INTERNATIONAL SOCIETY FOR PHOTOGRAMMETRY AND REMOTE SENSING

16. Int. Congr. ISPRS
Kyoto/Japan, July 1 - 10, 1988

SOME EXPERIENCE WITH THE DETERMINATION OF THE OPTIMUM SAMPLING DENSITY

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Int. Arch. Phot. Rem. Sens., 27, Kyoto

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Summary

The optimum sampling density for data acquisition of digital terrain models is mostly derived by analyses of dense terrain profiles. Different procedures have been proposed in the recent past based on deterministic and stochastic formulations, respectively.

The paper demonstrates three methods to solve this task: a trial-and-error formulation, a Fourier-series-like approach and a variogram evaluation. As far as random observations are concerned, different approaches can be applied to solve the profile smoothing. For that reason, also a smoothing procedure by Wiener filtering is integrated into the data analysis. The influence of noisy observations on the determination of the optimum sampling interval is proven by some examples.

A critical comparison between the methods demonstrated provides for further experience in this field.

1. Introduction

The determination of the optimum sampling interval for data acquisition of grid digital terrain models becomes more and more important especially with regard to the set up of country-wide DTM's. Its purpose is to acquire DTM data with the least number of sample points and yet produce DTM products with sufficient accuracy. In practice the DTM data are either under- or over-sampled leading to an inaccurate DTM, which requires resampling, or needs unnecessary observation time, extra data processing and storage problems to be avoided by an optimum sampling strategy.

Hence it is undispensable to prove the sampling strategy on the optimum sampling interval, what can be done by analyses of profiles recorded from the stereo model or portions of it. For the data analysis different strategies may be applied, but all need an a priori accuracy number σ_s indicating the error budget one is willing to pay for. This number can be estimated using the knowledge of similar projects, in which the error behaviour of the whole DTM process has been computed (model setup, derivation of contour lines, etc.). Therefore, some experience must be presupposed to come to reliable estimates on σ_s. A detailled description on error transfer functions has been given by K. Tempfli (1982) using the method of spectral analysis - a stochastic approach.

The strategies for the analyses of profiles can be either deterministic or stochastic. Different procedures are known, which have to be compared with each other to have some knowledge on the advantages and disadvantages. A first comparison has been made by A.E. Balce (1986, 1987) using a trial-and-error approach, the program SPECTRA (D. Fritsch, 1984) and the program LOGKV (P. Frederiksen et al., 1983, 1984, 1986).

The aim of this paper is to study the influence of noisy observations on the profile analyses. For that reason, the same programs have been implemented with slightly changes (E. Dirscherl, 1987), but were supplemented by a smoothing procedure to eliminate the observation noise. This implicates a two-step procedure

(i) smooth the profiles
(ii) analyse the smoothed profiles.

A quite similar approach is proposed by J. Lindenberger (1986, 1987), in which ARIMA processes are used to model terrain profiles.

2. Data Smoothing

Let x(s) be a continous profile of the stereo model or portions of it to observe continously in any distance interval $s \in (s_0, s_{m-1})$. The profile measurement samples x(s) into the sample set $(x(s=s_0), x(s=s_1), \ldots x(s=s_{m-1}))$ with M samples in all.

This data set represents a ' digital signal ', which is causal and of finite length because of the definition of the distance interval; it can be described by

$$x(m) \qquad \forall m = 0,1,2, \ldots ,M-1 \tag{1}$$

equidistant data sampling being assumed with Δ_p as sampling interval.

With the digital unit sample (impulse)

$$d(m) = \begin{cases} 1 & m=0 \\ 0 & m\neq 0 \end{cases} \quad \text{and} \quad d(m-k) = \begin{cases} 1 & m=k \\ 0 & m\neq k \end{cases} \tag{2}$$

the signal x(m) is given as discrete convolution

$$x(m) = \sum_{k=0}^{M-1} x(k)d(m-k) = x(m)*d(m) \tag{3}$$

with x(k) as the magnitude of the signal x(m); this convolution is usually symbolized by an asterisk * .

2.1 Wiener Filtering

The concept of Wiener filtering can be transferred into digital filtering using some definitions of the signal processing discipline.

Let be given the filtered signal (profile) y(m) represented by the convolution of x(m) and a kernel h(m)

$$y(m) = \sum_{k=0}^{M-1} h(k)x(m-k) = h(m)*x(m) \tag{4}$$

whereby the kernel h(m) describes the filter behaviour; it is also called ' impulse response ' because it is the output (reaction) of the filter when the unit impulse d(m) is its input (D. Fritsch, 1982b).

In order to compute the kernel h(m) the following signal model is introduced

$$\boxed{x(m) = y(m) + n(m)} \tag{5}$$

that is, the profile measurements $x(m)$ are disturbed by any errors or noise $n(m)$, whereby $y(m)$ is the true or errorless signal we are interested in. Hence, the filter kernel has to be derived such, that the additive noise of (5) will be removed leading to an estimate $\hat{y}(m)$. Using the Wiener filtering notation its objective function minimizes the variance σ^2 of the estimation error $e(m)=y(m)-\hat{y}(m)$ defined as

$$\sigma^2 = E((e(m) - E(e(m)))^2) = \min \tag{6}$$

in which E is the expectation. For stationary $x(m)$ and the mean values $E(y(m))=0$, $E(n(m))=0$ as well as $E(e(m))=0$ the solution of (6) results into the discrete form of the well-known Wiener-Hopf integral equation

$$R_{yx}(\lambda) = \sum_{k=0}^{M-1} h(\lambda) R_{xx}(\lambda-k) = h(\lambda) * R_{xx}(\lambda) \tag{7}$$

with $R_{yx}(\lambda)$ as crosscorrelation function between $y(m)$ and $x(m)$ depending on the correlation lag $\lambda \, \forall \lambda = 0,1,2, \ldots, M-1$; $R_{xx}(\lambda)$ is the autocorrelation of $x(m)$ and $h(\lambda)$ the kernel of the Wiener filter. This convolution could be solved as linear equation system, but for larger data sets the inversion of the autocorrelation matrix $R_{xx}(\lambda)$ costs a lot of computing time if it has not a special structure as Toeplitz matrices have.

