Partial Procrustes solution of the threedimensional orientation problem from GPS/LPS observations

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Abstract

The need for a direct procedure for the determination of the threedimensional orientation parameters of type $\{\Lambda_\Gamma, \Phi_\Gamma\}$ being the direction of the local gravity vector Γ (i.e. the *astronomical longitude* Λ_Γ and the *astronomical latitude* Φ_Γ) and the *"orientation unknown"* Σ_Γ in the horizontal plane is still a fundamental task of geodesy. The *partial Procrustes procedure presented* provides such a *direct solution*. Further, the desire to avoid tiresome and costly astronomical observations in the determination of the deflection of the vertical can be supplemented by integrating the GPS and LPS systems as illustrated in the test example.

0. Introduction

Other than its capability to position in threedimension, the GPS satellites find use in the solution of other geodetic problems. With this realisation, *E. Grafarend et al.* (1989) by solving the *threedimensional resection problem* demonstrated in a series of papers that the *threedimensional orientation problem* could be solved in a closed form by the integration of GPS and LPS observations.

Following the determination of the threedimensional GPS Cartesian coordinates of the unknown point $P \in \mathbb{E}^3$ by the *threedimensional resection* technique, it may be desirable to obtain the *threedimensional orientation parameters* $\{\Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma\}$ of the sensor (theodolite in the case of geodetic observations, camera for photography, and CCD sensor for robotics). The threedimensional orientation parameters of type $\{\Lambda_\Gamma, \Phi_\Gamma\}$ being the direction of the local gravity vector Γ have traditionally been obtained by astronomical observation to stars and related to the geodetic coordinates $\{\lambda, \phi\}$ being the *ellipsoidal longitude* and the *ellipsoidal latitude* respectively to obtain the deflection of the vertical. Through the integration of the GPS and the LPS observations, the astronomical observations of type $\{\Lambda_\Gamma, \Phi_\Gamma\}$ are obtained by the solution of the *threedimensional orientation problem* and thus alleviating the tiresome and expensive night astronomic observations. The greatest desire in the solution of the threedimensional orientation problem has been to obtain a *direct closed* form procedure that would relate the observations of type *horizontal directions* T_i, *vertical directions* B_i, and the *spatial distances* S_i in the *Local Level Reference Frame* $\mathbb{F}^*$ to the Cartesian GPS coordinates in the *Global Reference Frame* $\mathbb{F}^\bullet$ in order to obtain the *threedimensional orientation parameters* $\{\Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma\}$.

In contrast to the geodesist, *the threedimensional orientation problem* has been part and parcel of the life of the photogrammetrist. This is evident in their attempts to obtain a *direct solution* to the problem. Earlier procedures were iterative based and were later upgraded to procedures that parameterized the unknowns without linearization such as the *Hamilton-Quaternion approach* described by *E. Grafarend* (1989). Similar studies in the same direction include the works of *G. H. Schut* (1958), *E. H.*

Thompson (1959a, 1959b), and *S. Zhang* (1994). The present paper considers the applicability of the *partial Procrustes procedure* in providing a *direct solution* to the *threedimensional orientation problem*. The first use of *Procrustes analysis* in geodesy may be attributed to the work of *F. Crosilla* (1983a, 1983b) where the *Procrustes analysis* is used in the creation of the criterion matrix used for deformation analysis.

In the test network of Stuttgart Central, 8 GPS stations are used with the aim of determining the *threedimensional orientation parameters* $\{\Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma\}$ and the deflection of the vertical $\{\xi, \eta\}$. In *section 1*, the *partial Procrustes procedure* is presented, while in *section 2*, the procedure for the computation of the vertical deflection is presented. *In section 3,* the test network "Stuttgart Central" is described. *Section 4* presents the results and conclusion of the study. This work is a special presentation to *Prof. Dr.-Ing. habil. E. Grafarend* on his 60th birthday. His effort to have a direct solution to the threedimensional orientation problem is evident in his introduction of the Hamilton-Quaternion procedure to geodesy. "*The partial Procrustes procedure from the bed of Procrustes in the solution of the threedimensional orientation problem is your brainchild!*"

1. The Partial Procrustes Approach

The *general Procrustes algorithm* being a technique of matching one configuration into another and producing a measure of the match, seeks the *isotropic dilation and the rigid translation, reflection and rotation* needed to best match one configuration to another *(T. F. Cox and M. A. Cox* 1994 p. 92*).* The term *partial Procrustes algorithm* in the present paper refers to the optimal rotation in contrast to other definitions as given e.g. by *M. Gulliksson* (1995a, 1995b) and *I. Dryden* (1998). The two configurations i.e. the *Local Level Reference Frame* $\mathbb{F}^*$ and the *Global Reference Frame* $\mathbb{F}^\bullet$ are related as in *Box 1.1* below, with $X, Y, Z, X_i, Y_i, Z_i \ \forall_i \in \mathbb{N}$ being the GPS co-ordinates in the *Global Reference Frame* $\mathbb{F}^\bullet$, while $S_i, T_i, B_i \forall_i \in \mathbb{N}$ are the spherical coordinates of the local gravity vector of type *spatial distances, horizontal and vertical directions* respectively in the *LPS Frame* $\mathbb{F}^*$.

