for obtaining a specific solution of the relevant differential equations.

We will not pursue this matter any further, but we will follow a different approach motivated by the methods used in the theoretical study of earth rotations.

4. Tisserand axes

The rotation of the earth is governed by the differential equation $\frac{d\vec{h}}{dt}=\vec{l}$, where $\vec{h}=\int_E \vec{x}\times\vec{v}\, dm$ is the angular momentum, $\vec{l}=\int_E \vec{x}\times\vec{a}\, dm$ is the acting torque, $\vec{v}=\frac{d\vec{x}}{dt}$ are the velocities of earth points and $\vec{a}$ the corresponding acting accelerations. The rotational equation which in the inertial frame becomes simply $\frac{d\mathbf{h}_I}{dt}=\mathbf{l}_I$, obtains a more complicated form when expressed with respect to the terrestrial frame. Differentiation of $\vec{\mathbf{e}}=\vec{\mathbf{e}}^I\mathbf{Q}$, yields $\frac{d\vec{\mathbf{e}}}{dt}=\vec{\mathbf{e}}^I\frac{d\mathbf{Q}}{dt}=\vec{\mathbf{e}}\mathbf{Q}^T\frac{d\mathbf{Q}}{dt}\equiv\vec{\mathbf{e}}[\boldsymbol{\omega}\times]$, where $\vec{\omega}=\vec{\mathbf{e}}\boldsymbol{\omega}$ is the instantaneous rotation vector of the earth, so that $\vec{v}=\frac{d\vec{x}}{dt}=\frac{d}{dt}(\vec{\mathbf{e}}\mathbf{x})=\vec{\mathbf{e}}(\frac{d\mathbf{x}}{dt}+[\boldsymbol{\omega}\times]\mathbf{x})$ and similarly $\frac{d\vec{h}}{dt}=\vec{\mathbf{e}}(\frac{d\mathbf{h}}{dt}+[\boldsymbol{\omega}\times]\mathbf{h})=\vec{l}=\vec{\mathbf{e}}\,\mathbf{l}$. The components of $\vec{h}=\vec{\mathbf{e}}\mathbf{h}$ in the terrestrial system become

$$\mathbf{h}=\int_E[\mathbf{x}\times]\left(\frac{d\mathbf{x}}{dt}+[\boldsymbol{\omega}\times]\mathbf{x}\right)dm=\left(-\int_E[\mathbf{x}\times][\mathbf{x}\times]dm\right)\boldsymbol{\omega}+\int_E[\mathbf{x}\times]\frac{d\mathbf{x}}{dt}\,dm\equiv\mathbf{C}\boldsymbol{\omega}+\mathbf{h}_R \tag{24}$$

where $\mathbf{C}$ is the inertia matrix and $\mathbf{h}_R$ the relative angular momentum of the earth. Replacing $\mathbf{h}=\mathbf{C}\boldsymbol{\omega}+\mathbf{h}_R$ in the rotational equations $\frac{d\mathbf{h}}{dt}+[\boldsymbol{\omega}\times]\mathbf{h}=\mathbf{l}$ yields the *Liouville equations*

$$\mathbf{C}\frac{d\boldsymbol{\omega}}{dt}+\frac{d\mathbf{C}}{dt}\boldsymbol{\omega}+\frac{d\mathbf{h}_R}{dt}+[\boldsymbol{\omega}\times](\mathbf{C}\boldsymbol{\omega}+\mathbf{h}_R)=\mathbf{l}\,. \tag{25}$$

