Diffusion with space memory

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Abstract

For a better resolution of the gravity values monitored on the surface of the Earth or underground is needed to analyze the time variation of the elevation at the measurement site. An important variation of the elevation is due to the effect of the pore filling of the ground caused by the migration processes of the underground water often associated to the ocean tides.
In order to obtain a better representation of the diffusion processes of fluids the Darcy's law has been modified introducing a general *time* memory formalism represented by fractional derivatives which imply a time filtering of the pressure gradient without singularities (Caputo, 1998a, 1999); a model which is particularly valid when considering the local phenomenology. In this note we introduce in Darcy's law the *space* fractional derivatives of the pressure which seems appropiate when considering a half space in order to represent the effect of the medium previously affected by the fluid.
We find the Green function for the general boundary and initial value problem. In particular, we discuss the initial value problem when the pressure and its space derivatives are nil on the boundary at any time while the pressure in the medium is constant at the initial time and also the problem when on the boundary the pressure is constant while its first and second order derivative are there nil at any time and the initial value of the pressure in the medium is nil.
Keywords: Porous media, Diffusion, Memory, Fractional derivative.

Glossary

k ($s^{-2}m^2$)	ratio of the fluid pressure to the fluid density [see Eq.(2)].
n	fractional order of differentation [see Eq.(3) and Eq.(4)] (dimensionless).
$p(x,t)$ ($kg\ s^{-2}m^{-1}$)	fluid pressure.
$q(x,t)$ ($kg\ s^{-1}m^{-2}$)	fluid mass flow rate in the porous medium.
r, R	radius of the inner and outer circles, respectively, of the integration path of Eq.(A1) shown in Fig.3.
t (s)	time.
x (m)	distance from the boundary plane.
α ($s\ m^n$)	coefficient of the Darcy's law modified [see Eq.(3)].
αk ($s^{-1}m^{2+n}$)	pseudodiffusivity.
β (s)	coefficient of the classic Darcy's law [see Eq.(3)].
$\rho(x,t)$ ($kg\ m^{-3}$)	fluid density.
ω, ε	imaginary and real parts in the plane of the integral in Eq.(A1).

1. Introduction.

In monitoring the local values of the gravity at measurement sites located on the surface of the Earth or underground the knowledge of the time variation of the elevation of the site has become specially important. The principal periodic variations of elevation are due to the solid Earth tide but other important variations are the secular variations due to tectonic activity and those due to the indirect effect of the pore filling of the ground caused by the migration of underground water. The latter phenomenon, at some sites, is due to the tidal variation of the sea level in the near coast; the water load causes a migration of the fluids which is governed by the equations of diffusion.

Some data on the flow of fluids in rocks exhibit properties which may not be interpreted with the classic theory of the propagation of pressure and of fluids in porous media (Bell and Nur, 1978; Roeloffs, 1988) based on the classic Darcy's law which states that the flux is proportional to the pressure gradient.

Memory has been used previously in studying electromagnetic phenomena by (e.g., Graffi, 1936), diffusion (e.g., Kabala and Sposito, 1991; Hu and Cushman, 1994; Indelman and Abramovich, 1994) and biological phenomena (e.g., Volterra, 1930). In this note we shall use space memory represented by fractional order derivative operating on the pressure.

Classic cases of use of time fractional order derivatives as memory operators are those of energy dissipation in anelastic media (e.g., Caputo, 1969; Caputo an Mainardi, 1972; Bagley and Torvik, 1983, 1986; Körnig and Müller, 1989), of dispersion in dielectrics (e.g., Le Mehaute an Crepy, 1983; Jacquelin 1984, 1991; Pelton et al.; 1983; Caputo and Plastino, 1998) of population growth (e.g., Caputo, 1984) and of diffusion in financial (e.g., Mainardi et al., 1998; Caputo, 1998b) and hydrologic phenomena (e.g., Caputo, 1999).

The time derivative of fractional order used in the former cases is also presented and discussed (Caputo, 1969; Lucko and Gorenflo, 1998); in the present note, we shall use in the space domain.

Among other memory models developed in the research on the diffusion of fluids in rocks must be considered the use of the fractional derivative introduced in the Darcy's law operating on the flow as well as on the pressure gradient which imply a filtering of the pressure gradient without singularities (Caputo, 1998a).