Box 1.1: Relating the *Local Level Reference Frame* $\mathbb{F}^*$ and the *Global Reference Frame* $\mathbb{F}^\bullet$

$$S_i \begin{bmatrix} \cos B_i \cos T_i \\ \cos B_i \cos T_i \\ \sin B_i \end{bmatrix}_{\mathbb{F}^*} = \begin{bmatrix} x_i - x \\ y_i - y \\ z_i - z \end{bmatrix}_{\mathbb{F}^*} \quad \forall_i \in \{1,2,\ldots,n\} \tag{1.1}$$

$$S_i = S(\mathbf{X}, \mathbf{X}_i) = \sqrt{(X_i - X)^2 + (Y_i - Y)^2 + (Z_i - Z)^2} \tag{1.2}$$

$$\begin{bmatrix} x_i - x \\ y_i - y \\ z_i - z \end{bmatrix}_{\mathbb{F}^*} = \mathbf{R_E}(\Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma) \begin{bmatrix} X_i - X \\ Y_i - Y \\ Z_i - Z \end{bmatrix}_{\mathbb{F}^\bullet} \tag{1.3}$$

$$\begin{bmatrix} x_1 - x & x_2 - x \ldots & x_n - x \\ y_1 - y & y_2 - y \ldots & y_n - y \\ z_1 - z & z_2 - z \ldots & z_n - z \end{bmatrix}_{\mathbb{F}^*} = \mathbf{R} \begin{bmatrix} X_1 - X & X_2 - X \ldots & X_n - X \\ Y_1 - Y & Y_2 - Y \ldots & Y_n - Y \\ Z_1 - Z & Z_2 - Z \ldots & Z_n - Z \end{bmatrix}_{\mathbb{F}^\bullet} \tag{1.4}$$

$$\mathbf{A} := \begin{bmatrix} x_1 - x & y_1 - y & z_1 - z \\ x_2 - x & y_2 - y & z_2 - z \\ . & . & . \\ . & . & . \\ . & . & . \\ x_n - x & y_n - y & z_n - z \end{bmatrix}_{\mathbb{F}^*} = \begin{bmatrix} X_1 - X & Y_1 - Y & Z_1 - Z \\ X_2 - X & Y_2 - Y & Z_2 - Z \\ . & . & . \\ . & . & . \\ . & . & . \\ X_n - X & Y_n - Y & Z_n - Z \end{bmatrix}_{\mathbb{F}^\bullet} \mathbf{R}^T =: \mathbf{BT} \tag{1.5}$$

$$\mathbf{A} \in \mathbb{R}^{nx3}, \mathbf{B} \in \mathbb{R}^{nx3}, \mathbf{R}^T = \mathbf{T} \in \mathbb{R}^{3x3}$$

The matrix on the left-hand side of equation (1.5) is denoted as $\mathbf{A}$, while the one on the right-hand side of equation (1.5) as $\mathbf{B}$. If the rotation matrix $\mathbf{R}^{\mathrm{T}}$ is denoted by $\mathbf{T}$, the *general Procrustes problem* is now concerned with fitting the configuration of $\mathbf{B}$ into $\mathbf{A}$ as close as possible. The simplest Procrustes case is one in which both configurations have the *same dimensionality* and the *same number of points*, which can be brought into a 1-1 correspondence by substantive considerations *(I. Borg and P. Groenen* 1997 p.339*)*. The present study considers such a case and the problem reduces to that of determination of the rotation matrix $\mathbf{T}$. With both $\mathbf{A}$ and $\mathbf{B}$ above having the same order *n* x 3, one writes

$$\mathbf{A} = \mathbf{BT} \tag{1.6}$$

The solution of $\mathbf{T}$ above entails measuring the distances between corresponding points in both configurations, squaring these values, and adding them to obtain the sum of squares $\|\mathbf{A}-\mathbf{BT}\|^2$. The transformation $\mathbf{T}$ that will minimize the sum of squares above is now sought using the *partial Procrustes procedure*. Proceeding by the *Frobenius-Norm* in *Box 1.2* below, and on using the property of invariance of the trace function under cyclic permutation, the simplification $tr\mathbf{T}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{BT} = tr\mathbf{B}^{\mathrm{T}}\mathbf{B}$ is obtained. Since $\mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{A})$ and $\mathrm{tr}(\mathbf{B}^{\mathrm{T}}\mathbf{B})$ are not dependent on $\mathbf{T}$, only the term $\mathbf{A}^{\mathrm{T}}\mathbf{BT}$ is considered.

Box 1.2: Frobenius-Norm

$$\|\mathbf{X}-\mathbf{YT}\| := \sqrt{tr(\mathbf{X}^{\mathrm{T}} - \mathbf{T}^{\mathrm{T}}\mathbf{Y}^{\mathrm{T}})(\mathbf{X}-\mathbf{YT})}$$
$$\mathbf{T}^{\mathrm{T}}\mathbf{T} = \mathbf{I}$$
$$\|\mathbf{A}-\mathbf{BT}\|^2 := tr(\mathbf{A}^{\mathrm{T}} - \mathbf{T}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}})(\mathbf{A}-\mathbf{BT})$$
$$\mathbf{T}^{\mathrm{T}}\mathbf{T} = \mathbf{I}$$
$$\begin{aligned}\min &= tr(\mathbf{A}^{\mathrm{T}}\mathbf{A} - 2\mathbf{A}^{\mathrm{T}}\mathbf{BT} + \mathbf{T}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{BT}) \\ &= tr\mathbf{A}^{\mathrm{T}}\mathbf{A} - 2tr\mathbf{A}^{\mathrm{T}}\mathbf{BT} + tr\mathbf{B}^{\mathrm{T}}\mathbf{B}\end{aligned}$$
$$tr\mathbf{T}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{BT} = tr\mathbf{T}\mathbf{T}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{B} = tr\mathbf{B}^{\mathrm{T}}\mathbf{B}$$
$$\|\mathbf{A}-\mathbf{BT}\|^2 = \min \Leftrightarrow tr(\mathbf{A}^{\mathrm{T}}\mathbf{BT}) = \max$$
$$\mathbf{T}^{\mathrm{T}}\mathbf{T} = \mathbf{T}\mathbf{T}^{\mathrm{T}} = \mathbf{I}_k$$

Box 1.3: Singular Value Decomposition (SVD)

$$\mathbf{A}^{\mathrm{T}}\mathbf{B} = \mathbf{U\Sigma V}^{\mathrm{T}}$$
$$\text{if } \mathbf{C} = \mathbf{U\Sigma V}^{\mathrm{T}},\ \mathbf{U},\mathbf{V}^{\mathrm{T}} \in \mathrm{SO}(3)$$
$$\Sigma = \mathrm{Diag}(\sigma_1, \ldots, \sigma_k)$$
then
$$\mathrm{tr}(\mathbf{CT}) \le \sum_{i=1}^{k} \sigma_k$$
mit
$$k = 3$$