The choice of the terrestrial frame in the study of earth rotation is dictated by the need to simplify the analytical work involved in solving the Liouville equations.
Two choices are under consideration (Munk & MacDonald, 1960, ch. 3.2, p. 10): the *principal axes* or *figure axes*, defined so that $\mathbf{C}$ becomes diagonal, and the *Tisserant axes* for which the relative angular momentum vanishes, $\mathbf{h}_R=\mathbf{0}$. The first choice is more appropriate for the theory of rotation of a rigid earth but it has a serious shortcoming when an elastic earth model is used: as a consequence of rotational elastic deformation the third (polar) axis of figure intersecting the earth at a point F , undergoes a diurnal rotation around the corresponding position F_0 of the rigid earth model with a radius of $F_0F=60\,\mathrm{m}$, while F_0 undergoes a rotation around the position O of third Tisserand axis, with radius of only $OF_0=2\,\mathrm{m}$ and a Chandler period of about 430 days (Moritz and Mueller, 1987, ch. 3.3.1). For this reason the Tisserand axes are the preferred ones for the description of the rotation of the deformable earth. Furthermore the Tisserand axes have the advantage that they minimize the relative to the terrestrial frame kinetic energy of the earth, i.e., $T_E=\int_E|\mathbf{v}|^2\,dm=\min$ (Moritz and Mueller, 1987, ch. 3.1). Both choices of figure and Tisserand axes, cannot determine a displacement but only the rotation from an initial arbitrary reference frame. In theory they are both considered to be geocentric.
The figure axes are uniquely defined for any body that has no axis of symmetry. They are therefore well defined for the real earth, but not for an ellipsoidal model-earth where only one direction (that of symmetry) coincides with one figure axes and the position of the other two must be arbitrarily chosen.
On the contrary the Tisserant axes are not uniquely defined. Indeed if $\mathbf{x}$ are coordinates with respect to a set of Tisserand axes and we consider a new set of axes defined by the transformation $\tilde{\mathbf{x}}=\mathbf{S}\mathbf{x}$, where $\mathbf{S}$ is a time-independent orthogonal matrix then

$$\tilde{\mathbf{h}}_R=\int_E[\tilde{\mathbf{x}}\times]\tilde{\mathbf{v}}\,dm=\int_E[(\mathbf{S}\mathbf{x})\times]\mathbf{S}\mathbf{v}\,dm=\int_E\mathbf{S}[\mathbf{x}\times]\mathbf{S}^T\mathbf{S}\mathbf{v}\,dm=\mathbf{S}\int_E[\mathbf{x}\times]\mathbf{v}\,dm=\mathbf{S}\mathbf{h}_R=\mathbf{S}\mathbf{0}=\mathbf{0} \tag{26}$$

and the $\tilde{\mathbf{x}}$ axes are also Tisserand axes. To choose a particular set of Tisserand axes we must fix their position $\mathbf{x}(t_0)$ at an initial epoch t_0.

For a discrete network of mass points we may define a set of "Tisserand" axes by setting the corresponding relative momentum equal to zero

$$\mathbf{h}_{x'} \equiv \sum_i m_i [\mathbf{x}'_i \times] \frac{d\mathbf{x}'_i}{dt} = \mathbf{0} \tag{27}$$

and try to find the transformation parameters $\boldsymbol{\theta}(t)$, $\mathbf{b}(t)$ which convert coordinates $\mathbf{x}(t)$ in an originally available reference frame into "Tisserand" coordinates $\mathbf{x}'(t) = \mathbf{R}(\boldsymbol{\theta}(t))\mathbf{x}(t) + \mathbf{b}(t)$.

Setting

$$[\boldsymbol{\omega}_k \times] = \frac{\partial \mathbf{R}}{\partial \theta_k} \mathbf{R}^T , \qquad \frac{\partial \mathbf{R}}{\partial \theta_k} = [\boldsymbol{\omega}_k \times] \mathbf{R} \tag{28}$$

we have

$$\frac{d\mathbf{R}}{dt} = \frac{\partial \mathbf{R}}{\partial \theta_1}\frac{d\theta_1}{dt} + \frac{\partial \mathbf{R}}{\partial \theta_2}\frac{d\theta_2}{dt} + \frac{\partial \mathbf{R}}{\partial \theta_3}\frac{d\theta_3}{dt} = \frac{d\theta_1}{dt}[\boldsymbol{\omega}_1 \times]\mathbf{R} + \frac{d\theta_2}{dt}[\boldsymbol{\omega}_2 \times]\mathbf{R} + \frac{d\theta_3}{dt}[\boldsymbol{\omega}_3 \times]\mathbf{R} \tag{29}$$

$$\frac{d\mathbf{x}'_i}{dt} = \frac{d\mathbf{R}}{dt}\mathbf{x}_i + \mathbf{R}\frac{d\mathbf{x}_i}{dt} + \frac{d\mathbf{b}}{dt} = \frac{d\theta_1}{dt}[\boldsymbol{\omega}_1 \times]\mathbf{R}\mathbf{x}_i + \frac{d\theta_2}{dt}[\boldsymbol{\omega}_2 \times]\mathbf{R}\mathbf{x}_i + \frac{d\theta_3}{dt}[\boldsymbol{\omega}_3 \times]\mathbf{R}\mathbf{x}_i + \mathbf{R}\frac{d\mathbf{x}_i}{dt} + \frac{d\mathbf{b}}{dt} =$$