Taking the Fourier transform $\mathcal{F}$ on both sides of (7) the convolution sum is transferred into the multiplication

$$S_{yx}(e^{jw}) = H(e^{jw}) S_{xx}(e^{jw}) \tag{8}$$

whereby $S_{yx}(e^{jw})$ and $S_{xx}(e^{jw})$ are the power spectra and $H(e^{jw})$ is the transfer function (frequency response) of the Wiener filter. Under the assumption that $y(m)$ and $n(m)$ are not correlated with each other, the following simplifications are valid

$$R_{yx}(\lambda) = R_{xx}(\lambda) \tag{9a}$$

$$R_{xx}(\lambda) = R_{yy}(\lambda) + R_{nn}(\lambda) \tag{9b}$$

which lead to

$$H(e^{jw}) = \frac{S_{yy}(e^{jw})}{S_{yy}(e^{jw}) + S_{nn}(e^{jw})} \tag{10}$$

Let us furthermore suppose a white noise process $n(m)$ with corresponding correlation function and power spectrum, respectively

$$R_{nn}(\lambda) = \sigma_n^2 d(\lambda) \qquad S_{nn}(e^{jw}) = \sigma_n^2 \tag{11}$$

as well as a 1st order Markov process for the signal y(m) (D. Fritsch, 1982a) with

$$R_{yy}(\lambda) = \sigma_y^2 \alpha^{|\lambda|} \qquad 0 \le \alpha < 1 \tag{12a}$$

$$\overset{\tilde{\mathcal{F}}}{\Longrightarrow}$$

$$S_{yy}(e^{jw}) = -\frac{2\sigma_y^2 \ln(\alpha)}{(\ln\alpha)^2 + w^2} \tag{12b}$$

Thus, it follows for the frequency response of the Wiener filter

$$\boxed{H(e^{jw}) = \frac{1}{1 - \dfrac{\sigma_n^2\{(\ln\alpha)^2 + w^2)\}}{2\sigma_y^2 \ln(\alpha)}}} \tag{13}$$

to be interpreted as optimal lowpass filter if $\sigma_y > \sigma_n$ what is predominantly the case in profile analyses. The filter has to be fitted onto the data to be filtered, so that for given variances σ_y^2 and σ_n^2 one needs the interpedendence of y(m) to be estimated if and only if y(m) has the ergodic property in addition to be stationary.

In order to compute the kernel $h(\lambda)$ of the Wiener filter (13) the following relation is valid (L.R. Rabiner/B. Gold, 1975)

$$H(e^{jw}) = e^{-j((K-1)/2)w} \sum_{k=0}^{(K-1)/2} a(k)\cos(kw) =: e^{-j\beta} |H(e^{jw})| \tag{14a}$$

if the length K is odd, otherwise it holds

$$H(e^{jw}) = e^{-j((K-1)/2)w} \sum_{k=1}^{K/2} b(k)\cos(((K-1)/2)w) =: e^{-j\beta} |H(e^{jw})| \tag{14b}$$

In both terms the kernel h(k) is obtained by simple substitutions

$$a(0) = h((K-1)/2), \quad a(k) = 2h((K-1)/2 \pm k) \tag{15a}$$

$$b(k) = 2h(K/2 \pm k) \tag{15b}$$

The final estimation of h(k) uses these relations in a least squares approximation with the objective function

$$\sigma_t^2 = \int_o^\pi (|H(e^{jw})| - |H^*(e^{jw})|)^2 dw = \min \tag{16}$$

in which $|H(e^{jw})|$ is the ideal and $|H^*(e^{jw})|$ is the approximated magnitude of the frequency response.

Using the samples $|H(l)|$ $\forall l=0,1,2,\ldots,L-1$ of (13) the evaluation of (16) results into the least squares approach: let be $v(l) := |H(l)| - |H^*(l)|$ the residual at the sampling point $w_l := lw/(L-1)$ then the following linear model will be introduced

$$\boxed{y + v = Ax, \quad D(y) = \sigma^2 I} \tag{17}$$

whereby y contains the samples $|H(l)|$ with L samples in all, v is its corresponding residual vector, A is the matrix of known coefficients considering (14) and x contains the substituted filter kernel a(k) or b(k). The minimization of $v'v$ corresponds with (16) and leads to the normal equations

$$A'A\hat{x} = A'y \tag{18}$$

to be solved for the estimates

$$\hat{x} = (A'A)^{-1} A'y \tag{19a}$$

$$\hat{v} = A\hat{x} - y \tag{19b}$$

As goodness of fit for the evaluation of (16) the maximum residual can be used

$$\hat{\delta} = \max |\hat{v}_i| \qquad \forall i=1,2, \ldots ,L-1 \tag{20}$$

The implementation of the filter is given by (4) but one has to consider different lenghts of the kernel and the signal to be filtered: the longer the kernel the more computation time is needed for the filtering process. Therefore, the following relation holds: $K \ll M$. Because of the symmetry of the kernel (4) can be rewritten to provide for zero phase

$$\boxed{\hat{y}(m) = \sum_{k=-(K-1)/2}^{(K-1)/2} h(k)x(m-k)} \tag{21}$$

if K is odd; otherwise for K even the zero phased signal y(m) is not defined at the sample x(m) but in the midst of x(m) and x(m+1).

3. Data Analysis

The following methods for the analysis of profiles are investigated

(i) a trial-and-error approach by linear interpolation
(ii) a Fourier-series-like procedure
(iii) a variogram evaluation

While (i) and (ii) can be seen as purely deterministic, the variogram approximation is based upon the concept of self-similarity - quite similar to the concepts of random processes using autocorrelation functions.

3.1 Linear Interpolation

The procedure consisting of linear interpolation between profile samples is quite simple but very robust as can be shown later on. The optimum sampling interval is found recursively by the use of coarser profiles with a simultaneous quality control.

Starting with the first sample the respective neighbour can be found by linear interpolation

$$\boxed{\hat{x}(m+n) = x(m) + \frac{x(m+k) - x(m)}{k} n \qquad \forall n=1,2,\ldots,k-1} \tag{22}$$

with $\hat{x}(m+n)$ as interpolated (estimated) sample, $x(m)$ is the 1st reference point and $x(m+k)$ the 2nd reference point situated at $k\Delta_p$ apart from $x(m)$ within the profile.