In *Box 1.3* above $\mathbf{U\Sigma V}^{\mathrm{T}}$ is considered as the *singular value decomposition* of $\mathbf{A}^{\mathrm{T}}\mathbf{B}$ and $\mathbf{C} = \mathbf{A}^{\mathrm{T}}\mathbf{B}$. Substituting for $\mathbf{C}$ with its *singular value decomposition* in *Box 1.3* and the approach of *R. Mathar* (1997 p.34) we have the following:

$$\begin{cases} \mathrm{tr}(\mathbf{CT}) = \mathrm{tr}(\mathbf{U\Sigma V}^{\mathrm{T}}\mathbf{T}) = \mathrm{tr}(\mathbf{\Sigma V}^{\mathrm{T}}\mathbf{TU}) \\ \text{taking} \\ \mathbf{R} = (ij)1 \le i, j \le k = \mathbf{V}^{\mathrm{T}}\mathbf{TU} \text{ orthogonal and } |r_{ii}| \le 1 \\ \text{then} \\ \mathrm{tr}(\mathbf{\Sigma V}^{\mathrm{T}}\mathbf{TU}) = \sum_{i=1}^{k} \sigma_i r_{ii} \le \sum_{i=1}^{k} \sigma_i \end{cases} \tag{1.7}$$

From (1.7)

$$\mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{BT}) = \max \Leftrightarrow \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{BT}) \le \sum_{i=1}^{k} \sigma_i \tag{1.8}$$

Subject to the *singular value decomposition*

$$\mathbf{A}^{\mathrm{T}}\mathbf{B} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}, \ \mathbf{U}, \mathbf{V} \in SO(3) \text{ and orthogonal} \tag{1.9}$$

Finally, the maximum value

$$\max(\mathrm{tr}\mathbf{A}^{\mathrm{T}}\mathbf{B}\mathbf{T}) = \sum_{i=1}^{k} \sigma_i \Leftrightarrow \mathbf{T} = \mathbf{V}\mathbf{U}^{\mathrm{T}} \tag{1.10}$$

Thus the solution of the rotation matrix by *the partial Procrustes solution* is

$$\mathbf{T} = \mathbf{V}\mathbf{U}^{\mathrm{T}} \tag{1.11}$$

The solution of $\mathbf{T}$ based on the partial derivatives approach as in (*P.H Schönemann* 1966) is presented in the Appendix in *section 5*. The operations involved are:

- Solution of $\mathbf{T}^* = \mathbf{V}\mathbf{U}^{\mathrm{T}}$
- Obtaining the rotation elements from $\mathbf{R} = (\mathbf{T}^*)^{\mathrm{T}}$

Where $\mathbf{T}^*$ is the best possible matrix out of the set of all orthogonal matrices $\mathbf{T}$ which is obtained by imposing the restriction $\mathbf{T}\mathbf{T}^{\mathrm{T}} = \mathbf{T}^{\mathrm{T}}\mathbf{T} = \mathbf{I}$, otherwise $\mathbf{T}$ could be *any matrix*, which means, geometrically, that $\mathbf{T}$ is some *linear transformation* which in general may not preserve the shape of $\mathbf{B}$. The Eulerian rotational angles, which are the orientation elements, are finally deduced from $\mathbf{R}$.

2. Computation of the deflection of the vertical

To be able to map the topographical surface which is embedded into a threedimensional Euclidean space $\mathbb{R}^3$ pointwise into a reference ellipsoid of revolution, *E. Grafarend and P, Lohse* (1991) define the optimisation problem as follows: *Minimize the Euclidean distance between points on the topographical surfaces and the reference ellipsoid of revolution subject to the constraint that the projection point is a point on the reference ellipsoid of revolution*. The problem is written with X,Y,Z being the GPS coordinates of point K1 as in *Box 2.1* below.

Box 2.1: Computation of the deflection of the vertical

$$\left[\begin{array}{l} 2L(x_1, x_2, x_3, x_4) := \{|\mathbf{X} - \mathbf{x}|^2 + x_4[b^2(x_1^2 + x_2^2) + ax_3^2 - a^2b^2]\} \\ x \in X := \left\{ x \in \mathbb{R}^3 \left| \dfrac{x_1^2 + x_2^2}{a^2} + \dfrac{x_3^2}{b^2} = 1 \right. \right\} \end{array}\right. \tag{2.1}$$

$$\left[\begin{array}{l} \dfrac{\partial L}{\partial x_1}(\hat{x}) = -(X - x_1) + b^2\hat{x}_1\hat{x}_4 = 0, \quad \forall_{i=1,2,3,4} \\ \dfrac{\partial L}{\partial x_2}(\hat{x}) = -(Y - x_2) + b^2\hat{x}_2\hat{x}_4 = 0 \\ \dfrac{\partial L}{\partial x_3}(\hat{x}) = -(Z - x_3) + a^2\hat{x}_3\hat{x}_4 = 0 \\ \dfrac{\partial L}{\partial x_4}(\hat{x}) = \dfrac{1}{2}[b^2(\hat{x}_1^2 + \hat{x}_2^2) + a^2\hat{x}_3 - a^2b^2] = 0 \end{array}\right. \tag{2.2}$$

$$\left[\begin{array}{l} \dfrac{\partial^2 L(\hat{x}_1, \hat{x}_2, \hat{x}_3)}{\partial x_i \partial x_j} \text{positive definite} \, \forall_{i,j=1,2,3} \\ \text{or } \mathbf{H} := \dfrac{\partial^2 L(\hat{x})}{\partial x_i \partial x_j} = \begin{bmatrix} 1 + b^2\hat{x}_4 & & 0 \\ & 1 + b^2\hat{x}_4 & \\ 0 & & 1 + a^2\hat{x}_4 \end{bmatrix} \end{array}\right. \tag{2.3}$$

$$x_1 = \frac{X}{1 + b^2\hat{x}_4}; \ x_2 = \frac{Y}{1 + b^2\hat{x}_4} \ x_3 = \frac{Z}{1 + a^2\hat{x}_4} \tag{2.4}$$