$$= -\frac{d\theta_1}{dt}[(\mathbf{R}\mathbf{x}_i)\times]\boldsymbol{\omega}_1 - \frac{d\theta_2}{dt}[(\mathbf{R}\mathbf{x}_i)\times]\boldsymbol{\omega}_2 - \frac{d\theta_3}{dt}[(\mathbf{R}\mathbf{x}_i)\times]\boldsymbol{\omega}_3 + \mathbf{R}\frac{d\mathbf{x}_i}{dt} + \frac{d\mathbf{b}}{dt} =$$

$$= -[(\mathbf{R}\mathbf{x}_i)\times]\left(\frac{d\theta_1}{dt}\boldsymbol{\omega}_1 + \frac{d\theta_2}{dt}\boldsymbol{\omega}_2 + \frac{d\theta_3}{dt}\boldsymbol{\omega}_3\right) + \mathbf{R}\frac{d\mathbf{x}_i}{dt} + \frac{d\mathbf{b}}{dt} =$$

$$= -[(\mathbf{R}\mathbf{x}_i)\times][\boldsymbol{\omega}_1 \boldsymbol{\omega}_2 \boldsymbol{\omega}_3]\begin{bmatrix} \frac{d\theta_1}{dt} \\ \frac{d\theta_2}{dt} \\ \frac{d\theta_3}{dt} \end{bmatrix} + \mathbf{R}\frac{d\mathbf{x}_i}{dt} + \frac{d\mathbf{b}}{dt} \equiv -[(\mathbf{R}\mathbf{x}_i)\times]\,\boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt} + \mathbf{R}\frac{d\mathbf{x}_i}{dt} + \frac{d\mathbf{b}}{dt} =$$

$$= -\mathbf{R}[\mathbf{x}_i \times]\mathbf{R}^T \boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt} + \mathbf{R}\frac{d\mathbf{x}_i}{dt} + \frac{d\mathbf{b}}{dt} \tag{30}$$

Therefore, the vanishing relative momentum becomes

$$\mathbf{h}_x = \sum_i m_i [\mathbf{x}'_i \times]\frac{d\mathbf{x}'_i}{dt} = \sum_i m_i \left(\mathbf{R}[\mathbf{x}_i \times]\mathbf{R}^T + [\mathbf{b}\times]\right)\left(-\mathbf{R}[\mathbf{x}_i \times]\mathbf{R}^T \boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt} + \mathbf{R}\frac{d\mathbf{x}_i}{dt} + \frac{d\mathbf{b}}{dt}\right) =$$

$$=\sum_i m_i\left(-\mathbf{R}[\mathbf{x}_i\times][\mathbf{x}_i\times]\mathbf{R}^T\boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt}+\mathbf{R}[\mathbf{x}_i\times]\frac{d\mathbf{x}_i}{dt}+\mathbf{R}[\mathbf{x}_i\times]\mathbf{R}^T\frac{d\mathbf{b}}{dt}\right)+$$

$$+\sum_i m_i\left(-[\mathbf{b}\times]\mathbf{R}[\mathbf{x}_i\times]\mathbf{R}^T\boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt}+[\mathbf{b}\times]\mathbf{R}\frac{d\mathbf{x}_i}{dt}+[\mathbf{b}\times]\frac{d\mathbf{b}}{dt}\right)=$$

$$=-\mathbf{R}\left(\sum_i m_i[\mathbf{x}_i\times][\mathbf{x}_i\times]\right)\mathbf{R}^T\boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt}+\mathbf{R}\sum_i m_i[\mathbf{x}_i\times]\frac{d\mathbf{x}_i}{dt}+\mathbf{R}\left(\sum_i m_i[\mathbf{x}_i\times]\right)\mathbf{R}^T\frac{d\mathbf{b}}{dt}-$$

$$-[\mathbf{b}\times]\mathbf{R}\left(\sum_i m_i[\mathbf{x}_i\times]\right)\mathbf{R}^T\boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt}+[\mathbf{b}\times]\mathbf{R}\sum_i m_i\frac{d\mathbf{x}_i}{dt}+\left(\sum_i m_i\right)[\mathbf{b}\times]\frac{d\mathbf{b}}{dt}=$$