The time fractional order derivative of the pressure represents the local variations and is particularly valid when considering local phenomena. In an infinite medium is more appropriate to introduce the space fractional order derivative instead of the time fractional derivative order to represent the effect of the medium previously affected by the fluid. Therefore, the flow is not directly related to the instantaneous pressure gradient in the measurement site but to the spatial fractional derivative i.e. to the pressure gradient investigated in the path from the starting point to the measurement site.

In this note we shall devote our attention particularly to find the Green function of the initial value and of the boundary value problems in a semi-infinite medium bounded by plane.

We will first find the general solution of the initial and boundary value problem; namely when the pressure is initially constant in the medium and nil with its first and second order derivatives at all times on the boundary.

Then we discuss separately the boundary value problem. Specifically we discuss the case when the pressure and its first and second order spatial derivatives are assigned on the boundary while, in the medium, is assigned the initial value of the pressure.

2. The model.

In order to find general solution of the problem, that is the pressure distribution in the porous media affected by space memory we begin setting the constitutive equations. The first equation is the classic continuity equation between the time variation of the density and the divergence of the flux

$$q_x + \rho_t = 0 \tag{1}$$

Another constitutive equation is that relating the pressure to the variation of the density from its undisturbed condition

$$p = k\rho \tag{2}$$

Successively, to take into account the observed deviations of the flow from those implied by the classic diffusion equation, we introduce, as follows, a space memory formalism in Darcy's law consistent with the flow dependence on the history of the pressure gradient.

$$q = \alpha \frac{\partial^{1+n}}{\partial x^{1+n}} p + \beta \frac{\partial}{\partial x} p \tag{3}$$

with $0 \leq n < 1$, where the definition of derivative of fractional order *1+n* is (Caputo, 1969)

$$\frac{\partial^{1+n}}{\partial x^{1+n}} p(x,t) = \left(\frac{1}{\Gamma(1-n)} \right) \int_0^x (x-v)^{-n} \left(\frac{\partial^2 p(x,v)}{\partial v^2} \right) dv$$

In the constitutive equation (3), for sake of generality (Caputo, 1999), the effect of the memory affects only the part of the pressure p with factor α, while the term with factor β represents the part of the pressure gradient not affected by the memory and behaving as in the classic Darcy's law.
Replacing p/k from Eq.(2) in Eq.(1) and taking into account the derivative respect to x variable in Eq.(2) we obtain a single equation in p

$$-\frac{p_t}{k} = \alpha \frac{\partial^{2+n}}{\partial x^{2+n}} p + \beta \frac{\partial^2}{\partial x^2} p \tag{4}$$

In order to solve Eq.(4) we take its Laplace Transform (LT) respect to x variable using the LT theorem (Caputo, 1969):

$$LT \left(\frac{\partial^{1-\alpha}}{\partial x^{1-\alpha}} p(x,t) \right) = -u^{\alpha} p(x,0) + u^{1-\alpha} LT\left(p(x,t)\right)$$

where u is the LT variable and obtain the equation

$$P_t + k \left[\alpha u^{2+n} + \beta u^2 \right] P = \alpha k \left[u^{1+n} p(0,t) + u^n p_x(0,t) + u^{n-1} p_{xx}(0,t) \right]$$
$$+ \beta k \left[u p(0,t) + p_x(0,t) \right] \tag{5}$$

where $P(u,t) = LT_{x,u} p(x,t)$. Proceeding to the solution, now we take place the LT of Eq.(5) respect to t variable and obtain

$$V(u,v) = \frac{P(u,0)}{v + k\left[\alpha u^{2+n} + \beta u^2\right]}$$
$$+ \frac{\alpha k}{v + k\left[\alpha u^{2+n} + \beta u^2\right]} LT_{t,v} \left[u^{1+n} p(0,t) + u^n p_x(0,t) + u^{n-1} p_{xx}(0,t) \right]$$
$$+ \frac{\beta k}{v + k\left[\alpha u^{2+n} + \beta u^2\right]} LT_{t,v} \left[u p(0,t) + p_x(0,t) \right] \tag{6}$$

where $V(u,v) = LT_{t,v} P(u,t)$ and $P(u,0) = LT_{x,u}(p(x,0))$. The solution p is then be obtained by inverting both LT. The inverse $LT_{t,v}$ of Eq.(6) is

$$P(u,t) = LT_{t,v}^{-1} V(u,v) = P(u,0) e^{-tku^2[\alpha u^n + \beta]}$$
$$+ \alpha k e^{-tku^2[\alpha u^n + \beta]} *_t \left[u^{1+n} p(0,t) + u^n p_x(0,t) + u^{n-1} p_{xx}(0,t) \right]$$
$$+ \beta k e^{-tku^2[\alpha u^n + \beta]} *_t \left[u p(0,t) + p_x(0,t) \right] \tag{7}$$

