Somigliana-Pizzetti Minimum Distance Telluroid Mapping

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Abstract

A minimum distance mapping from the physical surface of the earth to the telluroid under the normal filed of *Somigliana-Pizzetti* is constructed. The point-wise minimum distance mapping under the constraint that actual gravity potential at the a point of physical surface of the earth be equal to normal potential of *Somigliana-Pizzetti* leads to a system of four nonlinear equations. The normal equations of minimum distance mapping are derived and solved via *Newton-Raphson iteration*. The problem of the existence and uniqueness of the solution is addressed. As a case study the quasi-geoid for the state Baden-Württemberg (Germany) is computed.

0. Introduction

We start with the definition of telluroid, after *E. Grafarend* (1978), as the best approximate representation of the surface of the earth. Given the geometry and potential field of the earth surface the telluroid can be completely defined as soon as we define a projection scheme. Telluroid mapping from the known surface of the earth has already been studied by the *A. Bode and E. Grafarend* (1982). They have presented an *isoparametric mapping* from the surface of the earth onto the telluroid under the influence of the spherical normal field including the centrifugal term. *Isoparametric mapping* is based on three assumptions/constrains, namely (i) $\lambda_p = \Lambda_P$, (ii) $\phi_p = \Phi_P$, and (iii) $w_p = W_P$. The triple coordinates $\{\lambda_p, \phi_p, w_p\}$ are representing the known longitude, latitude and gravity potential on the surface of the earth, and $\{\Lambda_P, \Phi_P, W_P\}$ are the normal counterpart of the same quantities on the telluroid. The first two constraints in *A. Bode and E. Grafarend* (1982) approach are referring to the definition of the mapping from the surface of the earth onto the telluroid (in their case *isoparametric mapping*). The present approach differs from approach proposed by *A. Bode and E. Grafarend* (1982) in (i) the mapping scheme and (ii) the normal field. Here we will use the *Somigliana-Pizzetti* field as the normal field and employ minimal distance criterion for mapping the surface of the earth onto the telluroid. *E. Grafarend* and *P. Lohse* (1991) have already used the minimum distance mapping to map the points on the physical surface of the earth onto the reference ellipsoid.

Paragraph one deals with the definition of two types of ellipsoidal coordinate systems which are used throughout the sequel. The set up of the variational equations of *Somigliana-Pizzetti minimum distance telluroid mapping* is dealt with in paragraph two. Paragraph three is devoted to our case study, i.e., "*quasi-geoid* for the state Baden-Württemberg".

1. Ellipsoidal Coordinates

Somigliana-Pizzetti field as the gravity field of a level ellipsoid can be most easily described in terms of any type of ellipsoidal coordinates which has an ellipsoid-of-revolution as one of it coordinate surfaces. For this reason here we will use *Jacobi spheroidal* coordinates $\{\lambda, \phi, u\}$ to present the *Somigliana-Pizzetti field*. *Jacobi spheroidal* coordinates $\{\lambda, \phi, u\}$ is one of the four variants of ellipsoidal coordinates in which the *Laplace partial differential equation* separates. A résumé of basic geometry of *Jacobi spheroidal* coordinates $\{\lambda,\phi,u\}$ is given in *Definition 1-1*. Details on Jacobi spheroidal coordinates can be found in *P. Moon and D. Spencer* (1953, 1961), *N. Thong and E. Grafarend* (1989) for example. Besides since the GPS/GLONASS coordinate are normally given in terms of *Gauss* spheroidal coordinates $\{l,b,h\}$ *Definition1-2* is included which covers some basic properties of *Gauss* spheroidal coordinates $\{l,b,h\}$. *Definition 1-3* provides us with forward and backward transformation between *Gauss* and *Jacobi spheroidal coordinates* after *E. Grafarend, A. Ardalan, M. Sideris* (1999).

Definition 1-1: *Jacobi spheroidal* coordinates $\{\lambda,\phi,u\}$ in $\mathbb{R}^3$

(i) Conversion of Cartesian coordinates $\{x,y,z\}$ *into Jacobi spheroidal coordinates* $\{\lambda,\phi,u\}$

(a) Forward transformation from spheroidal coordinates $\{\lambda,\phi,u\}$ *to Cartesian coordinates* $\{x,y,z\}$

$$\begin{aligned} x &= \sqrt{u^2+\varepsilon^2}\cos\phi\cos\lambda \\ y &= \sqrt{u^2+\varepsilon^2}\cos\phi\sin\lambda \\ z &= u\sin\phi \end{aligned} \tag{1.1}$$

$\varepsilon := \sqrt{a^2-b^2}$ defines the absolute eccentricity.

(b) Backward transformation from Cartesian coordinates $\{x,y,z\}$ *to spheroidal coordinates* $\{\lambda,\phi,u\}$

$$l = \begin{cases} \arctan\dfrac{y}{x} & \text{for } x>0 \text{ and } y\geq 0 \\ \arctan\dfrac{y}{x}+\pi & \text{for } x<0 \text{ and } y\neq 0 \\ \arctan\dfrac{y}{x}+2\pi & \text{for } x>0 \text{ and } y<0 \\ \dfrac{\pi}{2} & \text{for } x=0 \text{ and } y>0 \\ 3\dfrac{\pi}{2} & \text{for } x=0 \text{ and } y<0 \end{cases} \tag{1.2}$$

$$\phi = (\operatorname{sgn} z)\arccos\left\{\frac{1}{2\varepsilon^2}[(x^2+y^2+z^2)+\varepsilon^2-\sqrt{(x^2+y^2+z^2+\varepsilon^2)^2-4\varepsilon^2(x^2+y^2)}]\right\}^{1/2} \tag{1.3}$$

$$u = \{\frac{1}{2}[x^2+y^2+z^2-\varepsilon^2+\sqrt{(x^2+y^2+z^2-\varepsilon^2)^2+4\varepsilon^2 z^2}]\}^{1/2} \tag{1.4}$$

(ii) Jacobi matrix of forward transformation $\{x,y,z\}\mapsto\{\lambda,\phi,u\}$

$$J = \begin{bmatrix} -\sqrt{u^2+\varepsilon^2}\cos\phi\sin\lambda & -\sqrt{u^2+\varepsilon^2}\sin\phi\cos\lambda & u/\sqrt{u^2+\varepsilon^2}\cos\phi\cos\lambda \\ \sqrt{u^2+\varepsilon^2}\cos\phi\cos\lambda & -\sqrt{u^2+\varepsilon^2}\sin\phi\sin\lambda & u/\sqrt{u^2+\varepsilon^2}\cos\phi\sin\lambda \\ 0 & u\cos\phi & \sin\phi. \end{bmatrix} \tag{1.5}$$

(iii) Length element

$$dS^2 = [d\lambda, d\phi, du]J^* J\begin{bmatrix} d\lambda \\ d\phi \\ du \end{bmatrix} \tag{1.6}$$

(iv) Metric tensor

$$G := J^* J = \begin{bmatrix} (u^2+\varepsilon^2)\cos^2\phi & 0 & 0 \\ 0 & u^2+\varepsilon^2\sin^2\phi & 0 \\ 0 & 0 & (u^2+\varepsilon^2\sin^2\phi)/(u^2+\varepsilon^2) \end{bmatrix} := [g_{nm}] \quad \forall \qquad n,m \in \{1,2,3\}$$

(1.7)

■

Definition1-2: *Gauss* spheroidal coordinates $\{l,b,h\}$ in $\mathbb{R}^3$

(i) Conversion of Cartesian coordinates $\{x,y,z\}$ *into Gauss spheroidal coordinates* $\{l,b,h\}$

(a) Forward transformation from spheroidal coordinates $\{l,b,h\}$ *to Cartesian coordinates* $\{x,y,z\}$

$$\begin{bmatrix} x = [\dfrac{a}{\sqrt{1-e^2\sin^2 b}} + h]\cos b\cos l \\ y = [\dfrac{a}{\sqrt{1-e^2\sin^2 b}} + h]\cos b\sin l \\ z = [\dfrac{a(1-e^2)}{\sqrt{1-e^2\sin^2 b}} + h]\sin b \end{bmatrix} \tag{1.8}$$

$e := \sqrt{a^2 - b^2}\,/\,a$ defines the relative eccentricity.