$$\left[\begin{array}{l} b^2(1+a^2x_4)^2(X^2+Y^2) + a^2(1+b^2x_4)^2Z^2 - a^2b^2(1+a^2x_4)^2(1+b^2x_4)^2 \\ C_4x_4^4 + C_3x_4^3 + C_2x_4^2 + C_1x_4 + C_o = 0 \\ C_4 := a^6b^6 \\ C_3 := a^2b^2(2b^2a^4 + 2a^2b^4) \\ C_2 := a^2b^2(b^4 + a^4 + 4a^2b^2) - a^4b^2(x^2 + y^2) - b^4a^2z^2 \\ C_1 := a^2b^2(2b^2 + 2a^2) - 2a^2b^2(x^2 + y^2) - 2b^2a^2z^2 \\ C_o := a^2b^2 - b^2(x^2 + y^2) - a^2z^2 \end{array}\right. \tag{2.5}$$

$$\left[\begin{array}{l} \lambda = \tan^{-1}\left(\dfrac{\Delta Y}{\Delta X}\right) \\ \phi = \tan^{-1}\left(\dfrac{\Delta Z}{\sqrt{\Delta X^2 + \Delta Y^2}}\right) \\ \xi = \Phi_\Gamma - \phi_\Gamma \\ \eta = (\Lambda_\Gamma - \lambda_\Gamma)\cos\phi_\Gamma \end{array}\right. \tag{2.6}$$

Where x_1, x_2, x_3 are the x, y, z coordinates of K1 when projected onto the ellipsoid of revolution with the constrain (2.1)ii. (2.2) and (2.3) provide the *necessary* and *sufficiency* conditions for the solution. From (2.2), x_1, x_2, x_3 are expressed as indicated in (2.4) and substituted in ((2.2)iv to give a single variable polynomial of fourth degree in (2.5)ii. The solution of this polynomial gives four roots for x_4 with two being real. Inserting x_4 in (2.4) give x_1, x_2, x_3.Whether the solution x_1, x_2, x_3, x_4 is admissible is tested using the Hesse matrix (2.3) to guarantee *positive definiteness*. Should the first solution vector x_1, x_2, x_3, x_4 fail the test, the second real root of x_4 is employed. The geodetic coordinates ϕ, λ and the deflection of the vertical $\{\xi, \eta\}$ are given by (2.6).

3. Test Example

The following experiment was performed at the centre of Stuttgart on one of the pillars of the University buildings along Kepler Strasse 11 as depicted by *Figure 3.1* below.

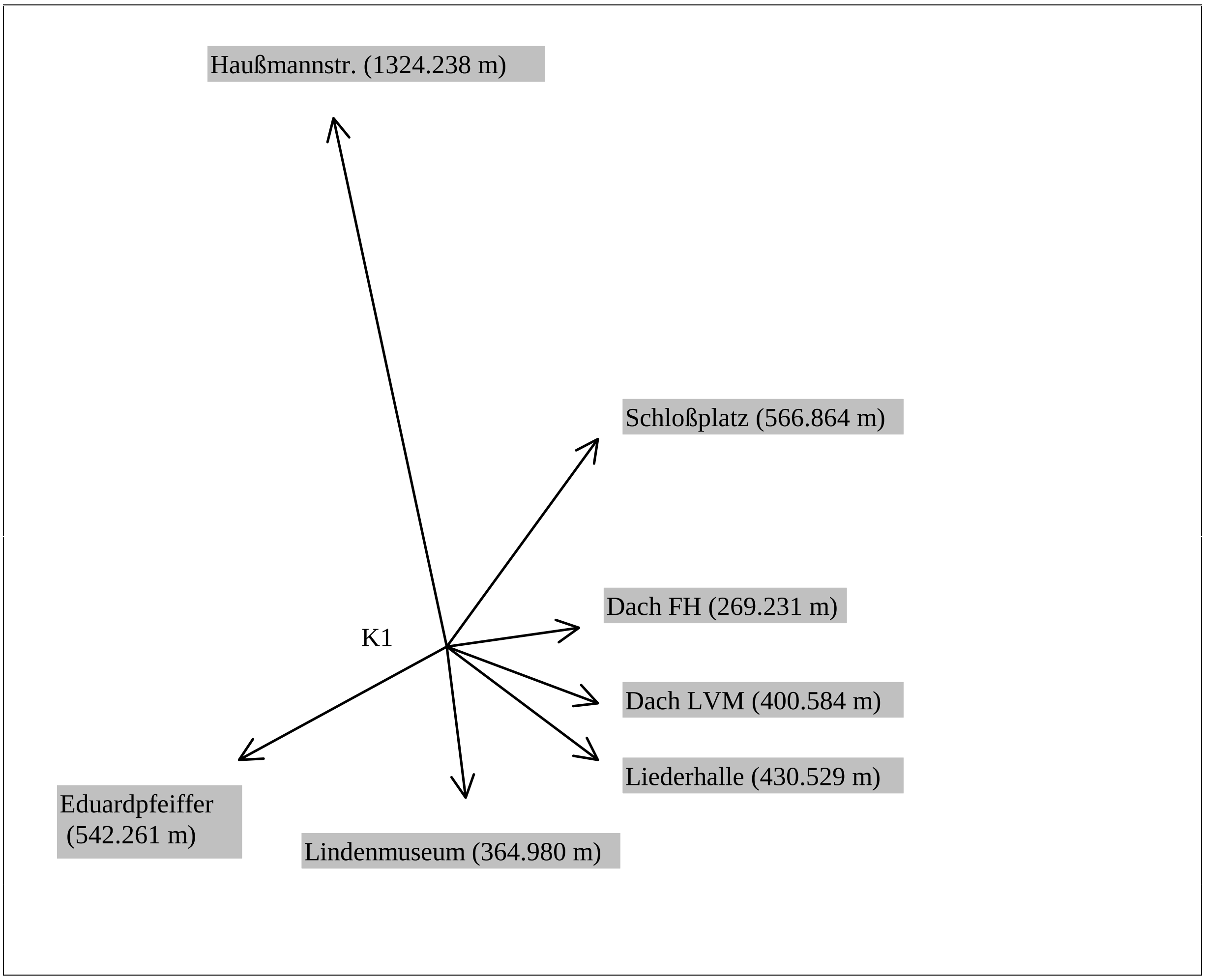

Figure 3.1: Graph of the “ Stuttgart Central”