$$=-\mathbf{R}\mathbf{C}_x\mathbf{R}^T\boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt}+\mathbf{R}\mathbf{h}_x+m\mathbf{R}[\bar{\mathbf{x}}\times]\mathbf{R}^T\frac{d\mathbf{b}}{dt}-m[\mathbf{b}\times]\mathbf{R}[\bar{\mathbf{x}}\times]\mathbf{R}^T\boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt}+m[\mathbf{b}\times]\mathbf{R}\frac{d\bar{\mathbf{x}}}{dt}+m[\mathbf{b}\times]\frac{d\mathbf{b}}{dt}=\mathbf{0}\,. \quad (31)$$

Under the feasible assumption that $\bar{\mathbf{x}}=\mathbf{0}$, and the consequent $\frac{d\bar{\mathbf{x}}}{dt}=\mathbf{0}$, the last equation simplifies to

$$\mathbf{h}_{x'}=-\mathbf{R}\mathbf{C}_x\mathbf{R}^T\boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt}+\mathbf{R}\mathbf{h}_x+m[\mathbf{b}\times]\frac{d\mathbf{b}}{dt}=\mathbf{0} \quad (32)$$

These are three equations in six unknowns, which is an underdetermined system. This reflects the fact that the Tisserant principle $\mathbf{h}_{x'}=\mathbf{0}$ can determine the orientation but not the position of the (geocentric) Tisserant axes with respect to the original working frame. For GPS-type networks where the original frame is already geocentric, we have $\mathbf{b}(t)=\mathbf{0}$, by definition. For VLBI-type networks we will set again $\mathbf{b}(t)=\mathbf{0}$ to obtain the orientation of a Tisserand frame parallel to the geocentric Tisserant frame, with the same origin as the original frame. We need a separate optimization principle for the determination of an optimal origin of the network, since the position of the geocenter remains undeterminable from the available data.

With the choice $\mathbf{b}(t)=\mathbf{0}$, the three transformation parameters $\boldsymbol{\theta}(t)$ to "Tisserand" coordinates should be determined from the solution of the three differential equations $\mathbf{h}_{x'}=-\mathbf{R}\mathbf{C}_x\mathbf{R}^T\boldsymbol{\Omega}\frac{d\boldsymbol{\theta}}{dt}+\mathbf{R}\mathbf{h}_x=\mathbf{0}$, which under the additional assumption that $|\boldsymbol{\Omega}|\neq 0$ take the form

$$\frac{d\boldsymbol{\theta}}{dt}=\boldsymbol{\Omega}^{-1}\mathbf{R}\mathbf{C}_x^{-1}\mathbf{h}_x\,. \quad (33)$$

These can be integrated to obtain a solution

$$\boldsymbol{\theta}(t)=\boldsymbol{\theta}(t_0)+\int_{t_0}^{t}\boldsymbol{\Omega}^{-1}\big(\boldsymbol{\theta}(\tau)\big)\mathbf{R}\big(\boldsymbol{\theta}(\tau)\big)\mathbf{C}_x^{-1}(\tau)\,\mathbf{h}_x(\tau)\,d\tau \quad (34)$$

which depends on the chosen initial value $\boldsymbol{\theta}(t_0)$ which determines one out of all the possible Tisserand frames. For example if $\boldsymbol{\theta}(t_0)=\mathbf{0}$ is chosen, the Tisserand axes will coincide with the original axes at the initial epoch, since $\mathbf{x}'(t_0)=\mathbf{R}\big(\boldsymbol{\theta}(t_0)\big)\mathbf{x}(t_0)=\mathbf{R}(\mathbf{0})\mathbf{x}(t_0)=\mathbf{I}\,\mathbf{x}(t_0)=\mathbf{x}(t_0)$.

The explicit form of the differential equations (33) depends on the chosen parametrization of the rotation matrix $\mathbf{R}$ in terms of three parameters $\boldsymbol{\theta}$.