(b) Backward transformation from Cartesian coordinates $\{x,y,z\}$ *to spheroidal coordinates* $\{l,b,h\}$

$$l = \begin{cases} \arctan\dfrac{y}{x} & \text{for } x > 0 \text{ and } y \geq 0 \\ \arctan\dfrac{y}{x} + \pi & \text{for } x < 0 \text{ and } y \neq 0 \\ \arctan\dfrac{y}{x} + 2\pi & \text{for } x > 0 \text{ and } y < 0 \\ \dfrac{\pi}{2} & \text{for } x = 0 \text{ and } y > 0 \\ 3\dfrac{\pi}{2} & \text{for } x = 0 \text{ and } y < 0 \end{cases} \tag{1.9}$$

$b(x,y,z)$, $h(x,y,z)$, can be derived either by *Newton* iteration *or* by solving a system of algebraic equations (*E. Grafarend and P. Lohse* (1991)), *or* by using closed formulae of *K. Borkowski* (1989), *H. Heikkinen* (1982) or *M. Paul* (1973), for instance.

(ii) Jacobi matrix of forward transformation $\{l,b,h\} \mapsto \{x,y,z\}$

$$J := \begin{bmatrix} x_l & x_b & x_h \\ y_l & y_b & y_h \\ z_l & z_b & z_h \end{bmatrix}, \tag{1.10}$$

subject to

$$x_l = D_l x = -[\frac{a}{\sqrt{1-e^2\sin^2 b}} + h]\cos b\sin l$$

$$x_b = D_b x = -[\frac{a}{\sqrt{1-e^2\sin^2 b}} + h]\sin b\cos l + \frac{ae^2\sin b\cos b}{(1-e^2\sin^2 b)^{3/2}}\cos b\cos l$$
$$x_h = D_h x = \cos b\cos l$$

$$y_l = D_l y = [\frac{a}{\sqrt{1-e^2\sin^2 b}} + h]\cos b\cos l$$
$$y_b = D_b y = -[\frac{a}{\sqrt{1-e^2\sin^2 b}} + h]\sin b\sin l + \frac{ae^2\sin b\cos b}{(1-e^2\sin^2 b)^{3/2}}\cos b\sin l$$
$$y_h = D_h y = \cos b\sin l$$

$$z_l = D_l z = 0$$
$$z_b = D_b z = [\frac{a(1-e^2)}{\sqrt{1-e^2\sin^2 b}} + h]\cos b + \frac{a(1-e^2)e^2\sin b\cos b}{(1-e^2\sin^2 b)^{3/2}}\sin b$$
$$z_h = D_h z = \sin b$$

(iii) Distance element "Metric of $\{\mathbb{R}^3, g_{kl}\}$*"*

$$ds^2 = [dl, db, dh]J^* J\begin{bmatrix} dl \\ db \\ dh \end{bmatrix} \tag{1.11}$$

(iv) Metric tensor

$$G := J^* J = \begin{bmatrix} (\frac{a}{\sqrt{1-e^2\sin^2 b}} + h)^2\cos^2 b & 0 & 0 \\ 0 & (\frac{a(1-e^2)}{(1-e^2\sin^2 b)^{3/2}} + h)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.12}$$

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Definition 1-3: *Forward* transformation of *Gauss* ellipsoidal coordinates into *Jacobi* spheroidal coordinates and vice versa

(i) Forward transformation equations $\{\lambda, \phi, u\} \mapsto \{l, b, h\}$

$$\lambda = l \tag{1.13}$$
$$\phi = \arctan(\sqrt{1-e^2}\tan b) \tag{1.14}$$
$$u = \frac{1}{\sqrt{1-e^2}}\cos b\left[\frac{a(1-e^2)}{(1-e^2\sin^2 b)^{1/2}} + h\right]\left[1 + (1-e^2)\tan^2 b\right]^{1/2} \tag{1.15}$$

(ii) Backward transformation equations $\{\lambda, \phi, u\} \mapsto \{l, b, h\}$

$$l = \lambda \tag{1.16}$$
$$b = \arctan(\frac{1}{\sqrt{1-e^2}}\tan\phi) \tag{1.17}$$
$$h = \sqrt{1-e^2}\,u\cos(\phi)[1 + \frac{1}{1-e^2}\tan^2\phi]^{1/2} - a(1-e^2)\left[1 - e^2\frac{\tan^2\phi}{1-e^2+\tan^2\phi}\right]^{-1/2} \tag{1.18}$$

■

2. Variational Equations of Somigliana-Pizzetti Minimum Distance Telluroid Mapping

We define the *Somigliana-Pizzetti minimum distance telluroid mapping*, as follows:

> Given the point $p(\mathbf{x})$ on the surface of the Earth $\mathbb{M}_h^2$, i.e. $p(\mathbf{x}) \in \mathbb{M}_h^2$, with potential value $w_p = w(\mathbf{x})$,
> find the point $P(\mathbf{X})$ such that;
> (i) the normal *Somigliana-Pizzetti* potential field $W_P = W(\mathbf{X})$ at point $P(\mathbf{X}) \in \mathbb{M}_H^2$ be equal to the actual potential at $p(\mathbf{x}) \in \mathbb{M}_h^2$,
> (ii) the point $P(\mathbf{X}) \in \mathbb{M}_H^2$ be at minimum (*Euclidean*) distance from the point $p(\mathbf{x}) \in \mathbb{M}_h^2$.on the physical surface of the earth.

The surface $\mathbb{M}_H^2$ is called *Molodensky telluroid*, or specifically in our case the *Molodensky telluroid of Somigliana-Pizzetti type*. *Figure 2-1* shows the points $p(\mathbf{x})$ on the earth's surface and its minimum distance projection $P(\mathbf{X})$ onto the telluroid.

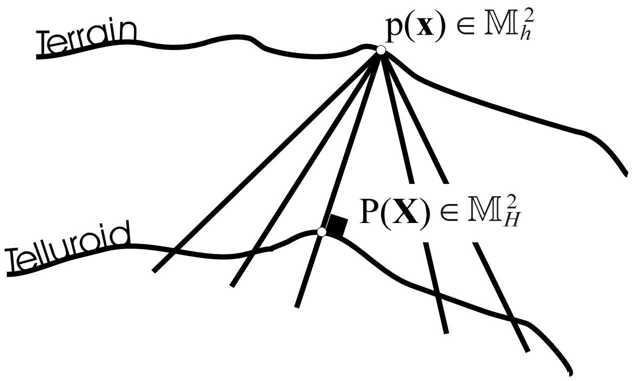

Figure 2-1: Point $p(\mathbf{x}) \in \mathbb{M}_h^2$ on the topographic surface and its minimum distance mapping onto the surface of type telluroid $\mathbb{M}_H^2$: Various projections, but one orthogonal projection of $\mathbb{M}_h^2$ onto $\mathbb{M}_H^2$.