The quality control of this approach is given by estimating the errors $\varepsilon(m)$

$$\hat{\varepsilon}(m+n) = x(m+n) - \hat{x}(m+n) \tag{23}$$

which lead to the quality measure (RMS-value)

$$\sigma^2_{int} = \frac{\|\hat{\varepsilon}\|^2}{r} =: RMS(k) \tag{24}$$

This value has to be compared with σ_s^2: if the RMS(k) is less than σ_s^2 the process is repeated by increasing the control spacing by Δ_p. Assuming that $k\Delta_p$ is the control spacing for each recursion, $k=2,3,\ldots,K$, where K is the least value of k whereby σ_s^2 is exeeded, then the optimum sampling interval has been found

$$\boxed{\Delta_{opt} = \left\{K-1 + \frac{\sigma_s^2 - RMS(K-1)}{RMS(K) - RMS(K-1)}\right\}\Delta_p} \tag{25}$$

when $K=2$, $\Delta_{opt}=\Delta_p$, assuming $RMS(1)=\sigma_s^2$.

3.2 Fourier Series Approach

The Fourier series approach has been given in D. Fritsch (1984, 1985) but should shortly be reviewed in the following:

Let $x(m)\forall m$ be the profile to be analyzed, its discrete Fourier transform (DFT) gives the complex set of functionals

$$X(k) = \sum_{m=0}^{M-1} x(m)e^{-j2\pi km/M} \tag{26}$$

to be transformed back by the inverse discrete Fourier transform (IDFT)

$$x(m) = \frac{1}{M}\sum_{k=0}^{M-1} X(k)e^{j2\pi km/M} \tag{27}$$

Using the Eulerian notation

$$X(k) = A(k)e^{j\beta(k)} \tag{28}$$

with $A(k):=\sqrt{X_{Re}^2(k)+X_{Im}^2(k)}$ as magnitude and $\beta(k):=\tan^{-1}(X_{Im}(k)/X_{Re}(k))$ as its corresponding phase, (28) may be rewritten to

$$x(m) = \frac{1}{M}\sum_{k=0}^{M-1} A(k)(\cos(mw(k)+\beta(k))+j\sin(mw(k)+\beta(k))) \tag{29}$$

For M even the profile can be reconstructed (D. Fritsch, 1985)

$$x(m) = \frac{A(0)}{M} + \frac{A(M/2)\cos(m\pi)}{M} + \frac{2}{M}\sum_{k=1}^{(M/2-1)} A(k)\cos(mw(k)+\beta(k)) \tag{30}$$

whereby for M odd the relation is valid

$$x(m) = \frac{A(0)}{M} + \frac{2}{M}\sum_{k=1}^{(M-1)/2} A(k)\cos(mw(k)+\beta(k)) \tag{31}$$

Assuming a coarser profil $x_k(m)\forall m$ with sampling distance $k\Delta_p$ its highest frequency is determined by the sampling theorem

$$\boxed{w(l) = \frac{\pi}{l} \qquad \forall l=1,2,3,\ \ldots.} \tag{32}$$

With this relation and the reconstruction formula (30) or (31) the computation of the sampling factor l can be done iteratively: start with all frequencies of the dense profile and reduce the frequency number going from the highest to the lower ones. Every reduction contributes to a RMS-value σ_k^2, so that with $\sigma_k^2 = \sigma_s^2$ the final frequency w(l) is found leading to

$$\boxed{\Delta_{opt} = l\,\Delta_p} \tag{33}$$

If different sections j within the profile i deliver factors l_{ij}, the final l may be defined as mean value

$$l = \frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N} l_{ij} \tag{34}$$

3.3 Variogram Evaluation

The variogram approach for profile analyses has been introduced by P. Frederiksen et al. (1983, 1984, 1986). It consists of a statistical analysis of the profile and is based upon the concept of self-similarity.

Let $x(m)\forall m$ be the profile, its variogram is the mean value of quadratic differences of two points with lag k

$$\boxed{V(k) = \frac{1}{M-1-k}\sum_{m=0}^{M-1-k} (x(m) - x(m+k)) \qquad \forall k=1,2,\ldots,M/2} \tag{35}$$

If V(k) versus k is plotted on a log-log-scale it can be modeled as

$$V(k) = ck^{\beta} \tag{36}$$

Investigations with real data have shown that the points tend to lose correlations at larger values of k. In many cases it is sufficient to use a straight line for the approximation of the log-log plot of the variogram, from which the constant logc and the slope β can be determined.

Starting with the first e.g. three points representing $\log V(k)\,\forall k=0,1,2$ a first straight line may be computed by least squares. The remaining residuals have to be compared with a threshold which corresponds with an acceptable graphical error. If residuals are less than this threshold, the process is repeated with the next point included, until at least one residual exceeds the threshold.

The optimum sampling interval is then given by

$$\boxed{\Delta_{opt} = L\Delta_{p}} \tag{37}$$

with L resulting from

$$\frac{\sigma_s^2}{c} = (L^{\beta})\left(\frac{2-2^{\beta-1}}{2^{\beta+1}}\right) \tag{38}$$

4. Applications

The two-step procedure consisting of Wiener filtering and profile analyses is applied on nine examples of real profiles. Because of the definition of the Wiener filter and also of the Fourier-series-like approach as well as the variogram evaluation the mean value of the profile to be analyzed should be

$$E(x(m)) = 0 \tag{39}$$

what means, that the real data have to be centered. Centering can be done using the strategies

(i) remove the mean value
(ii) remove a straight line

leading to the preprocessed profile

$$x(m) = x(m) - a_0 \tag{40a}$$

or

$$x(m) = x(m) - a_0 - a_1 m \tag{40b}$$

with a_0 and a_1 as parameters resulting from a least squares approximation. There are also other trend models such as a combination of (40b) with a Fourier series, but this should not be commented on in this paper.

4.1 Special Considerations

In order to show up the influence of noise on the determination of the optimum sampling interval the first profile was superimposed with Gaussian noise with $E(n(m))=0$ and $\sigma=0.5$. The following analyses were made

(i) compute the optimum sampling interval of the original profile
(ii) compute the optimum sampling interval of the noisy profile
(iii) smooth the profile and compute the optimum sampling interval of the smoothed profile.

All three profiles are represented by Fig. 1; the results of the profile analyses can be seen in Table 1.