The inverse $LT_{x,u}$ of Eq.(7) gives finally

$$p(x,t) = LT^{-1}_{x,u}P(u,t) = p(x,0) *_x LT^{-1}_{x,u}\left(e^{-tku^2[\alpha u^n+\beta]}\right)$$
$$+\alpha k\left[p(0,t) *_t LT^{-1}_{x,u}\left(u^{1+n}e^{-tku^2[\alpha u^n+\beta]}\right) + p_x(0,t) *_t LT^{-1}_{x,u}\left(u^n e^{-tku^2[\alpha u^n+\beta]}\right)\right.$$
$$\left.+p_{xx}(0,t) *_t LT^{-1}_{x,u}\left(u^{n-1}e^{-tku^2[\alpha u^n+\beta]}\right)\right.$$
$$+\beta k\left[p(0,t) *_t LT^{-1}_{x,u}\left(ue^{-tku^2[\alpha u^n+\beta]}\right) + p_x(0,t) *_t LT^{-1}_{x,u}\left(e^{-tku^2[\alpha u^n+\beta]}\right)\right] \quad (8)$$

which gives the formal general solution of the problem and includes the boundary conditions *p(0,t), p_x(0,t), p_{xx}(0,t)* in terms of the 2^{nd}, 3^{rd}, 4^{th} lines and also the initial condition *p(x,0)* in terms of 1^{st} line.

2.1. The explicit pressure solution.

Using Eq.(A3) of the appendix with $\gamma=$ *0, 1+n, n, n-1, 1* and substituting in Eq.(8) we obtain the solution reduced to simple integrations

$$p(x,t) = p(x,0) *_x \left(-\frac{1}{\pi}\int_0^\infty e^{-rx}e^{-tkr^2(\alpha r^n cos(n\pi)+\beta)}sin\left(\alpha kr^{2+n}tsin\left(n\pi\right)\right)dr\right)$$
$$+\alpha k\left[p(0,t) *_t \left(-\frac{1}{\pi}\int_0^\infty e^{-rx}r^{1+n}e^{-tkr^2(\alpha r^n cos(n\pi)+\beta)}sin\left(\alpha kr^{2+n}t\left(sin\left(n\pi\right)-\left(1+n\right)\pi\right)\right)dr\right)\right.$$
$$+p_x(0,t) *_t \left(-\frac{1}{\pi}\int_0^\infty e^{-rx}r^{n}e^{-tkr^2(\alpha r^n cos(n\pi)+\beta)}sin\left(\alpha kr^{2+n}t\left(sin\left(n\pi\right)-n\pi\right)\right)dr\right)$$
$$\left.+p_{xx}(0,t) *_t \left(-\frac{1}{\pi}\int_0^\infty e^{-rx}r^{n-1}e^{-tkr^2(\alpha r^n cos(n\pi)+\beta)}sin\left(\alpha kr^{2+n}t\left(sin\left(n\pi\right)-\left(n-1\right)\pi\right)\right)dr\right)\right]$$
$$+\beta k\left[p(0,t) *_t \left(-\frac{1}{\pi}\int_0^\infty e^{-rx}re^{-tkr^2(\alpha r^n cos(n\pi)+\beta)}sin\left(\alpha kr^{2+n}t\left(sin\left(n\pi\right)-\pi\right)\right)dr\right)\right.$$
$$\left.+p_x(0,t) *_t \left(-\frac{1}{\pi}\int_0^\infty e^{-rx}e^{-tkr^2(\alpha r^n cos(n\pi)+\beta)}sin\left(\alpha kr^{2+n}tsin\left(n\pi\right)\right)dr\right)\right] \quad (9)$$

where the values of the integrals depend on the variables x and t. We note that in Eq.(9) we have two types of convolution, one relative to the time variable and one relative to the space variable. We note again that the first term in Eq.(9) takes into account the initial values in the medium while the other terms take into account the boundary values. The computation of the initial value term implies the convolution relative to the space variable only while the computation of the terms relative to the boundary values imply convolutions relative to the time variable only. The boundary values consist of the boundary values of the function and of its first and second order space derivatives.

