A THOUGHT OF OPTIMIZATION AND DESIGN OF GEODETIC NETWORKS IN CONSIDERATION OF ACCURACY AND RELIABILITY

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ABSTRACT

Up to now the well known optimization and design of geodetic networks are based on accuracy criteron matrix / Grafarend and Sanso', 1985 /. Since the increased study of reliability the reliability parameters have been introduced into the optimization design of geodetic networks. But in most cases only the internal and external reliability (i. e. the redundancy number) are considered /Müller, 1986; Gu et al. 1989 etc.,/.

In this paper a new criterion matrix—reliability criterion matrix is suggested by author. Starting from both of accuracy and reliability criterion materices the mathematical formulae for determination of the first and second order design materices are derived by using matrix decomposition techniques. The correctness of the suggested optimization design method is proved by using a numerical example. Moreover the applicability and the problems to be solved are brifely discussed.

[Key words] Optimization design, Accuracy, Reliability, Criterion matrix, Matrix decomposition

1. GAUSS-MARKOV MODEL AND THE LEAST SQUARE SOLUTION

The linear G − M model

$$E(l) = A\tilde{x}, \quad D(l) = \sigma_0^2 Q_{ll} = \sigma_0 P^{-1} \qquad (1)$$

is given, in which weight matrix P is a symmetric positve matrix and can be decomposed as

$$P = G^T G \qquad (2)$$

with matrix G being a upper triangular matrix and inversable. If matrix P is a diagonal matrix, then

$$G = P^{1/2}$$

Taking matrix transformation as follows

$$\bar{A} = G A$$
$$\bar{l} = G l \qquad (3)$$

the model (1) is then written as

$$E(\bar{l}) = \bar{A}\tilde{x}, \qquad D(\bar{l}) = \sigma_0^2 I \qquad (4)$$

Obviously, model (4) and model (1) are equivalent. From model (4) the error equations of least Squares are

$$\bar{v} + \bar{l} = \bar{A}x \qquad P_{\bar{l}\bar{l}} = I \qquad (5)$$

For a full column rank matrix $\bar{A}$, i.e. $rg(\bar{A}) = rg(A) = u$, the normal equations and their solution are

$$(\overline{A}'\overline{A})\,x = \overline{A}'\overline{l}\,, \qquad x = (\overline{A}'\overline{A})^{-1}\overline{A}'\overline{l} \tag{6}$$

and

$$Q_{xx} = (\overline{A}'\overline{A})^{-1}\,, \qquad D_{xx} = \sigma_0^2 Q_{xx} \tag{7}$$

$$Q_{\overline{v}\,\overline{v}} = I - \overline{A}\,Q_{xx}\overline{A}' \tag{8}$$

Here matrix Q_{xx} reprensents weight coefficient matrix of unknown parameters and $Q_{\overline{v}\,\overline{v}}$ is weight coefficient matrix of residuals.

2. ACCURACY CRITERION MATRIX AND RELIABILITY CRITERION MATRIX

It's well known that the weight coefficient matrix Q_{xx} of unkowns or their variance-covariance matrix D_{xx} describe the precision character of unkown parameters. If we design a matrix Q_{xx} based on the requestion to the accuracy of a network, a accuracy criterion matrix (denoted below by Q_{xx} too) can be obtained. The optimization design of first and second order will be done starting from this criterion matrix / Grafarend, et al. 1985 /

According to extended theory of reliability / Li, 1986 / the matrix $Q_{\overline{v}\,\overline{v}}$ in Eq. (8) represents just the reliability of the observations. $Q_{\overline{v}\,\overline{v}}$ is a symmetric and idempotent matrix. i.e.

$$tr\,(Q_{\overline{v}\,\overline{v}}) = rg\,(Q_{\overline{v}\,\overline{v}}) = n - u = r$$

The i-th diagonal elements of matrix $Q_{\overline{v}\,\overline{v}}$

$$r_i = (Q_{\overline{v}\,\overline{v}})_{ii}$$

represents the redundancy number of reduced observation $\overline{l}_i$.