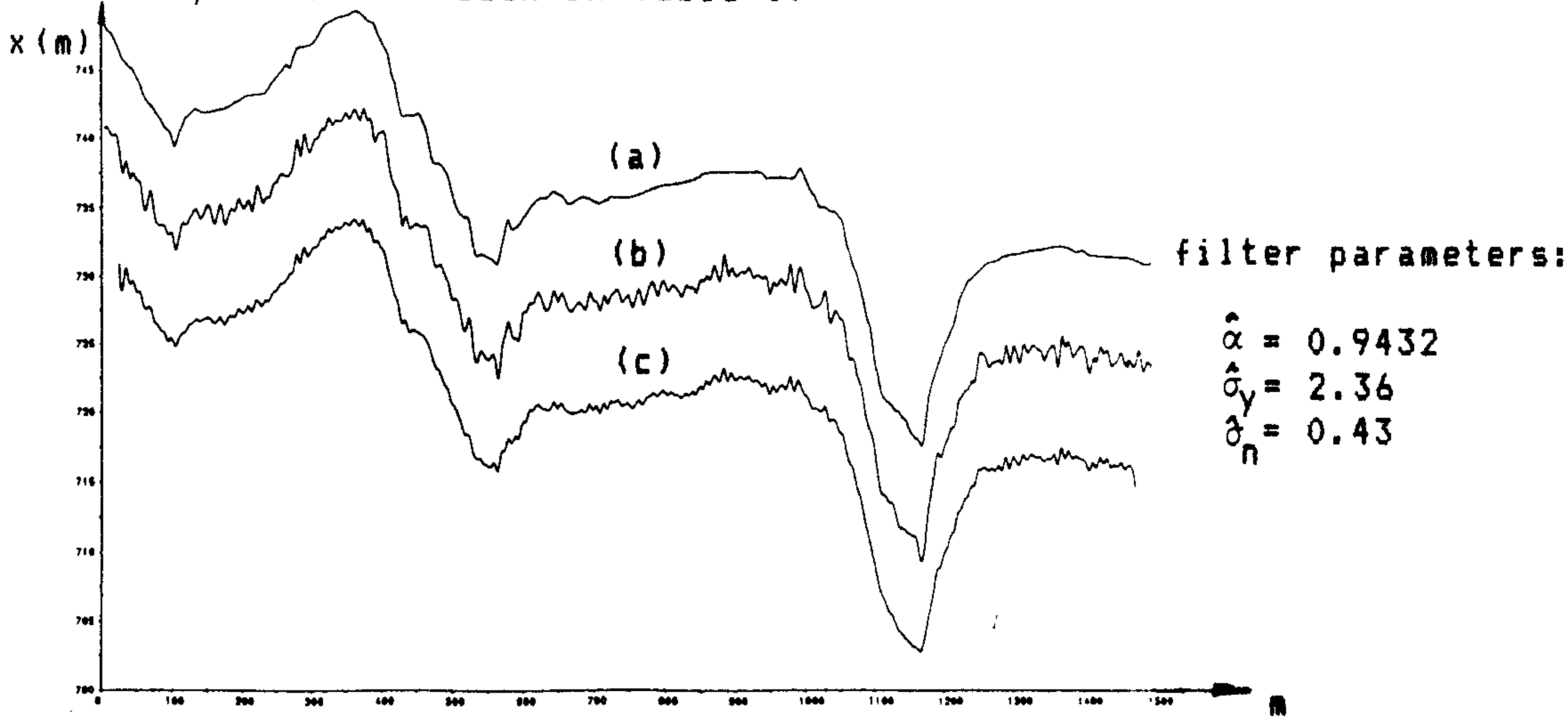


Fig. 1: Terrain profile No. 1 - (a) original, (b) noisy and (c) smoothed

Table 1: Profile analyses of terrain profile No. 1 (Δ_p=5(m), σ_s=1(m))

Typ	Trend	Lin. Int. Δ_{opt}	Fourier Δ_{opt}	Variogram
original	mean value	80 (m)	75	64
	straight line	80	107	65
noisy	m.v.	61	75	29
	s.l.	61	98	29
smoothed	m.v.	79	85	50
	s.l.	79	107	51

4.2 Further Analyses

The remaining profiles were analyzed without and with Wiener filtering (see Fig. 2). Because of the sensitivity of the Fourier approach against trend models the results of Table 2 and 3 have been obtained using a straight line to center the profiles. As can be seen in Table 3 the data smoothing provides in all but one case for larger optimum sampling intervals.

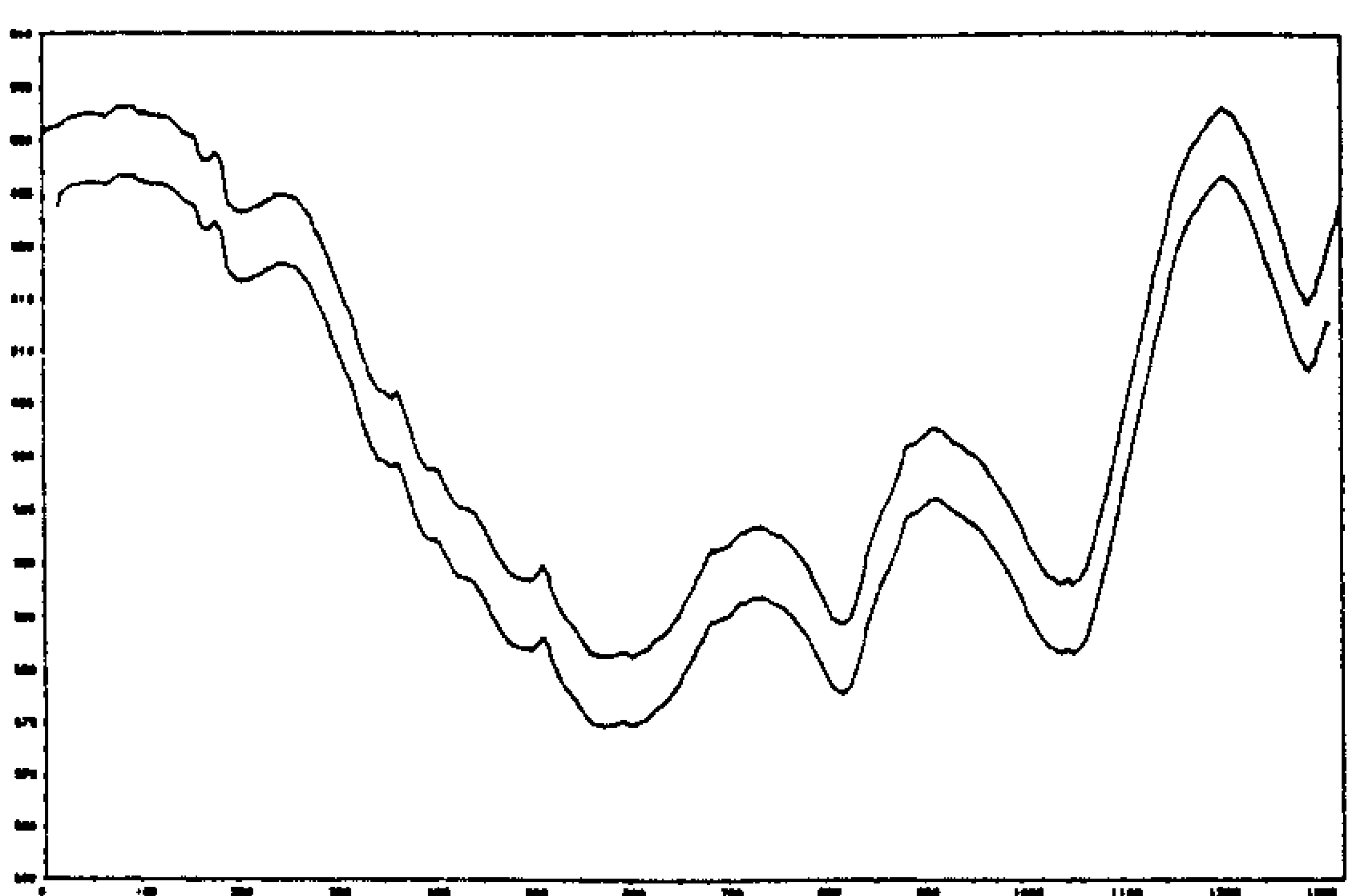
Table 2: Profile analyses of original profiles (σ_s=1(m), Dim.(Δopt=(m))

