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5. Methods of evaluating recent crustal movements

MULTIVARIATE HYPOTHESIS TESTS FOR DETECTING RECENT CRUSTAL MOVEMENTS

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ABSTRACT

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For detecting recent crustal movements repeated geodetic measurements are often observed in a net of control points. These data are evaluated in a multivariate statistical analysis, in order to separate the set of control points into two disjunctive sets, the set of fixed points and the set of variable points. The coordinates of the control points are introduced as unknown parameters for the analysis of the observations at different time epochs. However, the geodetic measurements often do not contain any information about the translation, the rotation or the scale of the coordinate system, so that restrictions have to be introduced to obtain estimable quantities. As shown, these restrictions must be formulated for all fixed points, in order to obtain a location of the control net which is optimal with respect to the detection of movements. But the fixed points have to be found first by means of univariate and multivariate hypothesis tests. This problem is solved by a successive estimation and test procedure, which can be automated, so that in a single analysis of the data the variable points are separated from the fixed points. As shown by an example, the univariate and multivariate test procedures enable the detection of horizontal movements in the order of magnitude of the lengths of the axes of the 95% confidence ellipses.

INTRODUCTION

To detect recent crustal movements or deformations of man-made constructions, geodetic measurements can be applied. Usually these observations establish a net of control points which are connected with that part of the surface of the earth or of the construction where movements are suspected to occur. Since only the relative movements of the points with respect to each other can be detected, one tries to set up the control net in such a way, that only a few points move compared to a majority of points which do not change their positions with respect to each other.These points will be called the fixed points and the points, which move, the variable points. Hence, the set of control points has to be separated into two disjunctive sets, the set of fixed points and the set of variable points.

To solve this problem, the measurements for the control net have to be repeated at different time epochs. If movements of control points between time epochs have occurred, they cannot be detected, if the movements are too small in comparison with the standard deviations of the observations, which establish the control net. The detection of movements therefore depends on the variances and covariances of the observations, on the kind of measurements, on the configuration of the net and on the amount of movements which occurred. As will be shown in the sequel, multivariate statistical analysis pushes the possibility of detecting movements to the limit set by the variances and covariances of the observations.

If the coordinates of the control points would be given in a three-dimensional coordinate system independent from any movements which occurred, it would be simple to find these movements by computing coordinate differences between different time epochs. It is therefore appropriate, to choose the coordinates of the control points as unknown parameters for the analysis of the observations. However, the measurements of the control net, usually distances and angles or bearings, do not contain any information about the translation, the rotation or the scale of the coordinate system. This information is introduced by the approximate coordinates of the control points which are assumed as given. With respect to this coordinate system the control net is translated, oriented or scaled. As will be shown, only the fixed points should be used for this procedure. But the fixed points must be detected first so that the estimation of coordinates together with hypothesis tests for fixed points are applied successively.

For the estimation of the unknown coordinates of the control points from the observations at different time epochs two models are available, the univariate model (Pelzer, 1971; Heck et al., 1977; Van Mierlo, 1980) and the multivariate model (Koch, 1976, 1978; Hein, 1978). The former combines the observations of all time epochs in one parameter estimation. The latter uses one parameter and observation vector for each time epoch. The advantage of the multivariate model over the univariate model lies in additional statistical tests which are very sensitive to the movements of the control points. For the multivariate model the setup of measurements must be the same over different time epochs. But if changes of observations are necessary, for instance if a control point has to be placed into a different location, it is possible by additional measurements to transform the original observations into observations which can be analyzed in a multivariate model. This is treated in more detail in Koch (1980b) together with a comparison of the univariate and multivariate models. In the following a multivariate model is applied to detect recent crustal movements.