5. Equivalence of Tisserant axes to a space-time generalization of Meissl's inner constraints

A different type of solution can be based on the extension of the well known concept of inner constraints, introduced by Meissl (1965, 1969). At any epoch t, when the network has coordinates $\mathbf{x}(t)$ in an original reference frame, the set of all coordinates $\mathbf{x}'(t)=\mathbf{R}(\boldsymbol{\theta}(t))\mathbf{x}(t)+\mathbf{b}(t)$, resulting as the parameters $\boldsymbol{\theta}(t)$ and $\mathbf{b}(t)$ take any possible value, form a manifold, i.e. a "curved" subspace M_t of the network coordinate space X. In fact M_t is the set of all network coordinates which give rise to the same network shape as the one defined by $\mathbf{x}(t)$. Obviously $\mathbf{x}'(t)\in\mathsf{M}_t$ for any particular epoch t. The idea is now to impose on the curve $\mathbf{x}'(t)$ to be such that its velocity $\frac{d\mathbf{x}'}{dt}(t)$ is perpendicular to the manifold M_t, or more precisely to the "flat" space, which is tangent to the (cutved) manifold M_t at the point $\mathbf{x}'(t)$. Since the parameters $\boldsymbol{\theta}(t)$ and $\mathbf{b}(t)$ comprise a set of curvilinear coordinates for M_t, the tangent space is the set of all liner combinations of the vectors tangent to the coordinate curves, namely $\frac{\partial\mathbf{x}'}{\partial\theta_1}$, $\frac{\partial\mathbf{x}'}{\partial\theta_2}$, $\frac{\partial\mathbf{x}'}{\partial\theta_3}$, $\frac{\partial\mathbf{x}'}{\partial b_1}$, $\frac{\partial\mathbf{x}'}{\partial b_2}$, $\frac{\partial\mathbf{x}'}{\partial b_3}$. The orthogonality conditions $\frac{d\mathbf{x}'}{dt}\perp\frac{\partial\mathbf{x}'}{\partial\theta_k}$, $\frac{d\mathbf{x}'}{dt}\perp\frac{\partial\mathbf{x}'}{\partial b_k}$ take the form $\left(\frac{d\mathbf{x}'}{dt}\right)^T\mathbf{M}\frac{\partial\mathbf{x}'}{\partial\theta_k}=0$, $\left(\frac{d\mathbf{x}'}{dt}\right)^T\mathbf{M}\frac{\partial\mathbf{x}'}{\partial b_k}=0$, $k=1,2,3$, or in compact matrix notation

$$\left(\frac{\partial\mathbf{x}'}{\partial\boldsymbol{\theta}}\right)^T\mathbf{M}\frac{d\mathbf{x}'}{dt}=\sum_{i=1}^{n}m_i\left(\frac{\partial\mathbf{x}'_i}{\partial\boldsymbol{\theta}}\right)^T\frac{d\mathbf{x}'_i}{dt}=\mathbf{0}\,,\qquad \left(\frac{\partial\mathbf{x}'}{\partial\mathbf{b}}\right)^T\mathbf{M}\frac{d\mathbf{x}'}{dt}=\sum_{i=1}^{n}m_i\left(\frac{\partial\mathbf{x}'_i}{\partial\mathbf{b}}\right)^T\frac{d\mathbf{x}'_i}{dt}=\mathbf{0} \tag{35}$$

Replacing $\frac{\partial\mathbf{x}'_i}{\partial\boldsymbol{\theta}}=\mathbf{R}[\mathbf{x}_i\times]\mathbf{R}^T\boldsymbol{\Omega}$, $\frac{\partial\mathbf{x}'_i}{\partial\mathbf{b}}=\mathbf{I}$, and $\frac{d\mathbf{x}'_i}{dt}$ from (30), implementing the usual assumption that $\bar{\mathbf{x}}=\mathbf{0}$, $\frac{d\bar{\mathbf{x}}}{dt}=\mathbf{0}$, we arrive at

$$\boldsymbol{\Omega}^T\mathbf{R}\left(\mathbf{C}_x\mathbf{R}^T\frac{d\boldsymbol{\theta}}{dt}-\mathbf{h}_x\right)=\mathbf{0}\,,\qquad \frac{d\mathbf{b}}{dt}=\mathbf{0}\,. \tag{36}$$

The first one of (36) is equivalent to (33) and therefore the *inner constraint* or *Meissl frame* is a Tisserant frame! The second of (36) yields $\mathbf{b}=\text{constant}$ and provides a solution to the origin determination for VLBI-type networks: If we chosse $\mathbf{b}=\mathbf{0}$, this means that, in relation to the assumption $\bar{\mathbf{x}}=\mathbf{0}$, the network origin should remain at the "center of mass" of the network.