From formulae /Ackermann, 1981 /

$$\nabla_0 \overline{l}_i = \sigma_0 \frac{\delta_0}{\sqrt{r_i}} \tag{9a}$$

and

$$\overline{\delta}_{0,i} = \delta_0 \sqrt{\frac{1 - r_i}{r_i}} \tag{9b}$$

we can find that the diagonal elements of matrix $Q_{\overline{v}\,\overline{v}}$ directly reflect the determinability of gross errors involving in observations (internal reliability) and the effect of non-determinable gross erros onto the adjusted results (external reliability).

According to the separability theory /Li, 1986/ the correlation coefficient ρ_{ij} between $\overline{v}_i$ and $\overline{v}_j$

$$\rho_{ij} = \frac{(Q_{\overline{v}\,\overline{v}})_{ij}}{\sqrt{(Q_{\overline{v}\,\overline{v}})_{ii}} \cdot \sqrt{(Q_{\overline{v}\,\overline{v}})_{jj}}} \tag{10}$$

will indicate the separability of gross error in $\overline{l}_i$ and $\overline{l}_j$(i.e. locatability of gross errors).

For multiple gross errors in observations the corresponding correlation coefficient is calculated by

$$\rho_{12}(s_1, s_2) = \frac{s_1^T (P_{ss})_{12} s_2}{\sqrt{s_1^T (P_{ss})_{12} s_2} \cdot \sqrt{s_2^T (P_{ss})_{12} s_2}} \tag{11}$$

where s_1 and s_2 represent two different directions in p-dimensional space of multiple gross errors respectively, and

$$(P_{ss})_{ij} = H_i^T Q_{\bar{v}\bar{v}} H_j \qquad (i, j = 1, 2)$$

In case the observations contaminated by gross errors are arranged together, we have

$$H_i = \begin{vmatrix} 0 \\ I_i \\ 0 \end{vmatrix}$$

Therefore $(P_{ss})_{ij}$ is in fact a corresponding submatrix of $Q_{\bar{v}\bar{v}}$.

Summarizing above discussion we can say that matrix $Q_{\bar{v}\bar{v}}$ fully describes the determinability and locatability of gross errors. An adjustment system having fair reliability and separability should satisfy at least the following two conditions:

a) The redundancy number of each observation r_i should be identical or $\sum (r_i - (r/n))^2 \rightarrow$ MIN, in which r_i should be greater than a lower bound value $r_{\min}$;

b) The correlation coefficient ρ_{ij} should be less than 75%, i.e. separability for gross errors should be greater than 95%.

In the same way as accuracy criterion matrix, the matrix $Q_{\bar{v}\bar{v}}$ will be called reliability criterion matrix, if we build it according to the requestions to the reliability of geodetic network to be designed.

Therefore, the optimization design of geodetic network in consideration of accuracy and reliability can be summed up as to analytically determine and find out a first order design matrix A and a second order design matrix P that can satisfy the given requestions of accuracy and reliability criterion matrices.

3. OPTIMIZATION DESIGN OF GEODETIC NETWORKS IN CONSIDERATION OF ACCURACY AND RELIABILITY REQUIREMENTS BY USING MATRIX DECOMPOSITION TECHNIQUES

Our proposition is to solve a first order and a second order design matrix (i.e. Matrix A and P) which can meet the requirement of the given accuracy and reliability criterion matrices. For this purpose the matrix decomposition techniques are used.

STEP I: Decomposition of matrix Q_{xx} with cholesky method

$$Q_{xx} = L^T L \tag{12}$$

in which L is an upper triangular matrix with $rg(L) = u$

STEP II: Decomposition of symmetrical, positive semidefinite matrix $(I - Q_{\bar{v}\bar{v}})$

$$(I - Q_{\bar{v}\bar{v}}) = \underset{n\cdot u}{D} \cdot \underset{u\cdot n}{D'} \quad (13)$$

in which $rg(D) = rg(I - Q_{\bar{v}\bar{v}}) = u$ and $D'D = \underset{u\cdot u}{I}$

<u>STEP III</u>: Computation of matrix $\bar{A}$ by use of theorem of orthogonal triangular decomposition of matrix.