MULTIVARIATE MODEL

Let a net of k control points be observed at p different time epochs without changing the number n and the kind of measurements and their weight matrices. From these repeated measurements the unknown three-dimensional coordinates of the k control points at each epoch have to be estimated. The nonlinear relationship between the observations and the coordinates is linearized by means of the approximate coordinates of the control points. Since the movements which occur are small, identical approximate coordinates are used for all time epochs, so that the $n \times 3k$ matrix $\boldsymbol{X}$ of partial derivatives is the same for all p epochs. Let the $3k \times 1$ vector $\boldsymbol{\beta}_i$ with $i \in \{1, ..., p\}$ contain the unknown three-dimensional coordinates x_{ji}, y_{ji}, z_{ji} with $j \in \{1, ..., k\}$ of the k control points at the epoch i, thus:

$$\boldsymbol{\beta}_i = |x_{1i}, y_{1i}, z_{1i}, ..., x_{ki}, y_{ki}, z_{ki}|' \quad \text{with } i \in \{1, ..., p\} \tag{1}$$

These unknown parameters represent the corrections to the approximate coordinates x_{j0}, y_{j0}, z_{j0} with $j \in \{1, ..., k\}$ of the k control points, which are collected in the $3k \times 1$ vector $\boldsymbol{\beta}_0$, hence:

$$\boldsymbol{\beta}_0 = |x_{10}, y_{10}, z_{10}, ..., x_{k0}, y_{k0}, z_{k0}|' \tag{2}$$

Let the $n \times 1$ vector $\boldsymbol{y}_i$ with $i \in \{1, ..., p\}$ contain the observations of the epoch i and the effect of the linearization. Because of the repetitions at different time epochs the observations are correlated, and it is assumed, that the covariance matrix of $\boldsymbol{y}_i$ and $\boldsymbol{y}_j$ is given by $C(\boldsymbol{y}_i, \boldsymbol{y}_j) = \sigma_{ij}\boldsymbol{I}_n$ where $\boldsymbol{I}_n$ denotes the $n \times n$ identity matrix. It is $i, j \in \{1, ..., p\}$ and the $p \times p$ unknown covariance matrix $\Sigma = (\sigma_{ij})$ is supposed to be positive definite. One could have assumed $C(\boldsymbol{y}_i, \boldsymbol{y}_j) = \sigma_{ij}\boldsymbol{P}^{-1}$ with $\boldsymbol{P}$ being the $n \times n$ positive, definite weight matrix of the observations, but by a simple transformation (Koch, 1980a, p. 146) the covariance matrix $\sigma_{ij}\boldsymbol{I}_n$ is obtained.

Under the assumptions given above the multivariate model for the unknown parameters $\boldsymbol{\beta}_i$ and the unknown covariances σ_{ij} is given with $\boldsymbol{\beta} = |\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_p|$ and $\boldsymbol{Y} = |\boldsymbol{y}_1, ..., \boldsymbol{y}_p|$ by

$$\boldsymbol{X\beta} = E(\boldsymbol{Y}) \quad \text{with } C(\boldsymbol{y}_i, \boldsymbol{y}_j) = \sigma_{ij}\boldsymbol{I}_n \text{ and } i, j \in \{1, ..., p\} \tag{3}$$

where the $3k \times p$ matrix $\boldsymbol{\beta}$ contains the parameter vectors $\boldsymbol{\beta}_i$ and the $n \times p$ matrix $\boldsymbol{Y}$ the observations $\boldsymbol{y}_i$. $E(\boldsymbol{Y})$ is the expected value of $\boldsymbol{Y}$. It can be shown (Koch, 1980a, p. 217), that the estimates of $\boldsymbol{\beta}_i$ are obtained without knowing the covariances σ_{ij}, which is a very advantageous feature of the multivariate model.

As already mentioned, the observations of a control net generally do not contain any information about the location of the coordinate system for the parameters $\boldsymbol{\beta}_i$. The coefficient matrix $\boldsymbol{X}$ in eq. 3 therefore has not full rank, but say $\text{rg}\boldsymbol{X} = q < 3k$. The rank defect for instance is $3k - q = 7$, if only angles are observed, since three translations, three rotations and the scale of the coordinate system are unknown. To obtain estimable quantities in the model (eq. 3) not of full rank, the parameters $\boldsymbol{\beta}_{\mathrm{b}i}$ are introduced, which are obtained by projecting the $3k$-dimensional Euclidean space E^{3k} onto the q-dimensional subspace E^q, where $\boldsymbol{\beta}_{\mathrm{b}i}$ is estimable. The projection is given by:

$$\boldsymbol{\beta}_{bi} = (X'X)^{-} X'X\boldsymbol{\beta}_i \tag{4}$$

where $(X'X)^{-}$ is a generalized inverse of the normal equation matrix $X'X$. The best linear unbiased estimate $\hat{\boldsymbol{\beta}}_{bi}$ of $\boldsymbol{\beta}_{bi}$ is given by (Koch, 1980a, p. 171):

$$\hat{\boldsymbol{\beta}}_{bi} = (X'X)^{-}_{rs} X'\boldsymbol{y}_i \tag{5}$$

where $(X'X)^{-}_{rs} = (X'X)^{-}X'X[(X'X)^{-}]'$ is a symmetrical reflexive generalized inverse of $X'X$.

An efficient way of computing $(X'X)^{-}_{rs}$ is given by means of the $(3k-q) \times 3k$ matrix E, whose rows contain a basis for the nullspace $N(X)$ of X, i.e. $XE' = 0$. If in addition one chooses a $(3k-q) \times 3k$ matrix B such that:

$$\mathrm{rg}|X', B'| = 3k \tag{6}$$

one gets (Koch, 1980 a, p.59):

$$(X'X)^{-}_{rs} = (X'X + B'B)^{-1} - E'(EB'BE')^{-1} E \tag{7}$$

The matrix B can be interpreted as matrix of the restrictions:

$$B\boldsymbol{\beta}_i = \boldsymbol{0} \tag{8}$$

to be imposed on the parameters in order to accomplish the projection (eq. 4) or to identify the parameters by removing the singularity from the normal equations. Depending on the choice of B, different generalized inverses $(X'X)^{-}_{rs}$ are obtained.

For control nets the matrix E can generally be given by purely geometrical considerations, since it contains the changes the unknown parameters $\boldsymbol{\beta}_i$ can undergo without affecting the observations $\boldsymbol{y}_i$ (Pope, 1971). With respect to the coordinate system defined for the approximate coordinates $\boldsymbol{\beta}_0$, the control net may undergo a translation, a differential rotation and a scale change if angles are observed. In such a case one gets (Koch, 1980a, p. 174):

$$E' = \begin{vmatrix} 1 & 0 & 0 & 0 & z_{10} & -y_{10} & x_{10} \\ 0 & 1 & 0 & -z_{10} & 0 & x_{10} & y_{10} \\ 0 & 0 & 1 & y_{10} & -x_{10} & 0 & z_{10} \\ 1 & 0 & 0 & 0 & z_{20} & -y_{20} & x_{20} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{9}$$

If the matrix E instead of B is introduced into eq. 7, one obtains the pseudoinverse $(X'X)^{+}$ of $X'X$ with:

$$(X'X)^{+} = (X'X + E'E)^{-1} - E'(EE'EE')^{-1} E \tag{10}$$

The projected parameters $\boldsymbol{\beta}_{ei}$ are given by:

$$\boldsymbol{\beta}_{ei} = (X'X)^{+} X'X\boldsymbol{\beta}_i \tag{11}$$

and the best linear unbiased estimate $\hat{\boldsymbol{\beta}}_{ei}$ of $\boldsymbol{\beta}_{ei}$ by:

$$\hat{\boldsymbol{\beta}}_{ei} = (X'X)^{+}X'y_i \tag{12}$$

CHOICE OF THE PROJECTION

It is a well known property of the pseudoinverse, that:

$$\hat{\boldsymbol{\beta}}'_{bi}\hat{\boldsymbol{\beta}}_{bi} \geqslant \hat{\boldsymbol{\beta}}'_{ei}\hat{\boldsymbol{\beta}}_{ei} \tag{13}$$

Thus, when estimating the unknown coordinates by means of the pseudoinverse, the control net is translated, differentially rotated or scaled in such a way, that the sum of the squares of the estimated coordinates, which are corrections to the approximate coordinates, are minimized. By this procedure the control net is located with respect to the coordinate system for the approximate coordinates, so that this coordinate system defines the system in which the parameters are estimated.