2.2. The initial value problem.

We consider nil the pressure and its derivative respect to the x variable on the boundary for any t while the pressure in the medium has initial (*t=0*) constant value $C \neq 0$. The solution is then readily obtained from Eq.(9) considering only the first integral

$$p(x,t) = \frac{C}{\pi}\int_0^\infty \frac{1-e^{-rx}}{r}e^{-tkr^2(\alpha r^n cos(n\pi)+\beta)}sin\left(\alpha kr^{2+n}tsin\left(n\pi\right)\right)dr \quad (10)$$

In order to tentatively explore the effect of the space memory we will first assume *β=0*, which excludes the portion of p following the classic Darcy's law in Eq.(3). The formula is then not difficult to compute for several values of x measuring the amplitude of the effect in units of C/π and measuring t, in all case considered, in units of αk (pseudodiffusivity). We considered for the curves shown in Fig.1 the values of *n=0.1, 0.2, 0.3, 0.4* which are sufficient to describe the dependence of the memory effect on the order of fractional derivation. The Fig.1 shows that the pressure at any point in the medium decreases during the time and the decrease diminishes with increasing of n. In the figure is also seen that at any given time the pressure increases with increasing distance from the boundary. The distances considered are in meters and cover a significant range of practical interest. It is easy to extend the range to distances of geodetic interest; however, we see that the effect of the memory is significant only in relatively short distances.

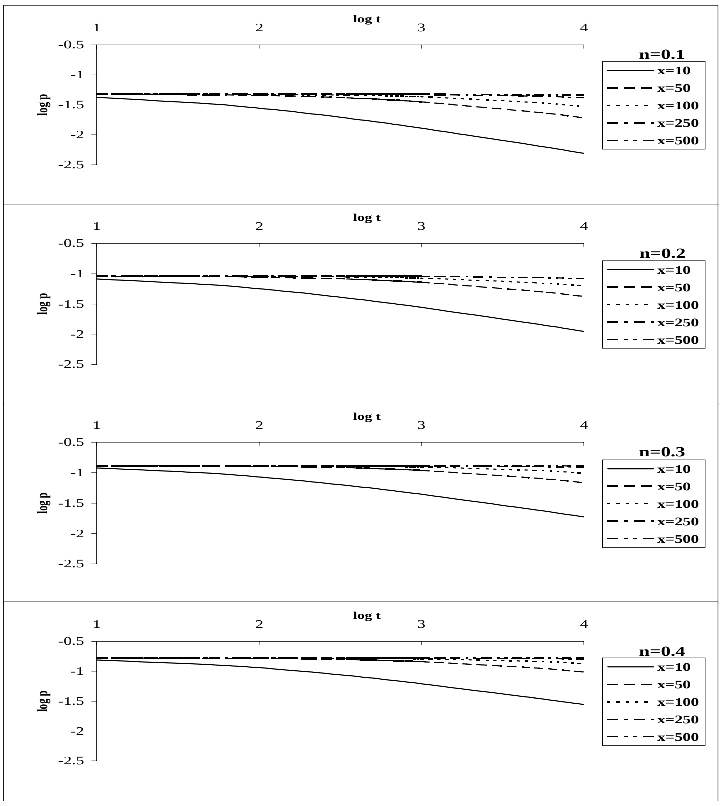

Fig.1. The initial value problem curves are related to the values of fractional derivative order *n=0.1, 0.2, 0.3, 0.4* and the distances from the boundary *x=10, 50, 100, 250, 500* meters. The amplitude is measured in units of C/π and the time in units of αk (pseudodiffusivity).