Any solution of (36) satisfies the geodesic or minimum energy equations (15), (16) and (17). Thus the Tisserand or Meissl frame solution $\mathbf{x}'(t)$ is a geodesic and even more it is a geodesic of minimum possible length among all geodesics, a property which follows from the fact that $\mathbf{x}'(t_0)\perp\mathsf{M}_{t_0}$ and $\mathbf{x}'(t_F)\perp\mathsf{M}_{t_F}$.

In order to see how the present solution is related to Meissl's concept of inner constraints, we must use $\frac{\partial\mathbf{x}'_i}{\partial\boldsymbol{\theta}}=[(\mathbf{R}\mathbf{x}_i)\times]\boldsymbol{\Omega}=[(\mathbf{x}'_i-\mathbf{b})\times]\boldsymbol{\Omega}$, $\frac{\partial\mathbf{x}'_i}{\partial\mathbf{b}}=\mathbf{I}$, $\bar{\mathbf{x}}=\mathbf{0}$ and $\frac{d\bar{\mathbf{x}}}{dt}=\mathbf{0}$, in order to rewrite the orthogonality conditions (35) in the form

$$\sum_i m_i[\mathbf{x}_i\times]\frac{d\mathbf{x}_i}{dt}=\mathbf{0}\,,\qquad \sum_i\frac{d\mathbf{x}_i}{dt}=\mathbf{0}\,. \tag{37}$$

Assume that the solution $\mathbf{x}'(t^0)$ has been determined at some epoch t^0 and we want to determine the solution at a slightly later epoch $t=t^0+\Delta t$, i.e. $\mathbf{x}'(t)=\mathbf{x}'(t^0+\Delta t)\approx\mathbf{x}'(t^0)+\frac{d\mathbf{x}'}{dt}(t^0)\,\Delta t$, using $\mathbf{x}'(t^0)=\mathbf{x}'^0$ as a starting approximate value. If $\mathbf{x}'_i$ is replaced by $\mathbf{x}_i^0$, $\frac{d\mathbf{x}'_i}{dt}$ is approximated by $\frac{\Delta\mathbf{x}_i}{\Delta t}=\frac{\mathbf{x}_i-\mathbf{x}_i^0}{\Delta t}$, and we choose $m_i=1$, equations (37) are converted to the well known inner constraints:

$$\sum_i [\mathbf{x}_i^0\times]\Delta\mathbf{x}_i=\mathbf{0}\,, \qquad \sum_i \Delta\mathbf{x}_i=\mathbf{0}\,. \tag{38}$$

6. An illustrative example

A particular choice of rotation parameters is

$$\mathbf{R}(\boldsymbol{\theta})=\mathbf{R}(\theta_1,\theta_2,\theta_3)=\mathbf{R}_3(\theta_3)\mathbf{R}_2(\theta_2)\mathbf{R}_1(\theta_1) \tag{39}$$

yielding

$$[\boldsymbol{\omega}_1\times]=\frac{\partial\mathbf{R}}{\partial\theta_1}\mathbf{R}^T=-\mathbf{R}_3(\theta_3)\mathbf{R}_2(\theta_2)\mathbf{R}_1(\theta_1)[\mathbf{i}_1\times]\mathbf{R}^T=-\mathbf{R}[\mathbf{i}_1\times]\mathbf{R}^T \tag{40}$$

$$[\boldsymbol{\omega}_2\times]=\frac{\partial\mathbf{R}}{\partial\theta_2}\mathbf{R}^T=-\mathbf{R}_3(\theta_3)\mathbf{R}_2(\theta_2)[\mathbf{i}_2\times]\mathbf{R}_1(\theta_1)\mathbf{R}^T=-\mathbf{R}\mathbf{R}_1(-\theta_1)[\mathbf{i}_2\times]\mathbf{R}_1(\theta_1)\mathbf{R}^T \tag{41}$$

$$[\boldsymbol{\omega}_3\times]=\frac{\partial\mathbf{R}}{\partial\theta_3}\mathbf{R}^T=-\mathbf{R}_3(\theta_3)[\mathbf{i}_3\times]\mathbf{R}_2(\theta_2)\mathbf{R}_1(\theta_1)\mathbf{R}^T=-\mathbf{R}\mathbf{R}_1(-\theta_1)\mathbf{R}_2(-\theta_2)[\mathbf{i}_3\times]\mathbf{R}_2(\theta_2)\mathbf{R}_1(\theta_1)\mathbf{R}^T \tag{42}$$