Although the choice of the projection (eq. 4) to obtain estimable parameters is arbitrary, the projection has to be selected such, that the differences of the coordinates estimated for different time epochs are caused by movements and not by the projection. If between the time epochs points of the control net have moved and if one uses the pseudoinverse to locate the control net, then according to eq. 13 the variable points also contribute to the translation, orientation or the scale of the control net, so that the estimated coordinates are affected by the movements of the variable points. Hence, fixed points only may be used to translate, rotate or scale the control net. The location of the control net between different time epochs is then influenced only by pseudomovements caused by the variances of the observations and by undetected movements. Thus, all fixed points have to be used to locate the control net, in order to keep the influence of these movements small.

Changes in the translation, orientation or scale of the control net are completely eliminated, if in a univariate model only three unknown coordinates for one fixed point are introduced for all time epochs. This is a common procedure in the analysis of crustal movements. But undetected movements of the fixed points are then treated as measuring errors which cause pseudomovements of the variable points. As will be shown in the following, in a multivariate analysis the assumption that the fixed points have identical coordinates for all time epochs is introduced as a hypothesis, to detect variable points, and not as an assumption at the beginning of the analysis as done in a univariate model.

In order to use fixed points only to locate the control net, one has to eliminate from the matrix $\boldsymbol{E}$ the variable points of the control net, so that the matrix $\boldsymbol{B}$ is obtained, formed exclusively by the fixed points. With the $3k \times 3k$ diagonal matrix $\boldsymbol{S}$, which contains on the diagonal zeros for the coordinates of the variable points and ones for the fixed points one gets:

$$\boldsymbol{B} = \boldsymbol{E}\boldsymbol{S} \tag{14}$$

The condition (eq. 6) for $\boldsymbol{B}$ is fulfilled, if there are at least $3k - q$ columns in $\boldsymbol{B}$ which are not zero vectors. By substituting eq. 14 in eq. 7 the best linear unbiased estimate $\hat{\boldsymbol{\beta}}_{bi}$ is obtained from eq. 5. It can be shown (Koch, 1980a, p. 175) that for these estimates the control net is fixed with respect to the coordinate system of the approximate coordinates such that the squares of the coordinates estimated for the fixed points are minimized.

HYPOTHESIS TESTING

To divide the set of control points into two disjunctive sets of fixed points and variable points, two hypotheses are introduced based on the assumption that certain points are fixed points.

Let the $r \times 1$ vector $\boldsymbol{\beta}_{fi}$ contain r projected coordinates of control points which are assumed to be fixed during the p time epochs. It is:

$$\boldsymbol{H}\boldsymbol{\beta}_{bi} = \boldsymbol{\beta}_{fi} \text{ and } \boldsymbol{H}\hat{\boldsymbol{\beta}}_{bi} = \hat{\boldsymbol{\beta}}_{fi} \text{ with } i \in \{1, ..., p\} \tag{15}$$

where each row of the $r \times 3k$ matrix $\boldsymbol{H}$ contains except zeros one number one at the appropriate place. In addition with $\boldsymbol{\beta}_b = |\boldsymbol{\beta}_{b1}, ..., \boldsymbol{\beta}_{bp}|$, $\hat{\boldsymbol{\beta}}_b = |\hat{\boldsymbol{\beta}}_{b1}, ..., \hat{\boldsymbol{\beta}}_{bp}|$, $\boldsymbol{\beta}_f = |\boldsymbol{\beta}_{f1}, ..., \boldsymbol{\beta}_{fp}|$ and $\hat{\boldsymbol{\beta}}_f = |\boldsymbol{\beta}_{f1}, ..., \boldsymbol{\beta}_{fp}|$ one gets:

$$\boldsymbol{H}\boldsymbol{\beta}_b = \boldsymbol{\beta}_f \text{ and } \boldsymbol{H}\hat{\boldsymbol{\beta}}_b = \hat{\boldsymbol{\beta}}_f \tag{16}$$