In order to solve the original optimisation problem we proceed as following:

> Minimise the *Euclidean* distance between the point $p(\mathbf{x}) \in \mathbb{M}_h^2$ on the physical surface of the earth and the point $P(\mathbf{X}) \in \mathbb{M}_H^2$ on the *telluroid*,
>
> such that
>
> the normal potential $W_P = W(\mathbf{X})$ on the telluroid, according to *Somigliana-Pizzetti* normal field, be equal to the actual potential $w_p = w(\mathbf{x})$ at the topographic surface of the earth.

Analytically we formulate the optimisation problem by minimising the *constraint Lagrangean*:

$$\begin{aligned} &L(x_1,x_2,x_3,x_4) := \|\mathbf{x}-\mathbf{X}\|^2 + x_4(W_P - w_p) \\ &= [x - X(x_1,x_2,x_3)]^2 + [y - Y(x_1,x_2,x_3)]^2 + [z - Z(x_1,x_2,x_3)]^2 + x_4[W(x_1,x_2,x_3) - w_p] \\ &= \min_{x_1,x_2,x_3,x_4} \end{aligned} \tag{2.1}$$

where $(x_1,x_2,x_3) = (\Lambda,\Phi,U)$ are *Jacobi spheroidal coordinates of the point* $P \sim \mathbf{X} \in \mathbb{M}_H^2$ on the telluroid, and x_4 is the unknown *Lagrange multiplier*.

Since the most suitable coordinate system to present the *Somigliana-Pizzetti field* is ellipsoidal coordinates, we formulate our minimisation problem in terms of *Jacobi spheroidal coordinates* $\{\lambda,\phi,u\}$.

Definition 2-1 presents the *Somigliana-Pizzetti* gravity potential field in terms of *Jacobi-spheroidal* coordinates $\{\lambda,\phi,u\}$. *Somigliana-Pizzetti* field has been developed by *P. Pizzetti* (1894) and *C. Somigliana* (1930) and recently extensively analysed by *E. Grafarend and A. Ardalan* (1999) in functional analytical terms.

Definition 2-1: *Somigliana-Pizzetti* field as developed by *P. Pizzetti* (1894), *C. Somigliana* (1930), and review by *E. Grafarend* and *A. Ardalan* (1999)

Somigliana-Pizzetti field as the gravity field of a rotational ellipsoid

$$\begin{aligned} W(\phi,u) &= \frac{GM}{\varepsilon}\operatorname{arc}\cot(\frac{u}{\varepsilon}) \\ &+\frac{1}{6}\Omega^2 a^2 \frac{(3\frac{u^2}{\varepsilon^2}+1)\operatorname{arc}\cot(\frac{u}{\varepsilon})-3\frac{u}{\varepsilon}}{(3\frac{b^2}{\varepsilon^2}+1)\operatorname{arc}\cot(\frac{b}{\varepsilon})-3\frac{b}{\varepsilon}}(3\sin^2\phi-1) \\ &+\frac{1}{2}\Omega^2(u^2+\varepsilon^2)\cos^2\phi \end{aligned} \tag{2.2}$$

■

Using the forward transformation relations of $\{\lambda,\phi,u\}\mapsto\{x,y,z\}$ (see *Equation* (1.1) in *Definition 1-1*) the functional $L(x_1,x_2,x_3,x_4)$ can be written as

$$\begin{aligned} L(\Lambda_P,\Phi_{P2},U_P,\lambda) :=& (x_p-\sqrt{{U_P}^2+\varepsilon^2}\cos\Phi_P\cos\Lambda_P)^2+(y_p-\sqrt{{U_P}^2+\varepsilon^2}\cos\Phi_P\sin\Lambda_P)^2+(z_p+U_P\sin\Phi_P)^2 \\ &+x_4(W(\Phi_P,U_P)-w_p) \end{aligned} \tag{2.3}$$

or

$$\begin{aligned} L(x_1,x_2,x_3,x_4) :=& (x_p-\sqrt{{x_3}^2+\varepsilon^2}\cos x_2\cos x_1)^2+(y_p-\sqrt{{x_3}^2+\varepsilon^2}\cos x_2\sin x_1)^2+(z_p+x_3\sin x_2)^2 \\ &+x_4(W(x_2,x_3)-w_p) \end{aligned} \tag{2.4}$$

where $\{x_1,x_2,x_3\}$ are unknown *Jacobi spheroidal coordinates* of the point $P(\Lambda,\Phi,U)=P(x_1,x_2,x_3)$ on the telluroid ($P(\mathbf{X}) \in \mathbb{M}_H^2$), $W(\Phi,U)=W(x_2,x_3)$ corresponds to *Somigliana-Pizetti* potential field at point $P(\Lambda,\Phi,U)\in \mathbb{M}_H^2$ according to (2.2), and w_p refers to actual gravity potential at point $p\{x,y,z\}$ on the surface of the earth.

The functional $L(\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4)$ is minimal if and only if following two conditions hold:

$$
\text{(i)}\quad \begin{cases}
f_1 := \dfrac{\partial L}{\partial x_1}(\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4) = 2\sqrt{\hat{x}_3{}^2+\varepsilon^2}\cos\hat{x}_2(x_p\sin\hat{x}_1 - y_p\cos\hat{x}_1) = 0 \\
f_2 := \dfrac{\partial L}{\partial x_2}(\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4) = 2(x_p - \sqrt{\hat{x}_3{}^2+\varepsilon^2}\cos\hat{x}_2\cos\hat{x}_1)\sqrt{\hat{x}_3{}^2+\varepsilon^2}\sin\hat{x}_2\cos\hat{x}_1 \\
\qquad +2(y_p - \sqrt{\hat{x}_3{}^2+\varepsilon^2}\cos\hat{x}_2\sin\hat{x}_1)\sqrt{\hat{x}_3{}^2+\varepsilon^2}\sin\hat{x}_2\sin\hat{x}_1 \\
\qquad -2(z_p - \hat{x}_3\sin\hat{x}_2)\hat{x}_3\cos\hat{x}_2 + \hat{x}_4(\dfrac{\partial W}{\partial x_2}(\hat{x}_2,\hat{x}_3)) = 0 \\
f_3 := \dfrac{\partial L}{\partial x_3}(\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4) = -2\dfrac{(x_p - \sqrt{\hat{x}_3{}^2+\varepsilon^2}\cos\hat{x}_2\cos\hat{x}_1)\hat{x}_3\cos\hat{x}_2\cos\hat{x}_1}{\sqrt{\hat{x}_3{}^2+\varepsilon^2}} \\
\qquad -2\dfrac{(y_p - \sqrt{\hat{x}_3{}^2+\varepsilon^2}\cos\hat{x}_2\sin\hat{x}_1)\hat{x}_3\cos\hat{x}_2\sin\hat{x}_1}{\sqrt{\hat{x}_3{}^2+\varepsilon^2}} \\
\qquad -2(z_p - \hat{x}_3\sin\hat{x}_2)\sin\hat{x}_2 + \hat{x}_4\dfrac{\partial W}{\partial x_3}(\hat{x}_2,\hat{x}_3) = 0 \\
f_4 := \dfrac{\partial L}{\partial x_4}(\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4) = W(\hat{x}_2,\hat{x}_3) - w_p = 0
\end{cases} \tag{2.5}
$$

$$
\text{(ii)}\quad \frac{\partial^2 L}{\partial x_i \partial x_j}(\hat{x}_1,\hat{x}_2,\hat{x}_3) \text{ be positive-semi-definite for } i,j = 1,2,3 \tag{2.6}
$$