2.3. The boundary value problem.

In this case we consider nil the pressure for *t=0* for any x in the medium while the pressure on the boundary (*x=0*) is constant with value $C \neq 0$ and its derivatives respect to the x variable are nil for *x=0*. The solution is then readily obtained from Eq.(9) considering the second and fifth integral

$$p(x,t) = \alpha \left(\frac{C}{\pi} \int_0^\infty e^{-rx} \frac{1 - e^{-tkr^2(\alpha r^n cos(n\pi)+\beta)}}{r^{1-n}\left(\alpha r^n cos(n\pi) + \beta\right)} sin\left(\alpha k r^{2+n} t \left(sin\left(n\pi\right) - \left(1+n\right)\pi\right)\right) dr \right)$$

$$+\beta\left(\frac{C}{\pi}\int_0^\infty e^{-rx}\frac{1-e^{-tkr^2(\alpha r^n cos(n\pi)+\beta)}}{r\left(\alpha r^n cos(n\pi)+\beta\right)}sin\left(\alpha kr^{2+n}t\left(sin\left(n\pi\right)-\pi\right)\right)dr\right) \tag{11}$$

Also in this case we exclude the portion of p following the classic Darcy's law in Eq.(3) and assume β*=0.* The amplitude is measured in units of $\alpha C/\pi$ and the time, in all case considered, in units of αk (pseudodiffusivity). The curves shown in Fig.2 are relatives to the values of n*=0.1, 0.2, 0.3, 0.4.* The Fig.2 shows that the pressure at any point at the boundary increases during the time and the increase diminishes with increasing of n. Besides, at any given time the pressure increases with decreasing distance from the boundary. The distances considered are in meters.

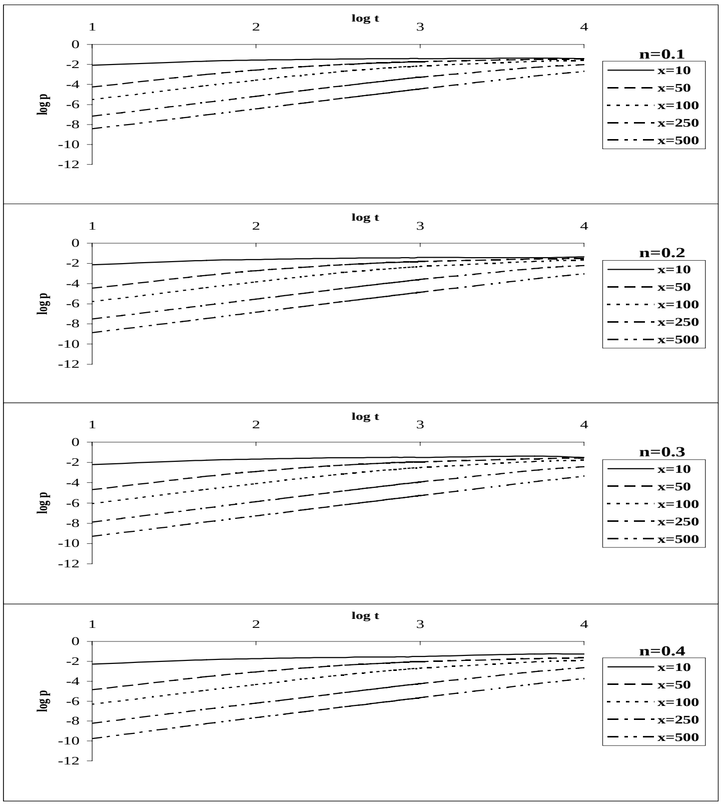

Fig.2. The boundary value problem curves are related to the values of fractional derivative order n*=0.1, 0.2, 0.3, 0.4* and the distances from the boundary x*=10, 50, 100, 250, 500* meters. The amplitude is measured in units of $\alpha C/\pi$ and the time in units of αk (pseudodiffusivity).

3. Conclusion.

The fluctuations in water level caused by Earth tides are not in complete agreement with the phases of the tides emphasizing that a memory mechanism could be the cause of this phenomenon. Particularly, the migration processes of the underground water near the coast are affected to this difference of phases. The Darcy's law modified by space derivative fractional order presented in this note may be a useful tool to describe the memory mechanism and to interpret part of the phenomenology also carachterized by anelasticity, inhomogeneity, anisotropy and of the medium.
Besides, we hope that the model of diffusion with memory in the space domain studied here be more usefull for applications to the study of the variations of the gravity field than that with memory in the time domain (Caputo, 1998a). Indeed, the latter seems more appropiate for

diffusion in layers of limited thickness, such as membranes or thin layers, while the former for diffusion in layers very thick such as in the case of water diffusion in thick layers of the Earth's crust which is of interest when studying the time variations of the gravity field.

