OPTICAL-MECHANICAL RECTIFICATION WITH NUMERICALLY DETERMINED SETTINGS

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1. INTRODUCTION

If a control sheet with 3, 4 or more control points is available, in one or another way, it is possible to measure the position of these points and also of the corresponding projected image points in a system of easel co-ordinates.

Knowing the co-ordinates of these points and the magnification of the rectifier, the rectifier elements of orientation must be determined in such a way that a geometrical condition as well as an optical one must be fulfilled. Because, optical-mechanical rectification means the reprojection of a tilted photograph into one which has:

a. The same properties as a map and at a desired scale (geometrical condition)
b. A sharp image (optical condition)

In the following observation an analytical orientation method will be described, which is based on both conditions.

2. NOTATION

Matrices will be denoted by capital letters.

U^T is the transposed matrix, U^{-1} is the inverse matrix, U^+ is the adjoint (cofactor) matrix and $|U|$ is the determinant of the matrix U.
I represents a unit matrix.

3. GEOMETRICAL CONDITION

When we seek a method of reprojecting a tilted photograph directly on the map, so that the correct position will be obtained for each point, there will be a relation between plate co-ordinates and map co-ordinates identical to that between plate co-ordinates and terrain co-ordinates.

Using a rectifier, the photograph is projected through a rectifying lens. The procedure is, therefore, essentially, a central projection.
That means: photo point, perspective centre of the lens and projected image point are collinear.

3.1 The relation between plate and map co-ordinates

The central projective relationship between the orthogonal plate co-ordinates (x,y) of an arbitrary photo point and the orthogonal map co-ordinates (r,s) of its projected point on the easel can be expressed as:

$$\begin{bmatrix} x - x_1 \\ y - y_1 \\ d \end{bmatrix} = e\, A \begin{bmatrix} r - r_1 \\ s - s_1 \\ - t_1 \end{bmatrix}$$

Where x_1, y_1 are the orthogonal plate co-ordinates of the perspective centre 1 of the rectifying lens, d is the z-co-ordinate of the negative plane, e is a length factor, A is a three by three orthogonal rotation matrix ($|A|=+1$),
r_1, s_1, t_1 are the orthogonal map co-ordinates of the perspective centre 1. The r,s-plane coincides with the easel plane.

The left side of the above relation may be eliminated by pre-multiplying both sides by the matrix

$$\begin{bmatrix} -d & o & x-x_1 \\ o & -d & y-y_1 \end{bmatrix}$$

and when we then divide by e (which is allowed, because e is never zero), we arrive at the formula:

$$o = \begin{bmatrix} -d & o & x-x_1 \\ o & -d & y-y_1 \end{bmatrix} A \begin{bmatrix} r - r_1 \\ s - s_1 \\ - t_1 \end{bmatrix} \qquad (1)$$

or, by grouping all unknown parameters together:

$$o = \begin{bmatrix} -1 & o & x \\ o & -1 & y \end{bmatrix} \begin{bmatrix} d & o & x_1 \\ o & d & y_1 \\ o & o & 1 \end{bmatrix} A \begin{bmatrix} 1 & o & -r_1 \\ o & 1 & -s_1 \\ o & o & -t_1 \end{bmatrix} \begin{bmatrix} r \\ s \\ 1 \end{bmatrix}$$

These equations are homogeneous. Therefore, one of the nine coefficients can be reduced to unity, by dividing each of them by that one which is never zero. This means that we can write the equations in the following form:

$$o = \begin{bmatrix} -1 & o & x \\ o & -1 & y \end{bmatrix} U \begin{bmatrix} r \\ s \\ 1 \end{bmatrix} \qquad (2)$$

Where $U = \frac{1}{\underline{1}} BAC$; $\underline{1} = -a_{31}r_1 - a_{32}s_1 - a_{33}t_1$

$$U= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & 1 \end{bmatrix}; \; B= \begin{bmatrix} d & o & x_1 \\ o & d & y_1 \\ o & o & 1 \end{bmatrix}; \; A= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \; C= \begin{bmatrix} 1 & o & -r_1 \\ o & 1 & -s_1 \\ o & o & -t_1 \end{bmatrix}$$

Thus, the equations contain eight coefficients, which are functions of the nine rectifier elements of orientation.

3.2 The relation between plate and terrain co-ordinates

It will be clear that there is a relation between the plate co-ordinates (x,y) of an arbitrary photo point, and the orthogonal terrain co-ordinates (u,v,w) of its corresponding terrain point, similar to that in the equations (1), but the coefficients may have different values.

$$o = \begin{bmatrix} -c & o & x - x_o \\ o & -c & y - y_o \end{bmatrix} D \begin{bmatrix} u - u_o \\ v - v_o \\ w - w_o \end{bmatrix} \qquad (3)$$

Where x_o, y_o are the orthogonal plate co-ordinates of the perspective centre o of the lens of the photogrammetric camera, c is the z-co-ordinate of the negative plane (c is the focal length), D is a three by three orthogonal rotation matrix ($|D|=+1$), u_o, v_o, w_o are the orthogonal terrain co-ordinates of the perspective centre o.