Partial derivatives $\partial W/\partial\phi$ and $\partial W/\partial u$ of (2.5) can be readily derive from (2.2) as follows

$$
\frac{\partial W}{\partial\phi} = \frac{\partial W}{\partial x_2} = a^2\Omega^2\frac{(3x_3{}^2+\varepsilon^2)\operatorname{arccot}(\dfrac{x_3}{\varepsilon}) - 3x_3\varepsilon}{[(3b^2+\varepsilon^2)\operatorname{arccot}(\dfrac{b}{\varepsilon}) - b\varepsilon]}\sin x_2\cos x_2 - \Omega^2(x_3{}^2+\varepsilon^2)\sin x_2\cos x_2 \tag{2.7}
$$

$$
\begin{aligned}
\frac{\partial W}{\partial u} = \frac{\partial W}{\partial x_3} &= -\frac{GM}{x_3{}^2+\varepsilon^2} \\
&-\frac{1}{3}\Omega^2 a^2\frac{\varepsilon(3x_3{}^2+2\varepsilon^2) + (-3x_3{}^3 - 3x_3\varepsilon^2)\operatorname{arccot}(\dfrac{x_3}{\varepsilon})}{(x_3{}^2+\varepsilon^2)[\operatorname{arccot}(\dfrac{b}{\varepsilon})\varepsilon^2 + (-3\varepsilon + 3\operatorname{arccot}(\dfrac{b}{\varepsilon})b)b]}(3\sin^2 x_2 - 1) \\
&+\Omega^2 x_3\cos^2 x_2
\end{aligned} \tag{2.8}
$$

Equations (2.5) builds up the variational equations of the optimisation problem (2.1). System of equations (2.5) is a nonlinear system; the *B. Taylor* expansion of it reads

$$
\begin{aligned}
\mathbf{F}(\mathbf{x}) &= \mathbf{F}(\mathbf{x}_0) + \frac{1}{1!}\mathbf{F}'(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) \\
&+\frac{1}{2!}\mathbf{F}''(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)\otimes(\mathbf{x}-\mathbf{x}_0) + \mathcal{O}_3\big((\mathbf{x}-\mathbf{x}_0)\otimes(\mathbf{x}-\mathbf{x}_0)\otimes(\mathbf{x}-\mathbf{x}_0)\big) \\
&= \mathbf{F}_0 + \mathbf{J}_0(\mathbf{x}-\mathbf{x}_0) + \frac{1}{2}\mathbf{H}_0(\mathbf{x}-\mathbf{x}_0)\otimes(\mathbf{x}-\mathbf{x}_0) + \mathcal{O}_3
\end{aligned} \tag{2.9}
$$

where

$$\mathbf{F}=\begin{bmatrix} f_1(x_1,x_2,x_3,x_4) \\ f_2(x_1,x_2,x_3,x_4) \\ f_3(x_1,x_2,x_3,x_4) \\ f_4(x_1,x_2,x_3,x_4) \end{bmatrix},\ \mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},\ \mathbf{F}':=\mathbf{J}=\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix},$$

$$\mathbf{F}'':=\mathbf{H}=$$

$$\begin{bmatrix}
\frac{\partial^2 f_1}{\partial x_1^2} & \frac{\partial^2 f_1}{\partial x_1 x_2} & \frac{\partial^2 f_1}{\partial x_1 x_3} & \frac{\partial^2 f_1}{\partial x_1 x_4} & \frac{\partial^2 f_1}{\partial x_2 x_1} & \frac{\partial^2 f_1}{\partial x_2^2} & \frac{\partial^2 f_1}{\partial x_2 x_3} & \frac{\partial^2 f_1}{\partial x_2 x_4} & \frac{\partial^2 f_1}{\partial x_3 x_1} & \frac{\partial^2 f_1}{\partial x_3 x_2} & \frac{\partial^2 f_1}{\partial x_3^2} & \frac{\partial^2 f_1}{\partial x_3 x_4} & \frac{\partial^2 f_1}{\partial x_4 x_1} & \frac{\partial^2 f_1}{\partial x_4 x_2} & \frac{\partial^2 f_1}{\partial x_4 x_3} & \frac{\partial^2 f_1}{\partial x_4^2} \\
\frac{\partial^2 f_2}{\partial x_1^2} & \frac{\partial^2 f_2}{\partial x_1 x_2} & \frac{\partial^2 f_2}{\partial x_1 x_3} & \frac{\partial^2 f_2}{\partial x_1 x_4} & \frac{\partial^2 f_2}{\partial x_2 x_1} & \frac{\partial^2 f_2}{\partial x_2^2} & \frac{\partial^2 f_2}{\partial x_2 x_3} & \frac{\partial^2 f_2}{\partial x_2 x_4} & \frac{\partial^2 f_2}{\partial x_3 x_1} & \frac{\partial^2 f_2}{\partial x_3 x_2} & \frac{\partial^2 f_2}{\partial x_3^2} & \frac{\partial^2 f_2}{\partial x_3 x_4} & \frac{\partial^2 f_2}{\partial x_4 x_1} & \frac{\partial^2 f_2}{\partial x_4 x_2} & \frac{\partial^2 f_2}{\partial x_4 x_3} & \frac{\partial^2 f_2}{\partial x_4^2} \\
\frac{\partial^2 f_3}{\partial x_1^2} & \frac{\partial^2 f_3}{\partial x_1 x_2} & \frac{\partial^2 f_3}{\partial x_1 x_3} & \frac{\partial^2 f_3}{\partial x_1 x_4} & \frac{\partial^2 f_3}{\partial x_2 x_1} & \frac{\partial^2 f_3}{\partial x_2^2} & \frac{\partial^2 f_3}{\partial x_2 x_3} & \frac{\partial^2 f_3}{\partial x_2 x_4} & \frac{\partial^2 f_3}{\partial x_3 x_1} & \frac{\partial^2 f_3}{\partial x_3 x_2} & \frac{\partial^2 f_3}{\partial x_3^2} & \frac{\partial^2 f_3}{\partial x_3 x_4} & \frac{\partial^2 f_3}{\partial x_4 x_1} & \frac{\partial^2 f_3}{\partial x_4 x_2} & \frac{\partial^2 f_3}{\partial x_4 x_3} & \frac{\partial^2 f_3}{\partial x_4^2} \\
\frac{\partial^2 f_4}{\partial x_1^2} & \frac{\partial^2 f_4}{\partial x_1 x_2} & \frac{\partial^2 f_4}{\partial x_1 x_3} & \frac{\partial^2 f_4}{\partial x_1 x_4} & \frac{\partial^2 f_4}{\partial x_2 x_1} & \frac{\partial^2 f_4}{\partial x_2^2} & \frac{\partial^2 f_4}{\partial x_2 x_3} & \frac{\partial^2 f_4}{\partial x_2 x_4} & \frac{\partial^2 f_4}{\partial x_3 x_1} & \frac{\partial^2 f_4}{\partial x_3 x_2} & \frac{\partial^2 f_4}{\partial x_3^2} & \frac{\partial^2 f_4}{\partial x_3 x_4} & \frac{\partial^2 f_4}{\partial x_4 x_1} & \frac{\partial^2 f_4}{\partial x_4 x_2} & \frac{\partial^2 f_4}{\partial x_4 x_3} & \frac{\partial^2 f_4}{\partial x_4^2}
\end{bmatrix}$$

and $\otimes$ stands for *Kronecker* tensor product.