No.	Δ_p	Lin. Int. Δ_{opt}	Fourier Δ_{opt}	Variogram Δ_{opt}	Mean
2	2.5	40	23	36	33.0
3	2.5	72	67	53	64.0
4	5.0	115	128	75	106.0
5	5.0	89	98	72	86.3
6	2.5	72	71	76	73.0
7	2.5	89	142	66	99.0
8	5.0	14	5	12	10.3
9	5.0	75	98	67	80.0

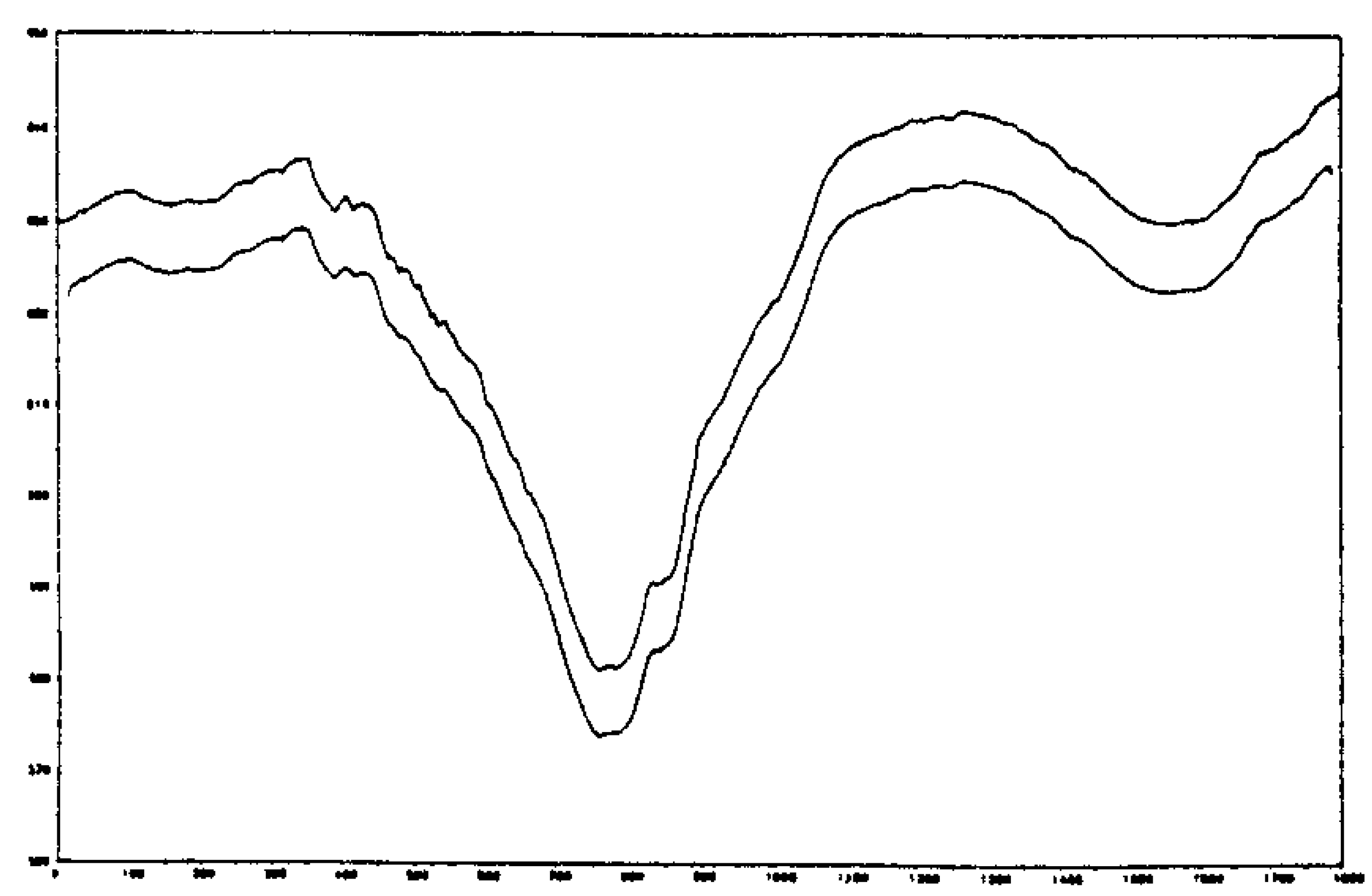
Table 3: Profile analyses of smoothed profiles (σ_s=1(m), Dim. (Δopt=(m))

No.	Δ_p	Lin. Int. Δ_{opt}	Fourier Δ_{opt}	Variogram Δ_{opt}	Mean
2	2.5	44	27	38	36.3
3	2.5	73	71	66	70.0
4	5.0	139	142	129	136.7
5	5.0	109	116	103	109.3
6	2.5	65	80	90	78.3
7	2.5	89	142	85	105.3
8	5.0	5	5	12	7.3
9	5.0	92	107	79	92.7