The test network of Stuttgart Central consisted of 8 GPS points listed in *Table 3.1* below. The theodolite is stationed at pillar K1 whose astronomical values are known from the previous observations made by the Department of Geodesy and GeoInformatics. Using the coordinates of the known GPS stations and the values of the local gravity vector Γ of station K1 on top of the University Building Keplerstrasse 11 the theoretical spherical coordinates of type horizontal and vertical directions from

K1 to the other known GPS stations were computed and are as presented in *Table 3.2*. The spherical coordinates $\{\Lambda_\Gamma, \Phi_\Gamma\}$ of the local gravity vector Γ were adopted from astronomical observations made by the Department of Geodesy and GeoInformatics, Stuttgart University, observed to an accuracy of $\pm 10''$ (*S. Kurz* 1996p. 46). Based on a 0.5" Wild T2000 theodolite, the observations of type horizontal and vertical directions are randomly generated with noise being injected to the theoretical values of *Table 3.2* to reflect their nature as field observations and are as presented in *Table 3.3* to *Table 3.13*. In total, 11 sets of observations were generated. Indeed the field observations contain errors of various kinds ranging from instrumental to observational. The simulated data are thus made to reflect these situation. The generated noise are normally distributed in the range $-6'' \le (\Delta T, \Delta B) \le 6''$.

The observations are thus designed such that by observing the other seven GPS stations, the orientation of the *Local Level Frame of Reference* $\mathbb{F}^*$ whose origin is station K1 to the *Global Frame of Reference* $\mathbb{F}^\bullet$ is obtained. The relationship between the $\mathbb{F}^*$ frame and the $\mathbb{F}^\bullet$ frame is given by *E. Grafarend* (1981p. 1-19, 1991p. 1-11) for instance

$$\begin{aligned}\left[\mathbb{F}_{1^*}, \mathbb{F}_{2^*}, \mathbb{F}_{3^*}\right] &= \left[\mathbb{F}_{1^\bullet}, \mathbb{F}_{2^\bullet}, \mathbb{F}_{3^\bullet}\right]\mathbf{R}_E^{\mathrm{T}}(\Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma) \\ &= \left[\mathbb{F}_{1^\bullet}, \mathbb{F}_{2^\bullet}, \mathbb{F}_{3^\bullet}\right]\mathbf{R}_3^{\mathrm{T}}(\Lambda_\Gamma)\mathbf{R}_2^{\mathrm{T}}(\frac{\pi}{2} - \Phi_\Gamma)\mathbf{R}_3^{\mathrm{T}}(\Sigma_\Gamma)\end{aligned} \tag{3.1}$$

The direction of Schloßplatz is chosen as the zero direction of the theodolite and this leads to the determination of the third component Σ_Γ of the threedimensional orientation parameters as in (3.1) above. To each of the target GPS points i , the observations of the type horizontal direction T_i and the vertical directions B_i are measured. The spatial distances $S_i^2(\mathbf{X}, \mathbf{X}_i) = \|\mathbf{X}_i - \mathbf{X}\|$ are readily obtained from the GPS coordinates in *Table 3.1*. Once we have the spherical coordinates of type spatial distances S_i , horizontal directions T_i , and the vertical directions B_i , a backward computation using the partial Procrustes procedure discussed in the next section is used to compute the direction parameters ($\Lambda_\Gamma, \Phi_\Gamma$) of the local vertical at K1 and the unknown orientation element Σ_Γ .
The obtained values are then compared to the starting values. The following symbols have been used. $\{\sigma_X, \sigma_Y, \sigma_Z\}$ are the standard errors of the GPS Cartesian coordinates. Covariances $\{\sigma_{XY}, \sigma_{YZ}, \sigma_{ZX}\}$ were neglected. $\{\sigma_T, \sigma_B\}$ are the standard deviation of horizontal and vertical directions respectively after an adjustment, $\{\Delta_T, \Delta_B\}$ the magnitude of the noise on the horizontal and vertical directions, respectively..

Table 3.1: GPS Coordinates

Station Name	**X** (m)	**Y** (m)	**Z** (m)	$\sigma_\mathbf{X}$ (m)	$\sigma_\mathbf{Y}$ (m)	$\sigma_\mathbf{Z}$ (m)
Dach K1	4157066.1116	671429.6655	4774879.3704	0.00107	0.00106	0.00109
Schloßplatz	4157246.5346	671877.0281	4774581.6314	0.00076	0.00076	0.00076
Haußmanstr.	4156749.5977	672711.4554	4774981.5459	0.00177	0.00159	0.00161
Edwardpfeiffer	4156748.6829	671171.9385	4775235.5483	0.00193	0.00184	0.00187
Lindenmuseum	4157066.8551	671066.9381	4774865.8238	0.00138	0.00129	0.00138
Liederhalle	4157266.6181	671099.1577	4774689.8536	0.00129	0.00128	0.00134
Dach LVM	4157307.5147	671171.7006	4774690.5691	0.00070	0.00240	0.00010
Dach FH	4157244.9515	671338.5915	4774699.9070	0.00280	0.00150	0.00310

Table 3.2: Theoretical Spherical Co-ordinates in the horizontal frame of reference

Station observed from K1	Spatial Distances (m)	Horizontal directions (gon)	Vertical directions (gon)
Schloßplatz	566.8635	52.320062	-6.705164
Haußmanstr.	1324.2380	107.160333	0.271038
Edwardpfeiffer	542.2609	224.582723	4.036011
Lindenmuseum	364.9797	293.965493	-8.398004
Liederhalle	430.5286	336.851237	-6.941728
Dach LVM	400.5837	347.702846	-1.921509
Dach FH	269.2306	370.832476	-6.686951