If we desire that the projection of the photograph on the easel has the same geometry as the orthogonal projection of the terrain points onto the map, a condition must be fulfilled. From the comparison of (1) and (3), we see that this is possible if, and only if, these equations are identical.

Identity exists if w is a linear function of u and v, thus if:
$w = k_1 u + k_2 v + k_3$ (in other words, if the terrain is a plane, not necessarily horizontal).

Let us assume that $u = g\ r$ and $v = g\ s$, where g is a scale factor, then the equations can be written as:

$$o = \begin{bmatrix} -c & o & x-x_o \\ o & -c & y=y_o \end{bmatrix} D \begin{bmatrix} gr & -r_o \\ gs & -s_o \\ gk_1 r + gk_2 s + k_3 - t_o \end{bmatrix}$$

or, by grouping all unknown parameters together:

$$o = \begin{bmatrix} -1 & o & x \\ o & 1 & y \end{bmatrix} \begin{bmatrix} c & o & x_o \\ o & c & y_o \\ o & o & 1 \end{bmatrix} D \begin{bmatrix} g & o & -r_o \\ o & g & -s_o \\ gk_1 & gk_2 & k_3-t_o \end{bmatrix} \begin{bmatrix} r \\ s \\ 1 \end{bmatrix}$$

These equations are also homogeneous. Therefore, we may write them in the following form:

$$o= \begin{bmatrix} -1 & o & x \\ o & -1 & y \end{bmatrix} V \begin{bmatrix} r \\ s \\ 1 \end{bmatrix} \qquad (4)$$

Where $V = \frac{1}{m}$ EDF; $m = -d_{31}r_o - d_{32}s_o + (k_3 - d_{33})t_o$

$$V= \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & 1 \end{bmatrix};\ E= \begin{bmatrix} c & o & x_o \\ o & c & y_o \\ o & o & 1 \end{bmatrix};\ D= \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix};\ F= \begin{bmatrix} g & o & -r_o \\ o & g & -s_o \\ gk_1 & gk_2 & k_3-t_o \end{bmatrix}$$

Let us assume also that the terrain is plane. In that case, we have identity between the equations (2) and (4), if the corresponding coefficients are proportional, i.e. with a proportionality factor h:

$$U = h\ V$$

Generally, in practice, the interior elements of orientation (x_o, y_o, c) of the photogrammetric camera are known. In that case, two independent relations can be derived between these elements and the coefficients of the first two columns of the matrix U.

According to (2) and (4)

$$U = \frac{h}{m} E\ D\ F$$

Pre-multiplying both sides of this condition by $E^{-1}(|E| \neq o)$ gives $E^{-1}U = \frac{h}{m} D\ F$

and, next, both sides by the transposed matrix of it.

$$U^T(E^{-1})^T E^{-1} U = \left(\frac{h}{m}\right)^2 F^T D^T D\ F = \left(\frac{h}{m}\right)^2 F^T F$$

(because $D^T D = I$)

From (2) and (4), after some manipulations, we get:

$$\frac{(u_{12} - x_o u_{32})^2 + (u_{22} - y_o u_{32})^2 + (c\ u_{32})^2}{(u_{11} - x_o u_{31})^2 + (u_{21} - y_o u_{31})^2 + (c\ u_{31})^2} = \frac{1 + (k_2)^2}{1 + (k_1)^2} \qquad (5)$$

$$\frac{(u_{11} - x_o u_{31})(u_{12} - x_o u_{32}) + (u_{21} - y_o u_{31})(u_{22} - y_o u_{32}) + (c)^2 u_{31} u_{32}}{(u_{11} - x_o u_{31})^2 + (u_{21} - y_o u_{31})^2 + (c\ u_{31})^2} = \frac{k_1 k_2}{1 + (k_1)^2}$$

4. OPTICAL CONDITION

As already mentioned, the equations (2) contain, in fact, eight coefficients which are functions of the nine elements of orientation. Consequently, we have an infinite number of central projections which will correctly rectify a tilted photograph of a given flat terrain (k_1, k_2, k_3).

That means, from all possible positions of the projector which are geometrically correct, the choice is limited to the particular position which gives a sharp image. Thus only one position exists in practice.

We will have a sharp image of the tilted photograph over the whole area which is projected, if for each of any three non-collinear points (or one line, which counts for two points, and one non-collinear point) the Gauss lens law of optics is fulfilled:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{f}$$

or (6)

$$b = \frac{af}{a - f}$$

Where a is the distance from the lens to the object, b is the distance from the image to the lens (measured along the optical axis) and f is the focal length of the rectifying lens.