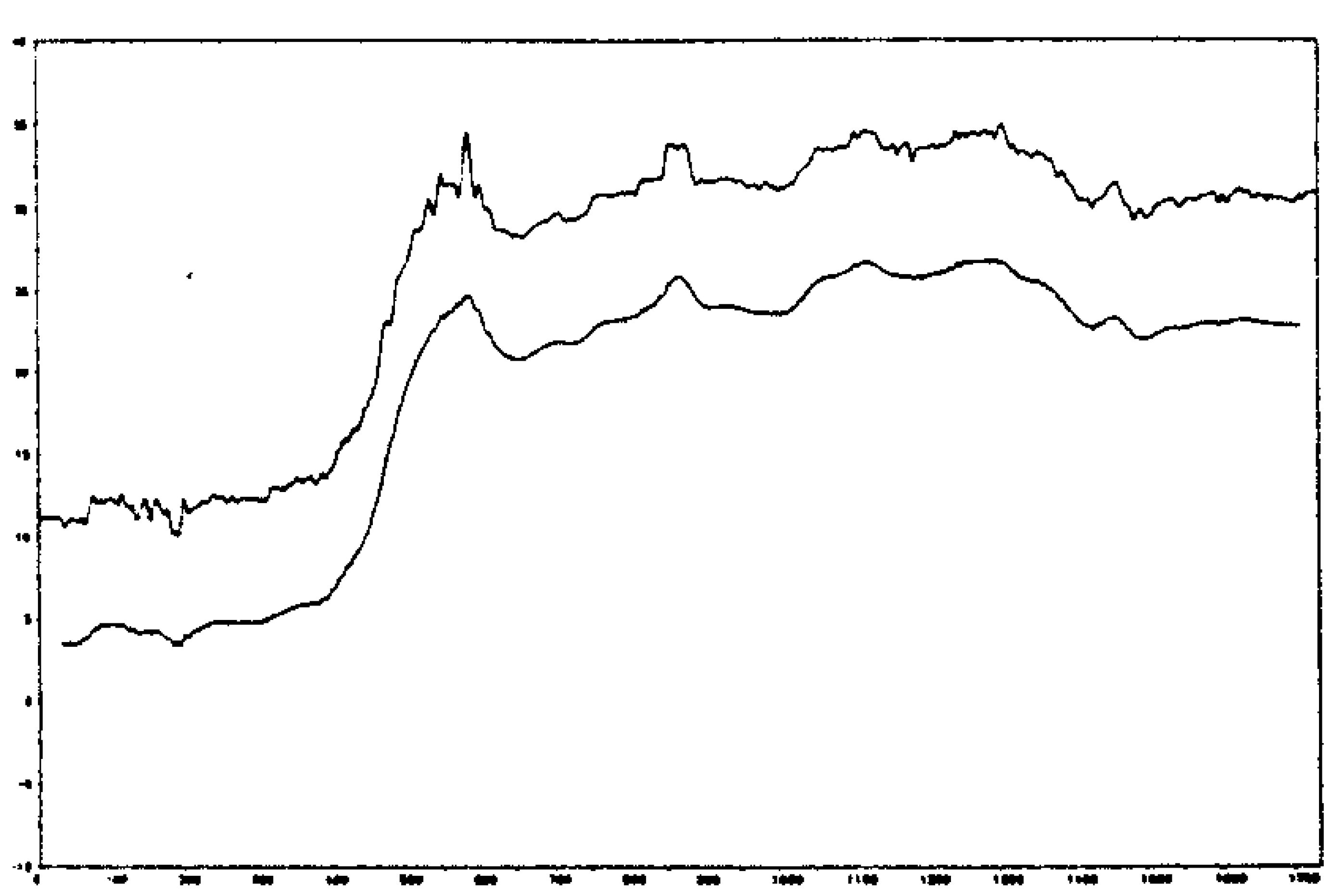
(2)



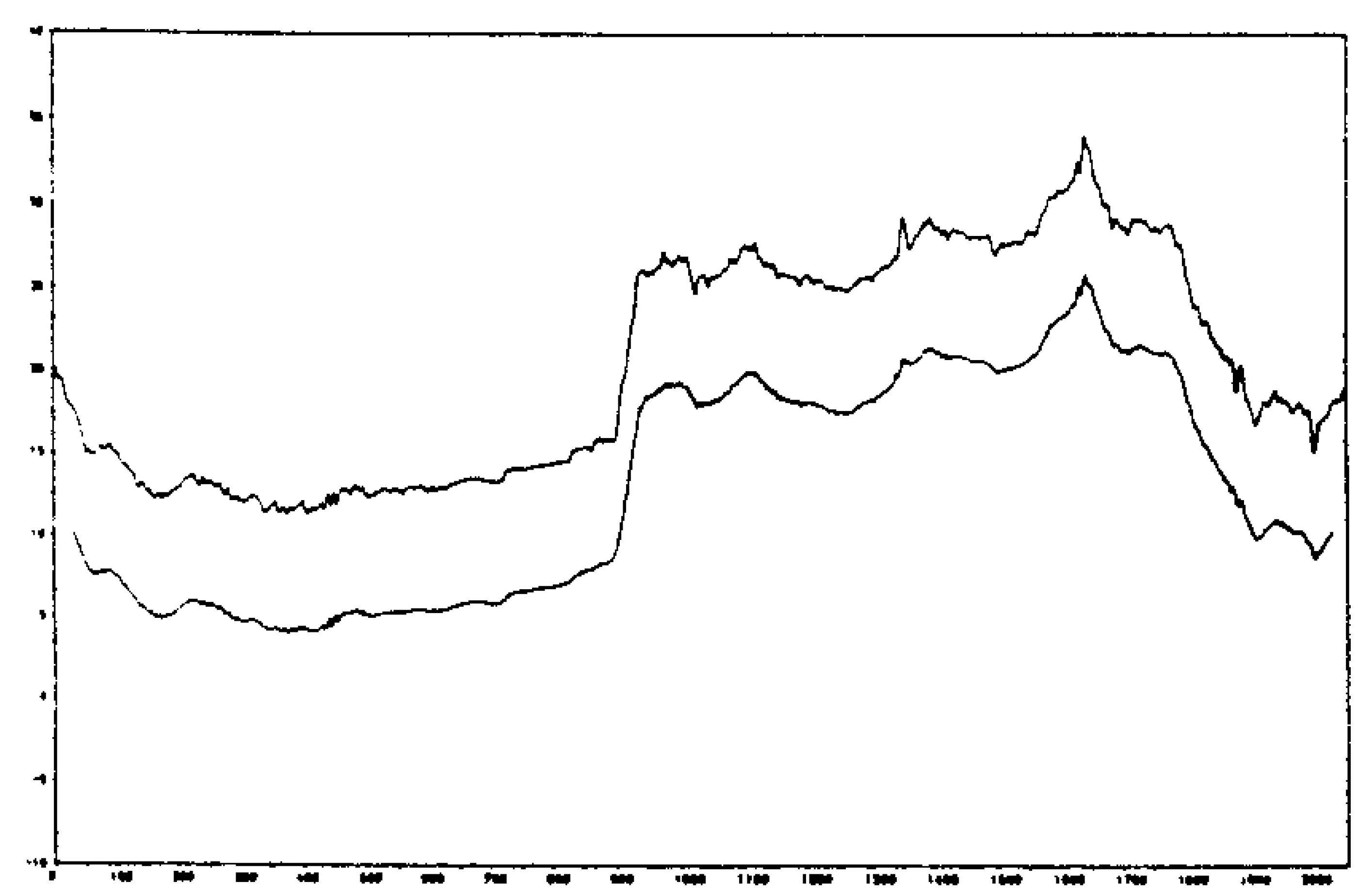
(3)