It leads to

$$\bar{A} = D \cdot (L^{-1})' \quad (14)$$

[Proof]

According to /wang, 1986/ a full column rank matrix $\bar{A}$ can be decompsed to an orthogonal matrix and an upper triangular matrix. i.e.

$$\underset{n\cdot u}{\bar{A}} = \underset{n\cdot u}{\bar{D}} \cdot \underset{u\cdot u}{\bar{L}} \quad (15)$$

in which $\bar{D}'\bar{D} = I$ and $\bar{L}$ is an inversaable upper triangular matrix.

Considering Eq.(15) and (7) the matrix Q_{xx} can be expressed as

$$Q_{xx} = (\bar{L}'\bar{D}'\bar{D}\,\bar{L})^{-1} = (\bar{L}'\bar{L})^{-1} = \bar{L}^{-1}(\bar{L}^{-1})'$$

In comparison with Eq.(12) and because of the uniqueness of orthogonal triangular decomposition of a matrix we have

$$\bar{L} \equiv (L^{-1})' \quad (16)$$

In the same way, considering Eq.(15) and (8) the matrix $Q_{\bar{v}\bar{v}}$ can be expressed as

$$Q_{\bar{v}\bar{v}} = I - \bar{D}\,\bar{L}\,(\bar{L}'\,\bar{L})^{-1}\bar{L}'\,\bar{D}' = I - \bar{D} \cdot \bar{D}'$$

i.e.

$$I - Q_{\bar{v}\bar{v}} = \bar{D} \cdot \bar{D}'.$$

Considering the character of orthogonal decomposion and comparing with the expression (13) we obtain

$$\bar{D} \equiv D \quad (17)$$

From Eqs.(16), (17) and (15) we get the final result

$$\bar{A} = D\,(L^{-1})'$$

which is just the to be proved Eq.(14) .

It means that the matrix $\bar{A}$, which consists of first and second order design matrix, can be uniquely computed from the given accuracy criterion and reliability criterion matrix.

<u>STEP IV</u> Deptermination of the first order design matrix A based on computed matrix $\bar{A}$ and given matrix P

According to Eq.(3) and (2) we have

$$G A = \overline{A} \quad \text{and} \quad P = G' G \tag{18}$$

If P is given, the inversable matrix G can be determined by decomposition of matrix P. Thus the first order design matrix A will be computed by

$$A = G^{-1}\overline{A} = G^{-1} \cdot D \, (L^{-1})' \tag{19}$$

in which matrices G, D and L are obtained from matrices P, $Q_{\overline{v}\overline{v}}$ and Q_{xx} respectively.

<u>STEP V</u>: Determination of the second order design matrix P based on computed matrix $\overline{A}$ and given matrix A

Our discussion here is restricted by the assumption of a diagonal weight matrix P. In this case the structure of matrix G should be

$$G = P^{1/2} = \begin{vmatrix} \sqrt{P_1} & & & & \\ & \sqrt{P_2} & & & \\ & & \cdots\cdots & & \\ & & & \cdots\cdots & \\ & & & & \sqrt{P_n} \end{vmatrix} \tag{20}$$

From Eq.(18) and (20) we obtain

$$\begin{vmatrix} \sqrt{P_1} & & & & \\ & \sqrt{P_2} & & & \\ & & \cdots\cdots & & \\ & & & \cdots\cdots & \\ & & & & \sqrt{P_n} \end{vmatrix} \begin{vmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1u} \\ a_{21} & \cdots & \cdots & \cdots & a_{2u} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nu} \end{vmatrix} = \begin{vmatrix} \overline{a}_{11} & \cdots & \cdots & \cdots & \overline{a}_{1u} \\ \overline{a}_{21} & \cdots & \cdots & \cdots & \overline{a}_{2u} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \overline{a}_{n1} & \cdots & \cdots & \cdots & \overline{a}_{nu} \end{vmatrix}$$

So the relationship

$$\sqrt{P_i} \cdot a_{ij} = \overline{a}_{ij} \qquad (j = 1, 2, \cdots\cdots u) \tag{21}$$

can be easily found. It means that the elements of each column of matrix A should be proportional to corresponding elements of matrix $\overline{A}$. As it is usually not true, the weight of each observation can be computed by arithmetic average:

$$P_i = \frac{1}{u'} \sum_{j=1}^{u} \left(\frac{\overline{a}_{ij}}{a_{ij}}\right)^2 \qquad (\text{for } a_{ij} \neq 0) \tag{22}$$

Where $u' = u - k$

k is the number of omitted elements with $a_{ij} = 0$.