Now the multivariate hypothesis is introduced, that r coordinates of fixed points of epoch i equal the r estimated coordinates of epoch $i + 1$:

$$|\boldsymbol{\beta}_{f1}, ..., \boldsymbol{\beta}_{f,p-1}, \boldsymbol{\beta}_{fp}| = |\hat{\boldsymbol{\beta}}_{f2}, ..., \hat{\boldsymbol{\beta}}_{fp}, \hat{\boldsymbol{\beta}}_{f1}| \tag{17}$$

or:

$$\boldsymbol{H}\boldsymbol{\beta}_b = \boldsymbol{W} \text{ with } \boldsymbol{W} = |\hat{\boldsymbol{\beta}}_{f2}, ..., \hat{\boldsymbol{\beta}}_{fp}, \hat{\boldsymbol{\beta}}_{f1}|$$

against the alternative hypothesis that for at least one coordinate the identity does not hold. Applying the likelihood ratio test, one obtains Wilk's likelihood ratio criterion (Koch, 1980a, p. 251):

$$\Lambda = \frac{\det \boldsymbol{\Omega}}{\det(\boldsymbol{\Omega} + \boldsymbol{R})} \tag{18}$$

with

$$\boldsymbol{\Omega} = (\boldsymbol{X}\hat{\boldsymbol{\beta}}_b - \boldsymbol{Y})'(\boldsymbol{X}\hat{\boldsymbol{\beta}}_b - \boldsymbol{Y})$$

and:

$$\boldsymbol{R} = (\boldsymbol{H}\hat{\boldsymbol{\beta}}_b - \boldsymbol{W})'(\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^-_{rs}\boldsymbol{H}')^{-1}(\boldsymbol{H}\hat{\boldsymbol{\beta}}_b - \boldsymbol{W})$$

In addition the trace criterion by Lawley and Hotelling:

$$T^2 = \mathrm{sp}(\boldsymbol{R}\,\boldsymbol{\Omega}^{-1}) \tag{19}$$

is used.

The p univariate hypotheses following from eq. 17 are given by:

$$\boldsymbol{\beta}_{f1} = \hat{\boldsymbol{\beta}}_{f2}, \quad ..., \quad \boldsymbol{\beta}_{f,p-1} = \hat{\boldsymbol{\beta}}_{fp}, \quad \boldsymbol{\beta}_{fp} = \hat{\boldsymbol{\beta}}_{f1} \tag{20}$$

The likelihood ratio test for these hypotheses leads with $\boldsymbol{\Omega} = (\omega_{ij})$ and $\boldsymbol{R} = (r_{ij})$ to the test statistic:

$$T_i = \frac{(r_{ii}/r)}{\omega_{ii}/(n-q)} \tag{21}$$

In addition the multivariate hypothesis is tested:

$$|\boldsymbol{\beta}_{f1} - \boldsymbol{\beta}_{f2}, \boldsymbol{\beta}_{f2} - \boldsymbol{\beta}_{f3}, ..., \boldsymbol{\beta}_{f,p-1} - \boldsymbol{\beta}_{fp}| = |\boldsymbol{0}, \boldsymbol{0}, ..., \boldsymbol{0}| \tag{22}$$

or:

$$\boldsymbol{H}\boldsymbol{\beta}_b \boldsymbol{U} = \boldsymbol{W} \quad \text{with } \boldsymbol{W} = |\boldsymbol{0}, \boldsymbol{0}, ..., \boldsymbol{0}|$$