Table 3.3: Observations Set 1

Station observed from K1	Horizontal directions (gon)	Vertical directions (gon)	σ_T (gon)	σ_B (gon)	Δ_T (gon)	Δ_B (gon)
Schloßplatz	0.000000	-6.705138	0.0025794	0.0024898	-0.000228	-0.000039
Haußmanstr.	54.840342	0.271005	0.0028756	0.0027171	-0.000298	0.000033
Edwardpfeiffer	172.262141	4.035491	0.0023303	0.0022050	0.000293	0.000520
Lindenmuseum	241.644854	-8.398175	0.0025255	0.0024874	0.000350	0.000171
Liederhalle	284.531189	-6.942558	0.0020781	0.0022399	-0.000024	0.000830
Dach LVM	295.382909	-1.921008	0.0029555	0.0024234	0.000278	-0.000275
Dach FH	318.512158	-6.687226	0.0026747	0.0024193	-0.000352	0.000500

Table 3.4: Observations Set 2

Station observed from K1	Horizontal directions (gon)	Vertical directions (gon)	σ_T (gon)	σ_B (gon)	Δ_T (gon)	Δ_B (gon)
Schloßplatz	0.0000000	-6.705636	0.0029467	0.0022479	0.000655	0.000459
Haußmanstr.	54.841828	0.270494	0.0023740	0.0018085	-0.000902	0.000544
Edwardpfeiffer	172.262016	4.035712	0.0025738	0.0025891	0.001300	0.000300
Lindenmuseum	241.645929	-8.397520	0.0025012	0.0027585	0.000156	-0.000484
Liederhalle	284.531106	-6.940833	0.0025388	0.0021120	0.000723	-0.000895
Dach LVM	295.382535	-1.920744	0.0024122	0.0022379	0.000904	-0.000765
Dach FH	318.512615	-6.686485	0.0024235	0.0027708	0.000453	-0.000466

Table 3.5: Observations Set 3

Station observed from K1	Horizontal directions (gon)	Vertical directions (gon)	σ_T (gon)	σ_B (gon)	Δ_T (gon)	Δ_B (gon)
Schloßplatz	0.000000	-6.704849	0.0025794	0.0023074	-0.001470	-0.000329
Haußmanstr.	54.839743	0.272062	0.0027316	0.0036383	0.000524	-0.001024
Edwardpfeiffer	172.261715	4.036063	0.0022680	0.0025318	0.000942	-0.000052
Lindenmuseum	241.645032	-8.398128	0.0031452	0.0030835	0.000395	0.000124
Liederhalle	284.530697	-6.941783	0.0025214	0.0024290	0.000473	0.000055
Dach LVM	295.382921	-1.922053	0.0024296	0.0028454	-0.000141	0.000544
Dach FH	318.511249	-6.686536	0.0024345	0.00227063	0.001174	-0.000415

Table 3.6: Observations Set 4

Station observed from K1	Horizontal directions (gon)	Vertical directions (gon)	σ_T (gon)	σ_B (gon)	Δ_T (gon)	Δ_B (gon)
Schloßplatz	0.000000	-6.704682	0.0023308	0.0027599	0.000862	-0.000496
Haußmanstr.	54.841145	0.271960	0.0026907	0.0021463	-0.000011	-0.000922
Edwardpfeiffer	172.264284	4.036170	0.0028699	0.0024486	-0.000760	-0.000159
Lindenmuseum	241.645972	-8.397035	0.0035089	0.0024921	0.000322	-0.000969
Liederhalle	284.532505	-6.941248	0.0026110	0.0033665	-0.000468	-0.000480
Dach LVM	295.384465	-1.921296	0.0027294	0.0026283	-0.000818	-0.000213
Dach FH	318.513839	-6.686125	0.0020477	0.0030185	-0.000562	-0.000826

Table 3.7: Observations Set 5

Station observed from K1	Horizontal directions (gon)	Vertical directions (gon)	σ_T (gon)	σ_B (gon)	Δ_T (gon)	Δ_B (gon)
Schloßplatz	0.000000	-6.705407	0.0023550	0.0026607	0.000275	0.000229
Haußmanstr.	54.839952	0.271814	0.0027139	0.0024570	0.000594	-0.000775
Edwardpfeiffer	172.262789	4.036099	0.0028628	0.0020811	0.000148	-0.000088
Lindenmuseum	241.645827	-8.398001	0.0027261	0.0027714	-0.000121	-0.000003
Liederhalle	284.530609	-6.940954	0.0029166	0.0024115	0.000840	-0.000774
Dach LVM	295.383197	-1.921506	0.0032741	0.0025684	-0.000138	-0.000003
Dach FH	318.513393	-6.686562	0.0031545	0.0028330	-0.000705	-0.000993

Table 3.8: Observations Set 6

Station observed from K1	Horizontal directions (gon)	Vertical directions (gon)	σ_T (gon)	σ_B (gon)	Δ_T (gon)	Δ_B (gon)
Schloßplatz	0.000000	-6.705699	0.0032227	0.0026362	-0.000230	0.000522
Haußmanstr.	54.841100	0.272198	0.0028716	0.0032300	-0.001059	-0.001160
Edwardpfeiffer	172.262254	4.035556	0.0027485	0.0022965	0.000177	0.000455
Lindenmuseum	241.645033	-8.398092	0.0028093	0.0030335	0.000167	0.000088
Liederhalle	284.531250	-6.941579	0.0022418	0.0023971	-0.000306	-0.000149
Dach LVM	295.383176	-1.921632	0.0028193	0.0031391	-0.000622	0.000123
Dach FH	318.512147	-6.687006	0.0026446	0.0018992	0.000037	0.000055

Table 3.9: Observations Set 7

Station observed from K1	Horizontal directions (gon)	Vertical directions (gon)	σ_T (gon)	σ_B (gon)	Δ_T (gon)	Δ_B (gon)
Schloßplatz	0.000000	-6.704710	0.0032501	0.0021664	0.000796	-0.000467
Haußmanstr.	54.840622	0.271205	0.0025500	0.0026468	0.000446	-0.000167
Edwardpfeiffer	172.262586	4.035479	0.0028646	0.0030243	0.000872	0.000532
Lindenmuseum	241.645766	-8.397192	0.0020303	0.0026158	0.000461	-0.000811
Liederhalle	284.533069	-6.940859	0.0026240	0.0022506	-0.001098	-0.000869
Dach LVM	295.383199	-1.920591	0.0029904	0.0026217	0.000381	-0.000918
Dach FH	318.512078	-6.686979	0.0024550	0.0023116	0.001132	0.000028