Because of the above treatment matrix A must be computed again (see step IV).

It is still an unsolved problem, how to make second order optimization design for correlated observations with full weight matrix P unter our conditions.

4. <u>A NUMERICAL EXAMPLE</u>

Below we take a very simple numerical example to prove the correctness of the above thought and derivation.

To determine the position (x , y) of new point P from six known points is assumed. The observations are azimuth angles from each known point to new point P . The unit of side length is kilometer. The designed accuracy and reliability criterion matrices are as follows

$$Q_{xx} = \begin{vmatrix} 1/3 & 0 \\ 0 & 1/3 \end{vmatrix}$$

and

$$Q_{\bar{v}\bar{v}} = \begin{vmatrix} 2/3 & -1/6 & 1/6 & -1/3 & 1/6 & -1/6 \\ -1/6 & 2/3 & -1/6 & 1/6 & 1/3 & 1/6 \\ 1/6 & -1/6 & 2/3 & -1/6 & 1/6 & 1/3 \\ -1/3 & 1/6 & -1/6 & 2/3 & -1/6 & 1/6 \\ 1/6 & 1/3 & 1/6 & -1/6 & 2/3 & -1/6 \\ -1/6 & 1/6 & 1/3 & 1/6 & -1/6 & 2/3 \end{vmatrix}$$

To be determined is the first design matrix A under the assumption of $P = I$.

[Solution]

STEP I : Decomposition of Q_{xx} for getting matrix $(L^{-1})^T$

$$Q_{xx} = L^T L = \begin{vmatrix} 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \end{vmatrix}$$

it leads to

$$(L^{-1})^T = \begin{vmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{vmatrix}$$

STEP II : Decomposition of $(I - Q_{\bar{v}\bar{v}})$ for getting matrix D

$$(I - Q_{\bar{v}\bar{v}}) = \begin{vmatrix} 1/3 & 1/6 & -1/6 & 1/3 & -1/6 & 1/6 \\ 1/6 & 1/3 & 1/6 & -1/6 & -1/3 & -1/6 \\ -1/6 & 1/6 & 1/3 & 1/6 & -1/6 & -1/3 \\ 1/3 & -1/6 & 1/6 & 1/3 & 1/6 & -1/6 \\ -1/6 & -1/3 & -1/6 & 1/6 & 1/3 & 1/6 \\ 1/6 & -1/6 & -1/3 & -1/6 & 1/6 & 1/3 \end{vmatrix}$$

It leads to

$$D = (1/\sqrt{3}) \begin{vmatrix} 0 & 1 \\ -\sqrt{3}/2 & 1/2 \\ -\sqrt{3}/2 & -1/2 \\ 0 & -1 \\ \sqrt{3}/2 & -1/2 \\ \sqrt{3}/2 & 1/2 \end{vmatrix} \quad \left(\text{Test : } D^T D = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right)$$

Because $P = I$, the first order design matrix A is

$$A = \overline{A} = D \cdot (L^{-1})^T = \begin{vmatrix} 0 & 1 \\ -\sqrt{3}/2 & 1/2 \\ -\sqrt{3}/2 & -1/2 \\ 0 & -1 \\ \sqrt{3}/2 & -1/2 \\ \sqrt{3}/2 & 1/2 \end{vmatrix}$$

From the geometry shown in Fig. 1 the coefficient of error equations are

$$a_i = -\sin\alpha_i / s_i \qquad (24)$$
$$b_i = \cos\alpha_i / s_i$$

Comparing the obtained first order design matrix A with Eq.(24) we obtain the final results:

$s_i = 1$ (Km)