Newton iteration solution (*X. Chen et al.* (1997)) can be performed by the n-sequence

$$\mathbf{x}-\ \mathbf{x}_0=\Delta\mathbf{x}=\mathbf{J}_0^{-1}(\mathbf{F}-\mathbf{F}_0)=\left(\mathbf{J}(\mathbf{x}_0)\right)^{-1}(\mathbf{F}-\mathbf{F}_0) \tag{2.10}$$

$$\mathbf{x}-\ \mathbf{x}_0\ =\mathbf{J}^{-1}(\mathbf{F}-\mathbf{F}_0)$$

$$\Rightarrow\ \ \mathbf{x}_1=\mathbf{x}_0+\mathbf{J}_0^{-1}(\mathbf{F}-\mathbf{F}_0) \tag{2.11}$$

$$\Rightarrow\ \ \mathbf{x}_2\ =\mathbf{x}_1+\mathbf{J}_1^{-1}(\mathbf{F}-\mathbf{F}_1) \tag{2.12}$$

$$\Rightarrow\qquad\cdots\qquad\Rightarrow\mathbf{x}_n=\mathbf{x}_{n-1} \tag{2.13}$$

where Jacobean matrix of linearized form of the variational equations (2.5) reads as

$$\mathbf{J}:=\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix} \tag{2.14}$$

$$\frac{\partial f_1}{\partial x_1}=2({x_3}^2+\varepsilon^2)^{1/2}\cos x_2(x_p\cos x_1+y_p\sin x_1)$$

$$\frac{\partial f_1}{\partial x_2}=2({x_3}^2+\varepsilon^2)^{1/2}\sin x_2(-x_p\sin x_1+y_p\cos x_1)$$

$$\frac{\partial f_1}{\partial x_3}=-2\frac{\cos x_2(-x_p\sin x_1+y_p\cos x_1)x_3}{({x_3}^2+\varepsilon^2)^{1/2}}$$

$$\frac{\partial f_1}{\partial x_4}=0$$

$$\frac{\partial f_2}{\partial x_1}=-2({x_3}^2+\varepsilon^2)^{1/2}\sin x_2(x_p\sin\ x_1-y_p\cos x_1)$$

$$\begin{aligned}\frac{\partial f_2}{\partial x_2}=&2({x_3}^2+\varepsilon^2)(\sin x_2)^2(\cos x_1)^2+2(x_p-({x_3}^2+\varepsilon^2)^{1/2}\cos(x_2)\cos x_1)({x_3}^2+\varepsilon^2)^{1/2}\cos x_2\cos x_1+2({x_3}^2\\ &+\varepsilon^2)(\sin x_2)^2(\sin x_1)^2+2(y_p-({x_3}^2+\varepsilon^2)^{1/2}\cos x_2\sin x_1)({x_3}^2+\varepsilon^2)^{1/2}\cos x_2\sin x_1+2{x_3}^2(\cos x_2)^2+2(z_p\\ &-x_3\sin x_2)x_3\sin x_2+x_4(\Omega^2a^2((3{x_3}^2/\varepsilon^2+1)\mathrm{acot}(x_3/\varepsilon)-3x_3/\varepsilon)/((3b^2/\varepsilon^2+1)\mathrm{acot}(b/\varepsilon)\end{aligned}$$

$$-3b/\varepsilon)\cos(x_2)^2-\Omega^2 a^2((3x_3^2/\varepsilon^2+1)\mathrm{acot}(x_3/\ )-3x_3/\ )/((3b^2/\varepsilon^2+1)\mathrm{acot}(b/\ )-3b/\ )(\sin x_2)^2$$

$$+\Omega^2(x_3^2+\varepsilon^2)(\sin x_2)^2-\Omega^2(x_3^2+\varepsilon^2)(\cos x_2)^2)$$

$$\frac{\partial f_2}{\partial x_3}=-2\sin x_2(\cos x_1)^2\cos(x_2)x_3+2(x_p-(x_3{}^2+\ {}^2)^{1/2}\cos x_2\cos x_1)/(x_3{}^2+\ {}^2)^{1/2}\sin x_2\cos x_1(x_3)$$

$$-2\sin x_2(\sin x_1)^2\cos x_2(x_3)+2(y_p-(x_3{}^2+\ {}^2)^{1/2}\cos x_2\sin x_1)/(x_3{}^2+\ {}^2)^{1/2}\sin x_2\sin x_1(x_3)$$

$$+2x_3\cos x_2\sin x_2-2(z_p-x_3\sin x_2)\cos x_2+x_4(\Omega^2a^2(6x_3\ \ {}^2\mathrm{acot}(x_3/\ )$$

$$-(3x_3{}^2/\ {}^2+1)/\ /(1+x_3{}^2/\ {}^2)-3/\ )/((3b^2/\ {}^2+1)\mathrm{acot}(b/\ )-3b/\ )\sin x_2\cos x_2$$

$$-2\Omega^2x_3\cos x_2\sin x_2)$$

$$\frac{\partial f_2}{\partial x_4}=\Omega^2a^2((3x_3{}^2/\ {}^2+1)\mathrm{acot}(x_3/\ )-3x_3/\ )/((3b^2/\ {}^2+1)\mathrm{acot}(b/\ )-3b/\ )\sin x_2\cos x_2$$

$$-\Omega^2(x_3{}^2+\ {}^2)\cos x_2\sin x_2$$

$$\frac{\partial f_3}{\partial x_1}=2\frac{\cos x_2x_3(\sin x_1x_p-\cos x_1y_p)}{(x_3{}^2+\varepsilon^2)^{1/2}}$$

$$\frac{\partial f_3}{\partial x_2}\quad 2\sin x_2(\cos x_1)^2\cos(x_2)x_3+2(x_p-(x_3{}^2+\ {}^2)^{1/2}\cos x_2\cos x_1)/(x_3{}^2+\ {}^2)^{1/2}\sin x_2\cos(x_1)x_3$$

$$-2\sin x_2(\sin x_1)^2\cos(x_2)x_3+2(y_p-(x_3{}^2+\ {}^2)^{1/2}\cos x_2\sin x_1)/(x_3{}^2+\ {}^2)^{1/2}\sin x_2\sin(x_1)x_3$$

$$+2x_3\cos x_2\sin x_2-2(z_p-x_3\sin x_2\cos x_2+x_4(\Omega^2a^2(6x_3/\ {}^2\mathrm{acot}(x_3/\ )-(3x_3{}^2/\ {}^2+1)/\ /(1$$