Let us assume that the negative plane n, the lens plane p, and the easel plane q intersect in a common line j; see figure 1. Then according to (6) a = o and b = o. Consequently, the common line j will be focussed in itself (Scheimpflug condition).

Since the common line j counts as two points, an extra non-collinear point is still required, e.g. the point of intersection of the optical axis and the negative plane (Gauss condition).

If both the Scheimpflug and the Gauss condition are fulfilled, then a sharply focussed image on the easel plane is obtained.

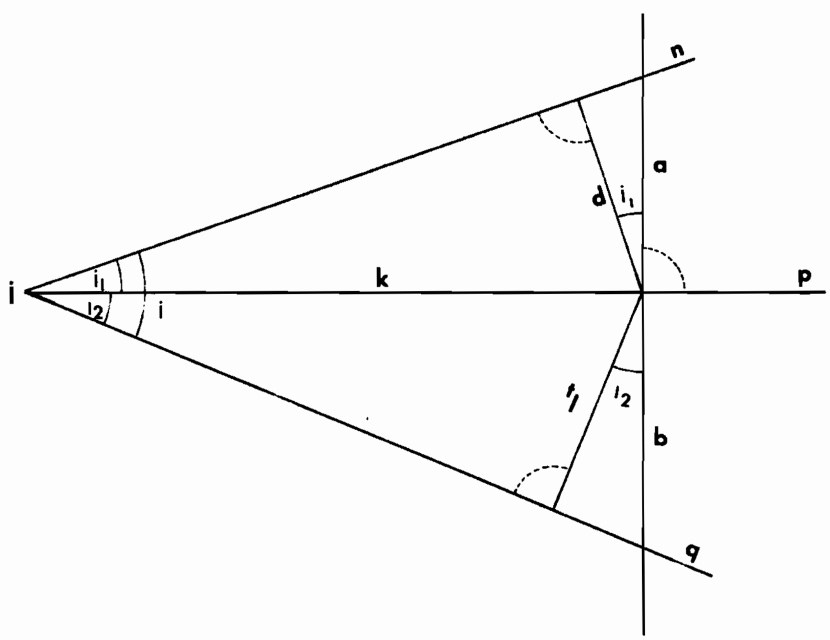

Figure 1: The Scheimpflug and Gauss conditions

It is easy to verify that the angle u is the angle of maximum tilt, i.e. the angle between the positive z-axis and t-axis and the cosines of this angle is equal to a_{33}.

According to Fig.1

$$\frac{\sin u_1}{d} = \frac{1}{k} ; \quad \frac{\sin u_2}{t_1} = \frac{1}{k} \qquad (7)$$

and

$$\frac{\cos u_1}{d} = \frac{1}{a} ; \quad \frac{\cos u_2}{t_1} = \frac{1}{b} \qquad (8)$$

Eliminating k from (7) and, next, a and b from (6) and (8) gives:

$$\frac{\sin u_1}{d} - \frac{\sin u_2}{t_1} = o \qquad (9)$$

$$\frac{\cos u_1}{d} + \frac{\cos u_2}{t_1} = \frac{1}{f} \qquad (10)$$

Taking the sum of squares of (9) and (10) we get, because $\cos (u_1 + u_2) = \cos u = a_{33}$:

$$\frac{1}{(d)^2} + \frac{2\, a_{33}}{d\, t_1} + \frac{1}{(t_1)^2} = \frac{1}{(f)^2} \qquad (11)$$

This equation is called "the focal equation for tilted images". It defines mathematically the Scheimpflug and the Gauss condition for tilted as well as for vertical photographs; because in the last case: $a_{33} = 1$, $d \equiv a$, $t_1 \equiv b$ and the Gauss equation is obtained.

5. DETERMINATION OF THE COEFFICIENTS U

Suppose

1. the interior elements of orientation x_o, y_o, c of the photogrammetric camera,
2. the plate co-ordinates x,y and the terrain co-ordinates u,v,w of 3, 4 or more corresponding points and
3. the scale factor g

are known.

Then a system of eight equations, consisting of six linear equations (2) and two non-linear equations of type (5) can be formed; for flat and sloping terrain (k_1, k_2, k_3) as well as for flat and horizontal terrain ($k_1 = k_2 = o$). The eight coefficients (the elements of the matrix U) can now be determined, under condition that the coefficient matrix of the (linearised) system is non-singular. This matrix is singular, for instance, if the exposure air-station is situated on the surface of the cylinder normal to the circumscribed circle of the three used terrain points. When there is flat and horizontal terrain and four points are given, a system of eight linear equations of type (2) can be formed.