$\alpha_i = 0, 60°, 120°, 180°, 240°, 300°$

It means that the six known points should be homogeneously distributed at a circumference with 1 Km long diameter and the new point P should be just the center of this circle (also see Fig.1)

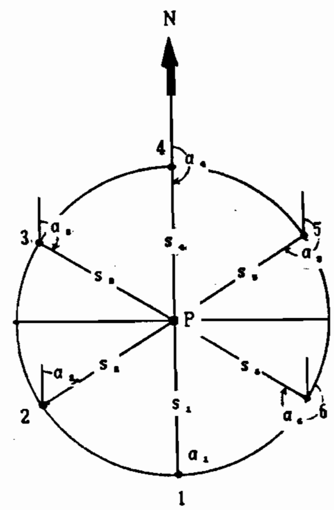

Fig.1 Forward intersection from six points

In this configuration the best precision of unknown coordinates for new point and the best reliability of the observations will be expected. The error ellipse of new point will be circle. The redundancy number, internal and external reliability are respectively

$r_i = r / n = 2/3$
$\overline{\nabla}_0 l_i = 5.06\sigma_0$
$\overline{\sigma}_{0,i} = 2.92$

Since the biggest correlation coefficient between two residuals is

$|\rho|_{max} = 50\%$,

the locatability of gross errors will be (see förstner,1983)

$(1 - \gamma_{ij}) \geqslant 99\% \qquad (K_a = 3.29 \ , \ \delta_0 = 4.13)$

In other words, any two gross errors can be separated with a probability greater than 99%.

If we neither change the reliability criterion matrix $Q_{\bar{v}\bar{v}}$, nor change the precision of azimuth observations, but only change the requestion to the precision of unknown point, saying for example:

$$Q_{xx} = (1/4) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

Through the decomposition of new Q_{xx} above, we get the matrix $(L^{-1})^T$

$$(L^{-1})^T = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$$

further

$$A = \overline{A} = D \cdot (L^{-1})^T = (2/\sqrt{3}) \begin{vmatrix} 0 & 1 \\ -\sqrt{3}/2 & 1/2 \\ -\sqrt{3}/2 & -1/2 \\ 0 & -1 \\ \sqrt{3}/2 & -1/2 \\ \sqrt{3}/2 & 1/2 \end{vmatrix}$$

According to Eq.(24) we have

$$s_i = \sqrt{3}/2\ (Km)$$
$$\alpha_i = 0,\ 60^\circ,\ 120^\circ,\ 180^\circ,\ 240^\circ,\ 300^\circ$$

This result shows that, the shorter the distance between known and unknown point, the better the precision of unknown point, when the configuration relationship keeps no change.

5. APPLICABILITY AND THE PROBLEMS

A new concept of optimization design of geodetic network in consideration of accuracy and reliability criterion matrices has been put forward and realized by means of matrix decomposition techniques. If both criterion matrices of accurecy and reliability are given, the first and second order design matrix will be analytically computed. In principle it can be used for optimization design in geodetic, photogrammetric and combined networks.

The only problem is how to build two criterion matrices, especially the reliability criterion matrix, for a very large network. We can solve this problem in two different ways. One way would be to find out some mathemathical techniques to set up the accuracy and reliability criterion matrix, but this is very difficult.

Another way is to use the priori knowledge about first and second order design matrix. We start from the given A^{0} and P^{0} matrices and compute matrices Q^{0}_{xx} and $Q^{0}_{\bar{v}\bar{v}}$, then check whether they satisfy the user requestions to precision and reliability. If not, we have to modify them by using some proper methods, and get $Q^{(1)}_{xx}$ and $Q^{(1)}_{\bar{v}\bar{v}}$. Now we can use the algorithm in this paper and obtain the modified design matrices A^{0} and P^{0}, then check again whether these two design matrices can be realized in practice. Repeat this procedure untill the aim of optimization design has been reached.

Another unsolved problem in this paper is how to compute the second order design matrix P from two criterion matrices, if matrix P is completely occupied. Like the variance-covarianct component estimation we can maybe only estimate a part of elements in matrix P.

All of these problems have to be deeply studied, discussed and investigated further.

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