$$+x_3{}^2/\ {}^2)-3/\ )/((3b^2/\ {}^2+1)\mathrm{acot}(b/\ )-3b/\ )\sin x_2\cos x_2-2\Omega^2x_3\cos x_2\sin x_2)$$

$$\frac{\partial f_3}{\partial x_3}=2/(x_3{}^2+\ {}^2)(\cos x_2)^2(\cos x_1)^2x_3{}^2+2(x_p-(x_3{}^2+\ {}^2)^{1/2}\cos x_2\cos x_1)/(x_3{}^2+\ {}^2)^{3/2}\cos x_2\cos(x_1)x_3{}^2$$

$$-2(x_p-(x_3{}^2+\ {}^2)^{1/2}\cos x_2\cos x_1)/(x_3{}^2+\ {}^2)^{1/2}\cos x_2\cos x_1+2/(x_3{}^2+\ {}^2)(\cos x_2)^2\sin(x_1)^2x_3{}^2$$

$$+2(y_p-(x_3{}^2+\ {}^2)^{1/2}\cos x_2\sin x_1)/(x_3{}^2+\ {}^2)^{3/2}\cos x_2\sin x_1x_3{}^2-2(y_p$$

$$-(x_3{}^2+\ {}^2)^{1/2}\cos x_2\sin x_1)/(x_3{}^2+\ {}^2)^{1/2}\cos x_2\sin x_1+2\sin x_2{}^2+x_4(2\mathrm{GM}/\ {}^4/(1+x_3{}^2/\ {}^2)^2x_3$$

$$+1/6\Omega^2a^2(6/\ {}^2\mathrm{acot}(x_3/\ )-12x_3/\ {}^3/(1+x_3{}^2/\ {}^2)+2(3x_3{}^2/\ {}^2+1)/\ {}^3/(1+x_3{}^2/\ {}^2)^2x_3)$$

$$/((3b^2/\ {}^2+1)\mathrm{acot}(b/\ )-3b/\ )(3\sin(x_2)^2-1)+\Omega^2(\cos x_2)^2)$$

$$\frac{\partial f_3}{\partial x_4}\quad \mathrm{GM}/\ {}^2/(1+x_3{}^2/\ {}^2)+1/6\Omega^2a^2(6x_3/\ {}^2\mathrm{acot}(x_3/\ )-(3x_3{}^2/\ {}^2+1)/\ /(1+x_3{}^2/\ {}^2)$$

$$-3/\ )/((3b^2/\ {}^2+1)\mathrm{acot}(b/\ )-3b/\ )(3(\sin x_2)^2-1)+\Omega^2x_3(\cos x_2)^2$$

$$\frac{\partial f_4}{\partial x_1}=0$$

$$\frac{\partial f_4}{\partial x_2}=\Omega^2a^2((3x_3{}^2/\ {}^2+1)\mathrm{acot}(x_3/\ )-3x_3/\ )/((3b^2/\ {}^2+1)\mathrm{acot}(b/\ )-3b/\ )\sin x_2\cos x_2$$

$$-\Omega^2(x_3{}^2+\ {}^2)\cos(x_2)\sin(x_2)$$

$$\frac{\partial f_4}{\partial x_3}\quad \mathrm{GM}/\ {}^2/(1+x_3{}^2/\ {}^2)+1/6\Omega^2a^2(6x_3/\ {}^2\mathrm{acot}(x_3/\ )-(3x_3{}^2/\ {}^2+1)/\ /(1+x_3{}^2/\ {}^2)-$$

$$3/\ )/((3b^2/\ {}^2+1)\mathrm{acot}(b/\ )-3b/\ )(3(\sin x_2)^2-1)+\Omega^2x_3(\cos x_2)^2$$

$$\frac{\partial f_4}{\partial x_4}=0$$

The solution set $(\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4)$ derived from final step of *Newton iteration* (2.13) provides the necessary condition (2.5) for having a minimal solution. This extremal solution is minimal if condition (ii) of (2.6) be also satisfied. Indeed, we must show that *Hesse matrix* $\mathbf{H}_L$ is *positive semi-definite*, i.e., the characteristic polynomials of $|\mathbf{H}_L-\lambda\mathbf{I}|=0$ are all *non-negative*. The *Hesse matrix* $\mathbf{H}_L$ of second derivatives is given below.

$$\mathbf{H}_L = \frac{\partial^2 L}{\partial x_i \partial x_j} = \begin{bmatrix} \dfrac{\partial^2 L}{\partial x_1^{\,2}} & \dfrac{\partial^2 L}{\partial x_1 x_2} & \dfrac{\partial^2 L}{\partial x_1 x_3} \\ \dfrac{\partial^2 L}{\partial x_2 x_1} & \dfrac{\partial^2 L}{\partial x_2^{\,2}} & \dfrac{\partial^2 L}{\partial x_2 x_3} \\ \dfrac{\partial^2 L}{\partial x_3 x_1} & \dfrac{\partial^2 L}{\partial x_3 x_2} & \dfrac{\partial^2 L}{\partial x_3^{\,2}} \end{bmatrix} \tag{2.15}$$

$$\frac{\partial^2 L}{\partial x_1^{\,2}} = 2(x_3^{\,2} + \varepsilon^2)^{1/2} \cos x_2 (x_p \cos x_1 + y_p \sin x_1)$$

$$\frac{\partial^2 L}{\partial x_1 x_2} = 2(x_3^{\,2} + \varepsilon^2)^{1/2} \sin x_2 (-x_p \sin x_1 + y_p \cos x_1)$$

$$\frac{\partial^2 L}{\partial x_1 x_3} = -2 x_3 \cos x_2 (-x_p \sin x_1 + y_p \cos x_1) / (x_3^{\,2} + \varepsilon^2)^{1/2}$$

$$\frac{\partial^2 L}{\partial x_2 x_1} = \frac{\partial^2 L}{\partial x_1 x_2}$$