The columns of the $p \times (p-1)$ matrix $\boldsymbol{U}$ contain zeros except a minus and a plus one at the appropriate places. The matrices $\boldsymbol{\Omega}$ and $\boldsymbol{R}$ for eqs. 18 and 19 are obtained by (Koch, 1980a, p. 252):

$$\boldsymbol{\Omega} = \boldsymbol{U}'(\boldsymbol{X}\hat{\boldsymbol{\beta}}_b - \boldsymbol{Y})'(\boldsymbol{X}\hat{\boldsymbol{\beta}}_b - \boldsymbol{Y})\boldsymbol{U}$$

and:

$$\boldsymbol{R} = (\boldsymbol{H}\hat{\boldsymbol{\beta}}_b \boldsymbol{U} - \boldsymbol{W})'(\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^-_{rs}\boldsymbol{H}')^{-1}(\boldsymbol{H}\hat{\boldsymbol{\beta}}_b \boldsymbol{U} - \boldsymbol{W})$$

The multivariate hypothesis (eq. 22) states, that r identical coordinates of fixed points can be introduced for all epochs.

The $p-1$ univariate hypotheses following from eq. 22 are given by:

$$\boldsymbol{\beta}_{f1} - \boldsymbol{\beta}_{f2} = \boldsymbol{0}\,,\ \boldsymbol{\beta}_{f2} - \boldsymbol{\beta}_{f3} = \boldsymbol{0}, ..., \boldsymbol{\beta}_{f,p-1} - \boldsymbol{\beta}_{fp} = \boldsymbol{0} \tag{23}$$

with the test statistics T_i given by eq. 21. These tests can be derived from a univariate model, if the measurements of two consecutive time epochs are combined to determine coordinate differences.

If the hypotheses (eq. 17 or eq. 22) are introduced for the coordinates of all fixed points which are used to form the matrix $\boldsymbol{B}$ in eq. 14, the matrix $|\boldsymbol{B}', \boldsymbol{H}'|$ is not of full column rank and the matrix $\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^-_{rs}\boldsymbol{H}'$ for eq. 18 and eq. 19 becomes singular (Koch, 1980a, p. 178). This is caused by the fact, that the coordinates $\boldsymbol{\beta}_{fi}$ and $\hat{\boldsymbol{\beta}}_{fi}$ in eqs. 17 and 22 fulfill the $3k-q$ linear restrictions given by eq. 8. If m points have been used to form the matrix $\boldsymbol{B}$ and if the same points are tested for being fixed points, only $r = 3m - (3k-q)$ restrictions can be introduced by eqs. 17 and 22, in order to obtain together with the restrictions of eq. 8 a set of linearly independent restrictions. With these $r = 3m - (3k-q)$ restrictions the matrix $\boldsymbol{H}(\boldsymbol{X}'\boldsymbol{X})^-_{rs}\boldsymbol{H}'$ in eqs. 18 and 19 is regular and it does not matter which r coordinates of the m points are used in eqs. 17 and 22, the remaining $3k-q$ restrictions are automatically fulfilled because of eq 8.

Hence, to apply the tests of eq. 17 or eq. 22 for the m points, used to form the matrix $\boldsymbol{B}$ in eq. 14, only $r = 3m - (3k-q)$ coordinates of the m points may be introduced in eq. 17 or eq. 22. If additional points are tested, all coordinates of these points have to enter into eq. 17 or eq. 22. If on the other hand less than r coordinates are tested in eq. 17 or eq. 22, because of the additional restrictions (eq. 8) these tests are difficult to interpret and should not be applied.

The multivariate hypothesis (eq. 17) together with the corresponding univariate hypotheses (eq. 20) are more sensitive to movements than the multivariate hypothesis (eq. 22) and the corresponding univariate hypotheses (eq. 23), since eqs. 17 and 20 test coordinates and eqs. 22 and 23 coordinate differences. To obtain comparable test results, a value of $\alpha = 0.01$ is therefore used as significance level for eqs. 17 and 20 and $\alpha = 0.05$ for eqs. 22 and 23. The univariate tests are less sensitive to movements than the multivariate test so that in cases where the univariate tests show inconclusive results, the multivariate test gives the final answer. The likelihood ratio criterion shows more resolution in the region of acceptance while the trace criterion has more resolution in the region of rejection with hardly any overlap, so that both test statistics complement each other very well.