Table 3.10: Observations Set 8

Station observed from K1	Horizontal directions (gon)	Vertical directions (gon)	σ_T (gon)	σ_B (gon)	Δ_T (gon)	Δ_B (gon)
Schloßplatz	0.000000	-6.705117	0.0019401	0.0025817	0.001199	-0.000060
Haußmanstr.	54.841233	0.271160	0.0020984	0.0028927	0.000238	-0.000122
Edwardpfeiffer	172.263880	4.036182	0.0026151	0.0022965	-0.000019	-0.000170
Lindenmuseum	241.645783	-8.397963	0.0029220	0.0022676	0.000847	-0.000041
Liederhalle	284.532564	-6.941989	0.0024886	0.0021962	-0.000188	0.000261
Dach LVM	295.383289	-1.920585	0.0025328	0.0025163	0.000694	-0.000924
Dach FH	318.514380	-6.686908	0.0028717	0.0028983	-0.000767	-0.000043

Table 3.11: Observations Set 9

Station observed from K1	Horizontal directions (gon)	Vertical directions (gon)	σ_T (gon)	σ_B (gon)	Δ_T (gon)	Δ_B (gon)
Schloßplatz	0.000000	-6.705021	0.0023099	0.0029693	-0.000872	-0.000156
Haußmanstr.	54.838686	0.271061	0.0025460	0.0024923	0.000714	-0.000023
Edwardpfeiffer	172.261443	4.035575	0.0027183	0.0026865	0.000347	0.000436
Lindenmuseum	241.645374	-8.397707	0.0024564	0.0024467	-0.000815	-0.000296
Liederhalle	284.530552	-6.941628	0.0034024	0.0027446	-0.000249	-0.000100
Dach LVM	295.381778	-1.921467	0.0022630	0.0027665	0.000134	-0.000042
Dach FH	318.511475	-6.686698	0.0022266	0.0025736	0.000068	-0.000253

Table 3.12: Observations Set 10

Station observed from K1	Horizontal directions (gon)	Vertical directions (gon)	σ_T (gon)	σ_B (gon)	Δ_T (gon)	Δ_B (gon)
Schloßplatz	0.000000	-6.705515	0.0024938	0.0032987	0.000299	0.000338
Haußmanstr.	54.841489	0.270932	0.0029717	0.0022950	-0.000918	0.000106
Edwardpfeiffer	172.263665	4.036147	0.0032672	0.0024499	-0.000704	-0.000136
Lindenmuseum	241.645336	-8.397823	0.0028515	0.0025473	0.000395	-0.000181
Liederhalle	284.531567	-6.941534	0.0022931	0.0021688	-0.000093	-0.000194
Dach LVM	295.383055	-1.922041	0.0033986	0.0028467	0.000028	0.000532
Dach FH	318.512017	-6.686773	0.0024359	0.0021356	0.000696	-0.000178

Table 3.13: Observations Set 11

Station observed from K1	Horizontal directions (gon)	Vertical directions (gon)	σ_T (gon)	σ_B (gon)	Δ_T (gon)	Δ_B (gon)
Schloßplatz	0.000000	-6.704889	0.0024962	0.0031604	0.000459	-0.000288
Haußmanstr.	54.840818	0.271340	0.0027559	0.0028895	-0.000088	0.001659
Edwardpfeiffer	172.263416	4.035779	0.0023929	0.0032068	-0.000296	0.000232
Lindenmuseum	241.645322	-8.398136	0.0031984	0.0019623	0.000568	0.000132
Liederhalle	284.532013	-6.942079	0.0027289	0.0032386	-0.000379	0.000351
Dach LVM	295.383571	-1.921888	0.0026898	0.0023682	-0.000328	0.000379
Dach FH	318.513029	-6.686424	0.0027481	0.0026191	-0.000157	-0.000527

Table 3.14: Elements of orientation parameters computed

Simulation Set Number	Λ_Γ	Φ_Γ	Σ_Γ^0 (gon)	$\Delta\Lambda_\Gamma$	$\Delta\Phi_\Gamma$	ξ	η	$\Delta\Sigma_\Gamma^0$ (gon)
0*	$09^{o}\ 10^{'}\ 29^{''}.8$	$48^{o}\ 46^{'}\ 54^{''}.9$	52.3200619	$0^{''}.0$	$0^{''}.0$	$0^{''}.0$	$0^{''}.1$	0.0000000
1	09 10 30.1	48 46 54.3	52.3200371	-0.3	0.6	0.6	0.3	-0.0000248
2	09 10 26.9	48 46 53.7	52.3198377	2.9	1.2	1.2	-1.8	0.0002243
3	09 10 33.9	48 46 55.4	52.3196156	-4.1	-0.5	-0.5	2.8	0.0004473
4	09 10 32.9	48 46 54.4	52.3184245	-3.1	0.5	0.5	2.2	0.0016375
5	09 10 32.3	48 46 55.2	52.3196519	-2.5	-0.3	-0.3	1.8	0.0004100
6	09 10 33.8	48 46 55.6	52.3186804	-4.0	-0.7	-0.7	2.8	0.0013815
7	09 10 30.2	48 46 52.6	52.3196222	-0.4	2.3	2.3	0.4	0.0004397
8	09 10 30.1	48 46 54.7	52.3191129	-0.3	0.2	0.2	0.3	0.0009491
9	09 10 30.1	48 46 54.0	52.3212011	-0.3	0.9	0.9	0.3	-0.0011394
10	09 10 29.1	48 46 55.7	52.3193629	0.7	-0.8	-0.8	-0.3	0.0006990
11	09 10 31.6	48 46 54.6	52.3191466	-1.8	0.3	0.3	1.3	0.0009154

4. Results and Conclusion

The results presented in *Table 3.14* indicate the threedimensional orientation parameters $\{\Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma\}$ and the deflection of the vertical $\{\xi, \eta\}$ to be determined to the range of $1^{''}$. With refined observations, the effects of the refraction properly taken care of, the *partial Procrustes procedure* adequately determines *the threedimensional orientation parameters* leading to the determination of the *deflection of the vertical*. From the 11 experiments performed, it was noted that a slight error in the vertical direction influences greatly the estimates. The influence in the astronomic longitude Λ_Γ was greater for an error in the vertical direction observed in the east-west direction e.g. Haußmannstr. as compared to the same error in magnitude in an observation in the north-south direction on the astronomic latitude Φ_Γ .