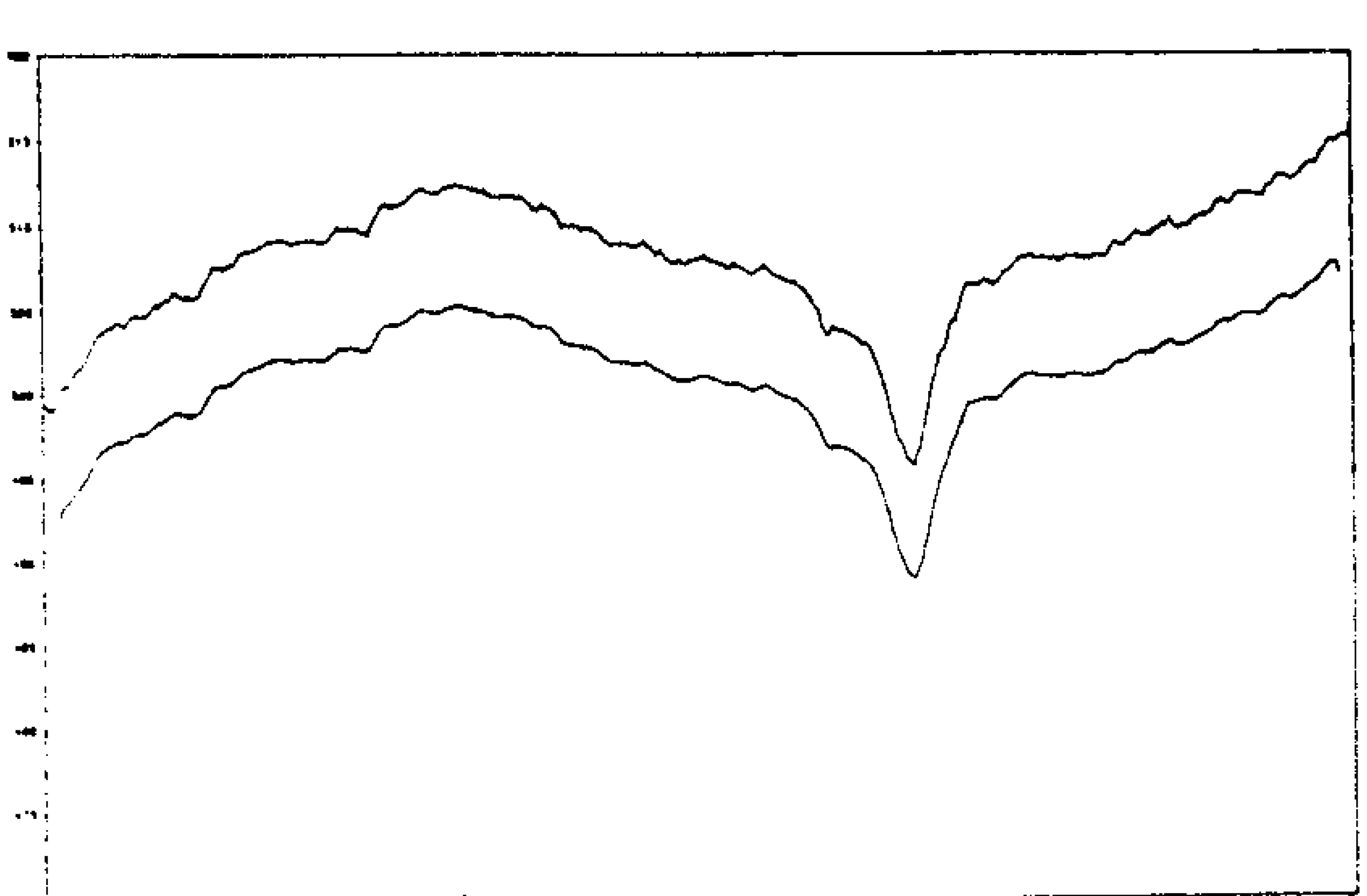
(4)