Appendix A

We calculate the $LT^{-1}_{x,u}\left(u^{\gamma}e^{-tku^2(\alpha u^n+\beta)}\right)$ of Eq.(8), where γ is real variable, integrating along the closed path shown in Fig.3 and taking the radius of the inner circle r to zero and that of the outer circle R to infinity.

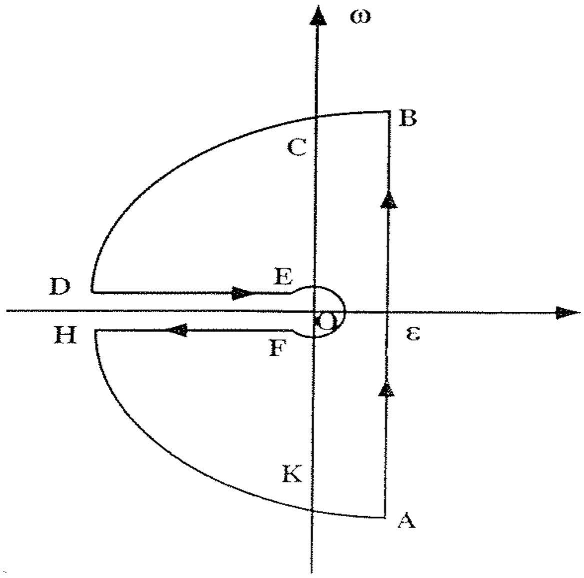

Fig.3. Path of the integration of Eq.(A1) in the complex plane. The path begins in A, follows the direction of the arrows, and return to A.

Inside the integration path there are no poles of the $LT^{-1}_{x,u}\left(u^{\gamma}e^{-tku^2(\alpha u^n+\beta)}\right)$ because this has no poles in the negative complex plane of u and then the integral is therefore nil because the residuals are nil. The integrals along BC, CD, HK, KA are nil when the outer radius R of the path is infinite; the integral on EF is nil when the inner radius r of the path is nil (Caputo, 1969) and finally we may write

$$LT^{-1}_{x,u}\left(u^{\gamma}e^{-tku^2(\alpha u^n+\beta)}\right)=\lim_{\omega\to\infty}\frac{1}{2\pi i}\left\{\int_{\varepsilon-i\omega}^{\varepsilon+i\omega}e^{ux}u^{\gamma}e^{-tku^2(\alpha u^n+\beta)}du\right.$$

$$\left.+\int_D^E e^{ux}u^{\gamma}e^{-tku^2(\alpha u^n+\beta)}du+\int_F^H e^{ux}u^{\gamma}e^{-tku^2(\alpha u^n+\beta)}du\right\} \tag{A1}$$

where ε and ω are the real and imaginary parts in the plane of integration shown in Fig.3.
We assume
$u = r\ e^{i\vartheta} = r\ (cos\ \vartheta\ +\ i\ sin\ \vartheta)$
$u^{\gamma}\ = r\ e^{i\gamma\vartheta} = r^{\gamma}\ (cos\ \gamma\theta\ +\ i\ sin\ \gamma\vartheta)$
on $\vartheta = \pm\pi$, u = -r, du = - dr, where r is the modulus of u and noting that the integration on DE: $\vartheta = \pi$ and on FH: ϑ = -π. We may write Eq.(10) as

$$LT^{-1}_{x,u}\left(u^{\gamma}e^{-tku^2(\alpha u^n+\beta)}\right)=\frac{1}{2\pi i}\left[-\int_{\infty}^{0}e^{-rx}r^{\gamma}e^{i\gamma\pi}e^{-tkr^2(\alpha r^n(cos(n\pi)+isin(n\pi))+\beta)}dr\right.$$

$$-\int_0^\infty e^{-rx} r^\gamma e^{-i\gamma\pi} e^{-tkr^2(\alpha r^n(cos(n\pi)-isin(n\pi))+\beta)} dr\Big]$$

(A2)

which may be simplified to

$$LT^{-1}_{x,u}\left(u^\gamma e^{-tku^2(\alpha u^n+\beta)}\right) = -\frac{1}{\pi}\int_0^\infty e^{-rx} r^\gamma e^{-tkr^2(\alpha r^n cos(n\pi)+\beta)} sin\left(\alpha k r^{2+n} t\left(sin\left(n\pi\right)-\gamma\pi\right)\right) dr \tag{A3}$$

Formula (A3) will be used to obtain the $LT^{-1}_{x,u}P(u,t)$.

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