$$\begin{aligned} \frac{\partial^2 L}{\partial x_2^{\,2}} =& 2(x_3^{\,2}+\varepsilon^2)(\sin x_2) x_3^{\,2} (\cos x_1)^2 + 2(x_p - (x_3^{\,2}+\varepsilon^2)^{1/2} \cos x_2 \cos x_1)(x_3^{\,2}+\varepsilon^2)^{1/2} \cos x_2 \cos x_1 \\ &+2(x_3^{\,2}+\varepsilon^2)(\sin x_2)^2 (\sin x_1)^2 + 2(y_p - (x_3^{\,2}+\varepsilon^2)^{1/2} \cos x_2 \sin x_1)(x_3^{\,2}+\varepsilon^2)^{1/2} \cos x_2 \sin x_1 \\ &+2 x_3^{\,2} \cos x_2^{\,2} + 2(z_p - x_3 \sin(x_2)) x_3 \sin x_2 + x_4 (\Omega^2 a^2 ((3 x_3^{\,2}/\varepsilon^2+1)\mathrm{acot}(x_3/\varepsilon) \\ &-3 x_3/\varepsilon)/((3b^2/\varepsilon^2+1)\mathrm{acot}(b/\varepsilon) - 3b/\varepsilon)(\cos x_2)^2 - \Omega^2 a^2 ((3 x_3^{\,2}/\varepsilon^2+1)\mathrm{acot}(x_3/\varepsilon) \\ &-3 x_3/\varepsilon)/((3b^2/\varepsilon^2+1)\mathrm{acot}(b/\varepsilon) - 3b/\varepsilon)(\sin x_2)^2 + \Omega^2 (x_3^{\,2}+\varepsilon^2)(\sin x_2)^2 \\ &-\Omega^2 (x_3^{\,2}+\varepsilon^2)(\cos x_2)^2) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 L}{\partial x_2 x_3} =& -2 \cos x_2 (\cos x_1)^2 x_3 \sin x_2 \\ &+2(x_p - (x_3^{\,2}+\varepsilon^2)^{1/2} \cos x_2 \cos x_1)/(x_3^{\,2}+\varepsilon^2)^{1/2} \sin x_2 \cos(x_1) x_3 \\ &-2 \cos x_2 (\sin x_1)^2 x_3 \sin x_2 + 2(y_p - (x_3^{\,2}+\varepsilon^2)^{1/2} \cos x_2 \sin x_1)/(x_3^{\,2} \\ &+\varepsilon^2)^{1/2} \sin x_2 \sin(x_1) x_3 + 2\sin(x_2) x_3 \cos x_2 - 2(z_p - x_3 \sin x_2)\cos x_2 \\ &+ x_4 (\Omega^2 a^2 (6 x_3/\varepsilon^2 \mathrm{acot}(x_3/\varepsilon) - (3 x_3^{\,2}/\varepsilon^2+1)/\varepsilon/(1+x_3^{\,2}/\mathrm{e}^2) \\ &-3/\varepsilon)/((3b^2/\varepsilon^2+1)\mathrm{acot}(b/\varepsilon) - 3b/\varepsilon)\sin x_2 \cos x_2 - 2\Omega^2 x_3 \cos x_2 \sin x_2) \end{aligned}$$

$$\frac{\partial^2 L}{\partial x_3 x_1} = \frac{\partial^2 L}{\partial x_1 x_3}$$

$$\frac{\partial^2 L}{\partial x_3 x_2} = \frac{\partial^2 L}{\partial x_2 x_3}$$

$$\begin{aligned} \frac{\partial^2 L}{\partial x_3^{\,2}} =& 2/(x_3^{\,2}+\varepsilon^2)(\cos x_2)^2 (\cos x_1)^2 x_3^{\,2} + 2(x_p - (x_3^{\,2}+\varepsilon^2)^{1/2} \cos x_2 \cos x_1)/(x_3^{\,2}+\varepsilon^2)^{3/2} \cos x_2 \cos(x_1) x_3^{\,2} \\ &-2(x_p - (x_3^{\,2}+\varepsilon^2)^{1/2} \cos x_2 \cos x_1)/(x_3^{\,2}+\varepsilon^2)^{1/2} \cos x_2 \cos x_1 \\ &+2/(x_3^{\,2}+\varepsilon^2)(\cos x_2)^2 (\sin x_1)^2 x_3^{\,2} + 2(y_p - (x_3^{\,2}+\varepsilon^2)^{1/2} \cos x_2 \sin x_1)/(x_3^{\,2} \\ &+\varepsilon^2)^{3/2} \cos x_2 \sin(x_1) x_3^{\,2} - 2(y_p - (x_3^{\,2}+\varepsilon^2)^{1/2} \cos x_2 \sin x_1)/(x_3^{\,2} \\ &+\varepsilon^2)^{1/2} \cos x_2 \sin x_1 + 2(\sin x_2)^2 + x_4 (2\mathrm{GM}/\varepsilon^4/(1+x_3^{\,2}/\varepsilon^2)^2 x_3 \\ &+1/6\, \Omega^2 a^2 (6/\varepsilon^2 \mathrm{acot}(x_3/\varepsilon) - 12 x_3/\varepsilon^3/(1+x_3^{\,2}/\varepsilon^2) \\ &+2(3 x_3^{\,2}/\varepsilon^2+1)/\varepsilon^3/(1+x_3^{\,2}/\varepsilon^2)^2 x_3)/((3b^2/\varepsilon^2+1)\mathrm{acot}(b/\varepsilon) - 3b/\varepsilon)(3(\sin x_2)^2 - 1) \\ &+\Omega^2 (\cos x_2)^2) \end{aligned}$$

We proved the *positive-defintenes*s of the *Hesse-matrix* H_L of second derivatives by a numerical test.

3. Case study quasi-geoid of Baden-Württemberg

Next, we shall present the results of the minimum distance mapping of the *physical surface of the earth* $\mathbb{M}_h^2$ to the *Somigliana-Pizzetti telluroid* $\mathbb{M}_H^2$ for 157 GPS stations in the state Baden-Württemberg/Germany. *Table 1* shows the ten first GPS points of the GPS file of Baden-Württemberg. The coordinates are given in terms of *Gauss ellipsoidal coordinates* $\{l,b,h\}$ with respect to the GRS80 reference ellipsoid. This set of points constitute the Baden-Württemberg part (BWREF) of the German GPS network (DREF) which itself is part to European GPS network (EUREF).

Table 1: A part of the GPS file of the state Baden-Württemberg/Germany

Point ID Number	Longitude (l_p) (deg)	Latitude (b_p) (deg)	Ellipsoidal height (h_p) (m)	Geopotential Number (m^2/s^2)
621707001	8.623833676111	49.71395228806	218.6128	2148.9295328735
631805402	8.812993394167	49.60717789944	587.0355	5785.3299183738
632000110	9.065490941944	49.65066941389	590.7425	5811.1345798656
632107425	9.215170690833	49.65909479083	234.5042	2305.4354809625
632302808	9.575139530833	49.67395950667	379.2613	3733.6137953571
632400308	9.808090562778	49.64845024611	412.8049	4058.8965149575
641400308	8.159236366944	49.59699675556	349.7134	3434.9047294867
641600108	8.470342882778	49.59000986444	136.4937	1339.6896647405
641701308	8.628513233333	49.52709190306	143.9359	1415.1545995409
642100208	9.250862697222	49.55034041111	523.2340	5160.5154883848

The *Gauss ellipsoidal coordinates* $\{l,b,h\}$ of 157 GPS stations are converted to *Jacobi* ellipsoidal coordinates $\{\lambda,\phi,u\}$ according to forward transformation equations (1.13)-(1.15). *Table 2* presents the *Jacobi* ellipsoidal coordinates of the sample stations of *Table 1*.

Table 2: Transferred *Jacobi* ellipsoidal coordinates $\{\lambda,\phi,u\}_p$ of *Table 1*

Point ID Number	λ_p	ϕ_p	u_p
62230002	9.617432869327	49.60898966016	6356968.4504330
62230009	9.549475653403	49.60842348799	6356966.7368298
62230010	9.552776702951	49.61342803807	6356955.3539607
62230029	9.518509890067	49.66515664873	6356940.4330151
62230035	9.584526828221	49.60509389361	6356960.5466633
62230051	9.535064264303	49.62428053013	6356953.7623961
62230054	9.526483136723	49.63746430741	6356947.9810204
62230058	9.525534139567	49.64979432997	6356958.1832061
62230060	9.512823290742	49.65372897068	6356948.4890495
62230064	9.544545338174	49.62132133267	6356961.9798826

Newton Raphson iteration solution of the normal equations (2.5) led to point-wise *telluroid mapping* of all GPS stations in the state of *Baden-Württemberg*. A portion of the results for first ten GPS stations is presented in *Table 3*. Columns 2-4 are referring to *Jacobi* spheroidal coordinates of telluroid projection points. Column 5 presents the difference between *u* component of the GPS stations and their telluroid projection. Finally, column 6 shows the projection of $u_p - U_p$ along the unit vector $\mathbf{E}_u$. The geometrical height $H=(u-U_p)\sqrt{G_{33}}$ presents the separation between the surface of the earth and *Molodensky telluroid*, specifically the minimum distance mapping of the physical surface of the earth to the *Somigliana-Pizzetti* telluroid. If this height be considered as the height above the *reference ellipsoid*, by definition we have a presentation of the *quasi-geoid*. *Figure 4-1* is the result of the *minimum distance mapping* described here for *Baden-Württemberg* in the form of a *quasi-geoid* map.