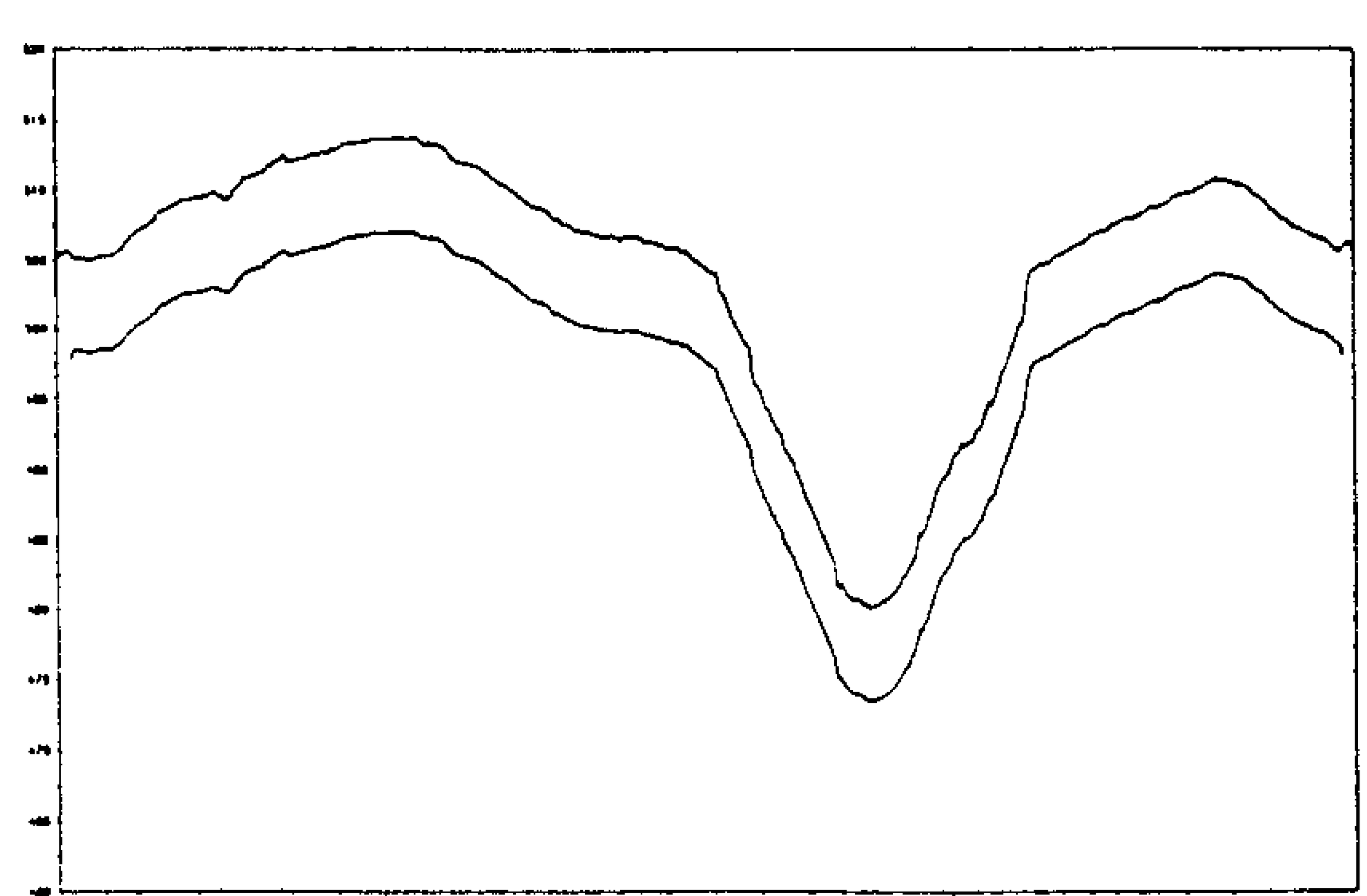
(5)



(6)



(7)



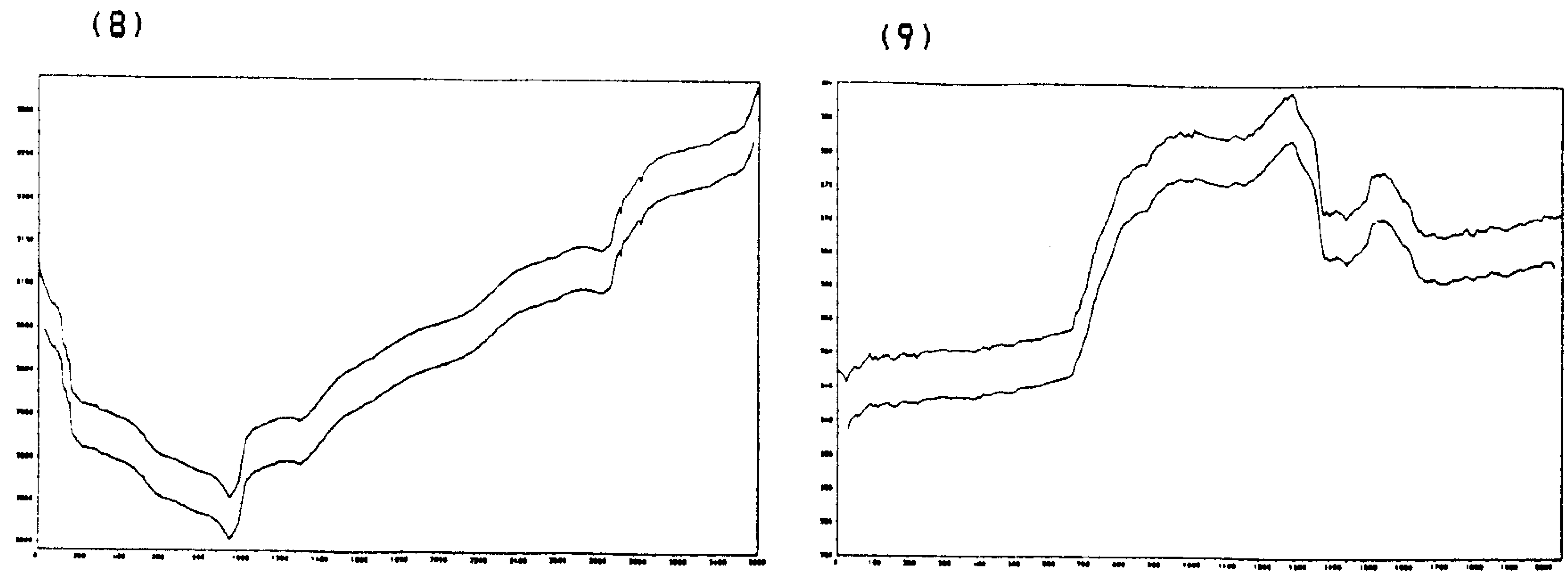


Fig. 2: Original and smoothed profiles being analyzed

5. Recommendations and Conclusions

A comparison of the results above shows the discrepancies of the methods presented. But first of all, the two-step procedure seems to be the right strategy, because observation noise leads to over-sampled DTM data to be avoided by an optimum data acquisition. Looking into further details, the trial-and error approach of linear interpolation gives promising results, if the deviations of the mean value are analyzed. It is very robust and very fast - in the contrary to the Fourier approach, which needs the most computation time. The variogram evaluation is very sensitive against observation noise and thus requires optimum data smoothing. Therefore, the linear interpolation is highly recommended but in combination with a smoothing procedure, which can be Wiener filtering, spline approximations or moving average (MA) procedures.

6. Acknowledgements

The author is indebted to E. Dirscherl, who made the computations and the drawings for this paper.

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