5. Appendix : Partial derivative solution of the *Partial Procrustes problem*.

Proceeding from the Frobenius Norm in *Box 1.2*,

$$d_1 = tr\,\mathbf{A}^{\mathrm{T}}\mathbf{A} - 2tr\,\mathbf{A}^{\mathrm{T}}\mathbf{BT} + \mathbf{T}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{BT}. \tag{D.1}$$

and from the condition that $\mathbf{T}^{\mathrm{T}}\mathbf{T} = \mathbf{I}$

$$d_2 = \Lambda(\mathbf{T}^{\mathrm{T}}\mathbf{T} - \mathbf{I}) \tag{D.2}$$

where Λ is the m x m unknown matrix of Lagrange multipliers . On adding (D.1) and (D.2)

$$d = d_1 + d_2 \tag{D.3}$$

we now find the partial derivative of (D.3) with respect to $\mathbf{T}$ as

$$\left[\begin{aligned}
&\frac{\partial d}{\partial \mathbf{T}} = \frac{\partial d_1}{\partial \mathbf{T}} + \frac{\partial d_2}{\partial \mathbf{T}} \\
&\quad = \frac{\partial\left(tr\,\mathbf{A}^{\mathrm{T}}\mathbf{A} - 2tr\,\mathbf{A}^{\mathrm{T}}\mathbf{BT} + tr\,\mathbf{T}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{BT}\right)}{\partial \mathbf{T}} + \frac{\partial\left(\Lambda\,\mathbf{T}^{\mathrm{T}}\mathbf{T} - \Lambda\,\mathbf{I}\right)}{\partial \mathbf{T}} \\
&\quad = -2\mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{B}^{\mathrm{T}}\mathbf{BT} + \mathbf{B}^{\mathrm{T}}\mathbf{BT} + \mathbf{T}\Lambda + \mathbf{T}\Lambda^{\mathrm{T}} \\
&\quad = \left(\mathbf{B}^{\mathrm{T}}\mathbf{B} + \mathbf{B}^{\mathrm{T}}\mathbf{B}\right)\mathbf{T} - 2\mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{T}\left(\Lambda + \Lambda^{\mathrm{T}}\right)
\end{aligned}\right. \tag{D.4}$$

From (D.4) let

$$\mathbf{B}^{\mathrm{T}}\mathbf{B} = \mathbf{B}^*,\ \ \mathbf{B}^{\mathrm{T}}\mathbf{A} = \mathbf{C}\ \text{ and }\ \left(\Lambda + \Lambda^{\mathrm{T}}\right) = 2\Lambda^* \tag{D.5}$$

for an extremum value of d , we set $\frac{\partial d}{\partial T} = 0$

$$\left[\begin{aligned}
&2\mathbf{C} = 2\mathbf{B}^*\mathbf{T} + 2\mathbf{T}\Lambda^* \\
&\mathbf{C} = \mathbf{B}^*\mathbf{T} + \mathbf{T}\Lambda^*
\end{aligned}\right. \tag{D.6}$$

we have that both $\mathbf{B}^*$ and Λ^* are symmetric hence

$$\Lambda^* = \mathbf{T}^{\mathrm{T}}\mathbf{C} - \mathbf{T}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{T} \tag{D.7}$$

But $\mathbf{B}^* \rightarrow$ symmetric and thus $\mathbf{T}^{\mathrm{T}}\mathbf{B}^*\mathbf{T}$ is also symmetric.
$\mathbf{T}^{\mathrm{T}}\mathbf{C}$ is therefore symmetric or

$$\left[\begin{aligned}
&\mathbf{T}^{\mathrm{T}}\mathbf{C} = \mathbf{C}^{\mathrm{T}}\mathbf{T} \\
&\qquad \text{from the side condition} \\
&\mathbf{T}^{\mathrm{T}}\mathbf{T} = \mathbf{TT}^{\mathrm{T}} = \mathbf{I}_3 \\
&\qquad \text{we have that} \\
&\quad \mathbf{C} = \mathbf{TC}^{\mathrm{T}}\mathbf{T}
\end{aligned}\right. \tag{D.8}$$

From *Box 1.3*, we have that $\mathbf{C} = \mathbf{A}^{\mathrm{T}}\mathbf{B} = \mathbf{U\Sigma V}^{\mathrm{T}}$ by SVD. In the present case we note that $\mathbf{C} = \mathbf{B}^{\mathrm{T}}\mathbf{A}$ thus $\mathbf{C} = \mathbf{B}^{\mathrm{T}}\mathbf{A} = \mathbf{V\Sigma U}^{\mathrm{T}}$. From equation (D.8) we have

$$\left[\begin{aligned}
&\text{with }\ \mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{UU}^{\mathrm{T}} = \mathbf{V}^{\mathrm{T}}\mathbf{V} = \mathbf{V}\mathbf{V}^{\mathrm{T}} = \mathbf{I}_3 \\
&\mathbf{C} = \mathbf{TC}^{\mathrm{T}}\mathbf{T} \\
&\mathbf{V}\,\Sigma\,\mathbf{U}^{\mathrm{T}} = \mathbf{TU\Sigma V}^{\mathrm{T}}\mathbf{T} \\
&\therefore\ \mathbf{V} = \mathbf{T}\,\mathbf{U} \\
&\text{Or }\ \ \mathbf{T} = \mathbf{VU}^{\mathrm{T}}
\end{aligned}\right. \tag{D.10}$$

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