$$\boldsymbol{\omega}_1=-\mathbf{R}\mathbf{i}_1=-\mathbf{R}_3(\theta_3)\mathbf{R}_2(\theta_2)\mathbf{i}_1 \tag{43}$$

$$\boldsymbol{\omega}_2=-\mathbf{R}\mathbf{R}_1(-\theta_1)\mathbf{i}_2=-\mathbf{R}_3(\theta_3)\mathbf{i}_2 \tag{44}$$

$$\boldsymbol{\omega}_3=-\mathbf{R}\mathbf{R}_1(-\theta_1)\mathbf{R}_2(-\theta_2)\mathbf{i}_3=-\mathbf{i}_3 \tag{45}$$

We may set

$$\mathbf{Q}=[\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3]\equiv-\mathbf{R}^T\boldsymbol{\Omega}=-\mathbf{R}^T[\boldsymbol{\omega}_1\boldsymbol{\omega}_2\boldsymbol{\omega}_3]\,, \qquad \mathbf{q}_k=-\mathbf{R}^T\boldsymbol{\omega}_k \tag{46}$$

$$\mathbf{q}_1=\mathbf{i}_1 \tag{47}$$

$$\mathbf{q}_2=\mathbf{R}_1(-\theta_1)\mathbf{i}_2=\begin{bmatrix}0\\ \cos\theta_1\\ \sin\theta_1\end{bmatrix} \tag{48}$$

$$\mathbf{q}_3=\mathbf{R}_1(-\theta_1)\mathbf{R}_2(-\theta_2)\mathbf{i}_3=\begin{bmatrix}1&0&0\\0&\cos\theta_1&-\sin\theta_1\\0&\sin\theta_1&\cos\theta_1\end{bmatrix}\begin{bmatrix}\sin\theta_2\\0\\ \cos\theta_2\end{bmatrix}=\begin{bmatrix}\sin\theta_2\\-\sin\theta_1\cos\theta_2\\ \cos\theta_1\cos\theta_2\end{bmatrix} \tag{49}$$

$$\mathbf{Q}\dot{\boldsymbol{\theta}}=\mathbf{C}_z^{-1}\mathbf{h}_z \tag{50}$$

or setting

$$\mathbf{C}_z^{-1}\mathbf{h}_z\equiv\mathbf{c}=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} \tag{51}$$

$$\begin{bmatrix} 1 & 0 & \sin\theta_2 \\ 0 & \cos\theta_1 & -\sin\theta_1\cos\theta_2 \\ 0 & \sin\theta_1 & \cos\theta_1\cos\theta_2 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{52}$$

$$\dot\theta_1 + \sin\theta_2 \dot\theta_3 = c_1 \tag{53}$$

$$\cos\theta_1 \dot\theta_2 - \sin\theta_1 \cos\theta_2 \dot\theta_3 = c_2 \tag{54}$$

$$\sin\theta_1 \dot\theta_2 + \cos\theta_1 \cos\theta_2 \dot\theta_3 = c_3 \tag{55}$$

Inversion of the matrix **Q** gives

$$\mathbf{Q}^{-1} = \begin{bmatrix} 1 & 0 & \sin\theta_2 \\ 0 & \cos\theta_1 & -\sin\theta_1\cos\theta_2 \\ 0 & \sin\theta_1 & \cos\theta_1\cos\theta_2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \sin\theta_1\tan\theta_2 & -\cos\theta_1\tan\theta_2 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\dfrac{\sin\theta_1}{\cos\theta_2} & \dfrac{\cos\theta_1}{\cos\theta_2} \end{bmatrix} \tag{56}$$

and the differential equations become

$$\dot{\boldsymbol{\theta}} = \mathbf{Q}^{-1}\mathbf{C}_z^{-1}\mathbf{h}_z = \mathbf{Q}^{-1}\mathbf{c} \tag{57}$$

or explicitly

$$\dot\theta_1 = c_1 + \sin\theta_1 \tan\theta_2 c_2 - \cos\theta_1 \tan\theta_2 c_3 \tag{58}$$

$$\dot\theta_2 = \cos\theta_1 c_2 + \sin\theta_1 c_3 \tag{59}$$

$$\dot\theta_3 = -\frac{\sin\theta_1}{\cos\theta_2} c_2 + \frac{\cos\theta_1}{\cos\theta_2} c_3 \tag{60}$$

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