6. DETERMINATION OF THE RECTIFIER ELEMENTS OF ORIENTATION

The matrix U (2) is a function of 15 unknowns:
d, x_1, y_1, r_1, s_1, t_1 and the nine elements of the orthogonal matrix A, but between these elements exist the following relations:

$$A^T A = I, \; A^+ = A^T \qquad (12)$$

However, six of these relations are independent.

As known

$$|U| = |\frac{B}{\underline{1}}| \, |A| \, |C|$$

so

$$|U| = \frac{-(d)^2 t_1}{\underline{1}^3} \qquad (13)$$

According to (2):

$$\underline{1}\, u_{31} = a_{31}; \; \underline{1}\, u_{32} = a_{32}; \text{ giving } \underline{1}^2 \{(u_{31})^2 + (u_{32})^2\} = 1-(a_{33})^2 \qquad (14)$$

Denoting the expression between brackets by p_1 we get:

$$(\underline{1})^2 p_1 = 1-(a_{33})^2 \qquad (15)$$

Further:

$$\begin{aligned} (\underline{1})^2 (u_{11} u_{32} - u_{12} u_{31}) &= d\,(a_{11} a_{32} - a_{12} a_{31}) = -d\, a_{31} \\ (\underline{1})^2 (u_{21} u_{32} - u_{22} u_{31}) &= d\,(a_{21} a_{32} - a_{22} a_{31}) = d\, a_{32} \end{aligned} \qquad (16)$$

or

$$\underline{1}^4 \{(u_{11} u_{32} - u_{12} u_{31})^2 + (u_{21} u_{32} - u_{22} u_{31})^2\} = d^2(1-a_{33}^2)$$

Denoting in the left side the expression between brackets by p_2, we get

$$\frac{(\underline{l})^2 p_2}{p_1} = (d)^2 \qquad (17)$$

so that (13) is

$$\frac{-\underline{l}\ |U|\ p_1}{p_2} = t_1 \qquad (18)$$

Eliminating d, t_1 and $\underline{l}$ from (17), (18), (15) and (11), we get

$$\frac{p_1}{p_2} - 2\ \frac{a_{33}}{|U|} \sqrt{\frac{p2}{p1}} + \frac{(p_2)^2}{(p_1)^2 |U|^2} = \frac{1-(a_{33})^2}{p_1 (f)^2} \qquad (19)$$

and a_{33}, $\underline{l}$, d, t_1, a_{31}, a_{32}, a_{13} and a_{23} are determined;
the angle i is relatively small, thus a_{33} is positive.

Further

$$(\underline{l})^2 (u_{11}u_{31} + u_{21}u_{32}) = -da_{31}a_{33} + (1-(a_{33})^2)\ x_1$$
$$(\underline{l})^2 (u_{22}u_{31} + u_{22}u_{32}) = da_{32}a_{33} + (1-(a_{33})^2)\ y_1 \qquad (20)$$

$$B^{-1}U = \frac{AC}{\underline{l}} \qquad (21)$$

$$A^{-1}B^{-1}U = \frac{C}{\underline{l}} \qquad (22)$$

By (20) are x_1, y_1, by (21) the remaining elements of the matrix A and, finally, by (22) r_1,s_1 and t_1 determined.

Some of the elements do not correspond with the settings of a particular rectifier. The necessary orientation data (settings) depend however on the mechanical design and automatic capabilities. Therefore only two relations will be dealt with here.

From (7), (8) and (6)

$$\frac{\sin u_1 \cos u_2}{d\ t_1} + \frac{\cos u_1 \sin u_2}{d\ t_1} = \frac{1}{k}\left(\frac{1}{a} + \frac{1}{b}\right)$$

or

$$\frac{\sin u}{d\ t_1} = \frac{1}{kf}$$

Eliminating k (7) gives

$$\frac{\sqrt{1-(\cos u)^2}}{d\ t_1} = \frac{\sin u_1}{d\ f} = \frac{\sqrt{1-(a_{33})^2}}{d\ t_1}$$

and

$$\frac{\sqrt{1-(\cos u)^2}}{d\ t_1} = \frac{\sin u_2}{t_1 f} = \frac{\sqrt{1-(a_{33})^2}}{d\ t_1} \qquad (23)$$

From (8) and (23)

$$a = \frac{d}{\cos u_1} = \frac{d}{\sqrt{1-(\sin u_1)^2}} = \frac{d\ t_1}{\sqrt{(t_1)^2 - (f)^2\ \{1-(a_{33})^2\}}}$$

$$b = \frac{t_1}{\cos u_2} = \frac{t_1}{\sqrt{1-(\sin u_2)^2}} = \frac{d\ t_1}{\sqrt{(d)^2 - (f)^2\{1-(a_{33})^2\}}}$$

REFERENCE

Van der Weele, A.J.: Meetkundige beschouwingen over ontschranking, Tijdschrift voor Kadaster en Landmeetkunde, 1951, p. 101-126.