Finally, the calculated quasi-geoid is compared with new *European Gravimetric Quasi-Geoid* (*EGG97*) (*H. Denker and W. Torge*, 1998). The summary of statistics of this comparison is given in Table 4.

Table 3: Telluroid mapping of the sample GPS stations of *Table 1*

Point ID Number	Λ_P	Φ_P	U_P	$u_p - U_p$	$(u - U_p)\sqrt{G_{33}}$ (m)
621707001	8.6238336761	49.619013350	6356923.6746	47.105907368	47.039684057
631805402	8.8129933941	49.512181451	6357295.0294	44.696132976	44.633028957
632000110	9.0654909419	49.555696232	6357297.6868	45.749522637	45.685046439
632107425	9.2151706908	49.564126141	6356939.7418	46.953250759	46.887093564
632302808	9.5751395308	49.578998873	6357085.8037	45.852377559	45.787813832
632400308	9.8080905627	49.553475871	6357118.3259	46.921681831	46.855543946
641400308	8.1592363669	49.501994892	6357054.1765	47.891587520	47.823938736
641600108	8.4703428827	49.495004292	6356840.8091	47.737670235	47.670215083
641701308	8.6285132333	49.432053186	6356848.6787	47.321327123	47.254288287
642100208	9.2508626972	49.455313888	6357231.6235	44.212039829	44.149472768

Table 4: Statistics of comparison between calculated height anomalies at 157 GPS station in Baden-Württemberg and EGG97

Statistics of $N_{EGG97} - \zeta$	(m)
mean	0.995013857205397
std	1.3227781145868
max	7.40296621419591
min	–0.921456458099193
number of sample points	157

4. Final Remarks and Conclusions

From a review of *Table 1 to Table 4* following conclusions can be made: (i) $\{\Lambda_P, \Phi_P\}$ of the telluroid point *P* is very close to $\{\lambda_p, \phi_p\}$ of point *p* on the surface of the earth. This reveals the fact that the minimum distance mapping of the physical surface of the earth to the *Somigliana-Pizzetti* telluroid is very close to the mapping along the coordinate line of *u*. (ii) The calculated quasi-geoid for GPS station based on minimum distance mapping of the physical surface of the earth to the *Somigliana-Pizzetti* deviates from *EGG97* by (0.995±1.322778)(m) on average. This difference can be mainly associated to the interpolations process involved in providing the GPS stations with geopotential numbers. Indeed, since the present GPS stations of *Baden-Württemberg* are not identical with the first order level stations, where we have the geopotential numbers, such a interpolation is unavoidable. However, the present results, which are based on a very simple interpolation process, are indicating the minimum distance mapping of the physical surface of the earth to the *Somigliana-Pizzetti* telluroid as an optimal method in quasi-geoid calculations. This is especially valid if the GPS stations are occupied at the first order levelling stations, which we recommend for the future national GPS campaigns.

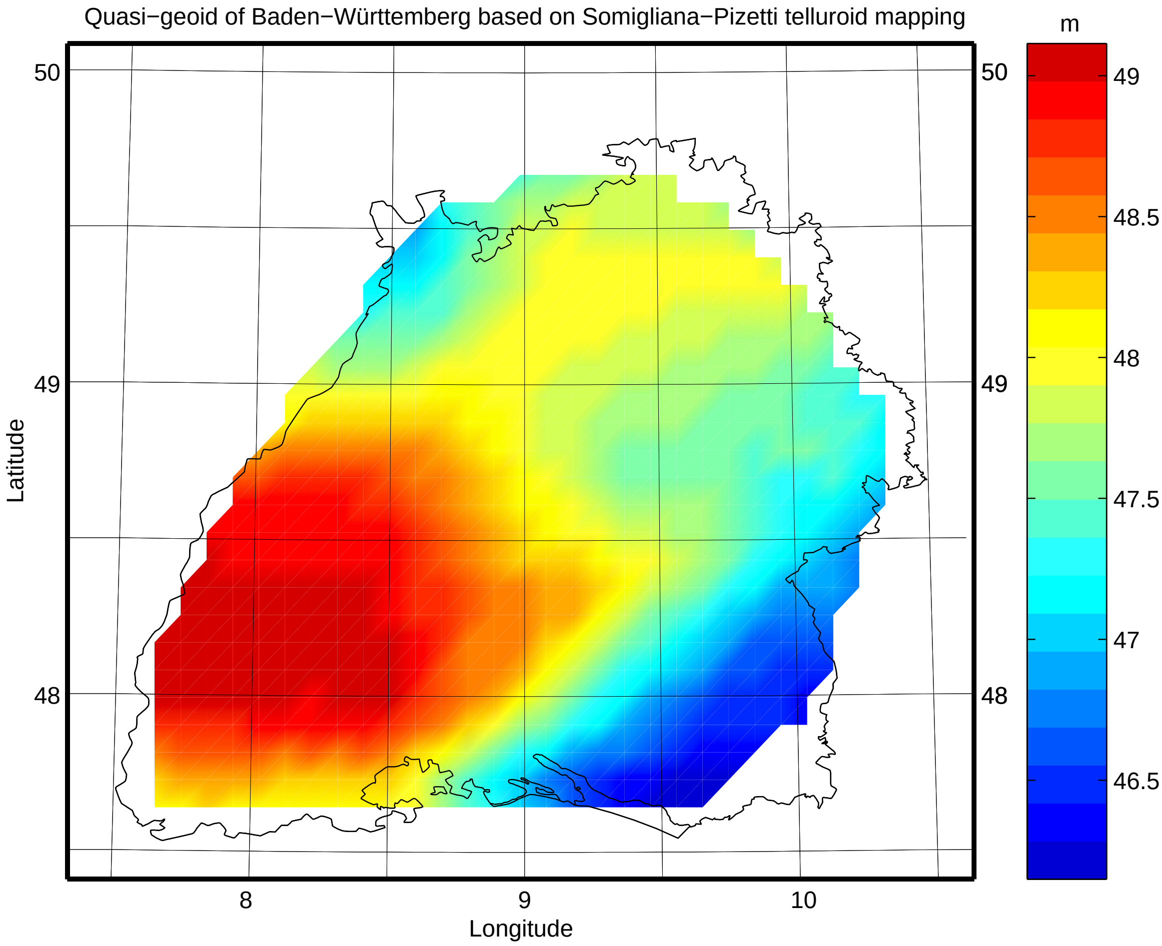

Figure 4-1: Quasi-geoid map of Baden-Württemberg, based on the minimum-distance mapping of the physical surface of the earth to the *Somigliana-Pizzetti* telluroid; variation: 46-49m.

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