SUCCESSIVE TEST PROCEDURE

To detect the variable points in the set of control points, one starts with the assumption, that all control points are fixed points. All control points are therefore used to form the matrix $\boldsymbol{B}$, so that $\boldsymbol{S}$ in eq. 14 becomes the identity matrix and the coordinates are estimated by means of the pseudoinverse (eq. 10). Then the tests of eqs. 17 and 22 are applied for all control points by keeping in mind that only $r = 3k - (3k - q) = q$ coordinates may be introduced. By means of the significance level α it is then decided to accept or to reject the hypothesis.

In the case of rejection one has to remember, as mentioned in the previous chapter, that the hypotheses eqs. 17 and 22 for detecting fixed points have to include all m points used to form the matrix $\boldsymbol{B}$ in eq. 14 or more than m points. Hence, a minimal number m_{m} of fixed points is selected which show the smallest amount of movements in their coordinates estimated by the pseudoinverse. The number m_{m} is determined by $3m_{\mathrm{m}} \geqslant 3k - q$. These m_{m} points are used to form the matrix $\boldsymbol{B}$ and the coordinates of the control points are estimated by eq. 5. Then additional fixed points are determined by eqs. 17 and 22 and can be used as an alternative choice for the minimal number of fixed points, to confirm by eqs. 17 and 22 the first selection of fixed points.

The fixed points which are found are then used again to form the matrix $\boldsymbol{B}$, since all fixed points have to be used, as mentioned in a previous chapter. Again the coordinates of the control points are estimated and again additional fixed points are sought by the hypotheses eqs. 17 and 22. The sequence of points to be tested is selected according to the increasing average ratio of the movements between the epochs and their standard deviations computed from the estimates of one epoch. This successive estimation of coordinates with the tests for fixed points stops when all fixed points are used to form B and all hypotheses for additional fixed points have to be rejected. The successive procedure can be easily automated, so that in one analysis of the data the set of control points is separated into the set of fixed points and the set of variable points.

EXAMPLE

As an example test results shall be given for a control net which is shown in Fig. 1 and which was established to detect horizontal crustal movements. The distances between the control points vary between 2.5 km to 8.8 km. The net is observed by distances with standard deviations of about 1 cm and by bearings with standard deviations of about 0.0003 gon. To find out, whether small movements are detected by the successive test procedure, observations for three time epochs are generated with the standard deviations given above and by introducing movements for certain control points.

For the results shown in Fig. 1 the points 3 and 7 were moved by 2.5 cm between epoch 1 and 2 and between epoch 2 and 3. The movements are of the order of magnitude of the axes of the 95% confidence ellipses given in Fig. 1 for the estimated coordinates of the control points. The directions of the displacements almost agree with the directions of movements obtained by the estimated coordinates and shown in Fig. 1 by solid lines for the movements between epoch 1 and 2 and by dashed lines for movements between epoch 2 and 3.

TABLE I

Test results when using the points 2, 4, 6, 8, 9, 10 for the translation and rotation

Additional fixed points	Test of eq. 17				Test of eq. 20			
	Λ	1%p	T^2	1%p	T_1	T_2	T_3	1%p
—	0.67	0.49	0.44	0.82	0.92	0.77	1.39	2.73
5	0.53	0.45	0.75	0.97	1.30	1.49	1.35	2.57
1, 5	0.44	0.40	1.02	1.10	1.33	1.67	1.78	2.44
1, 5, 7	0.24	0.37	2.14	1.24	2.83	3.38	2.41	2.34
1, 3, 5, 7	0.09	0.34	4.86	1.57	6.19	6.63	4.58	2.28
	Test of eq. 22				Test of eq. 23			
	Λ	5%p	T^2	5%p	T_1	T_2	5%p	
—	0.87	0.65	0.14	0.49	0.46	0.64	2.05	
5	0.80	0.61	0.24	0.57	0.89	0.62	1.96	
1, 5	0.74	0.57	0.33	0.66	0.99	0.82	1.89	
1, 5, 7	0.56	0.54	0.69	0.74	2.01	1.10	1.83	
1, 3, 5, 7	0.32	0.50	1.57	0.83	3.94	2.10	1.80	

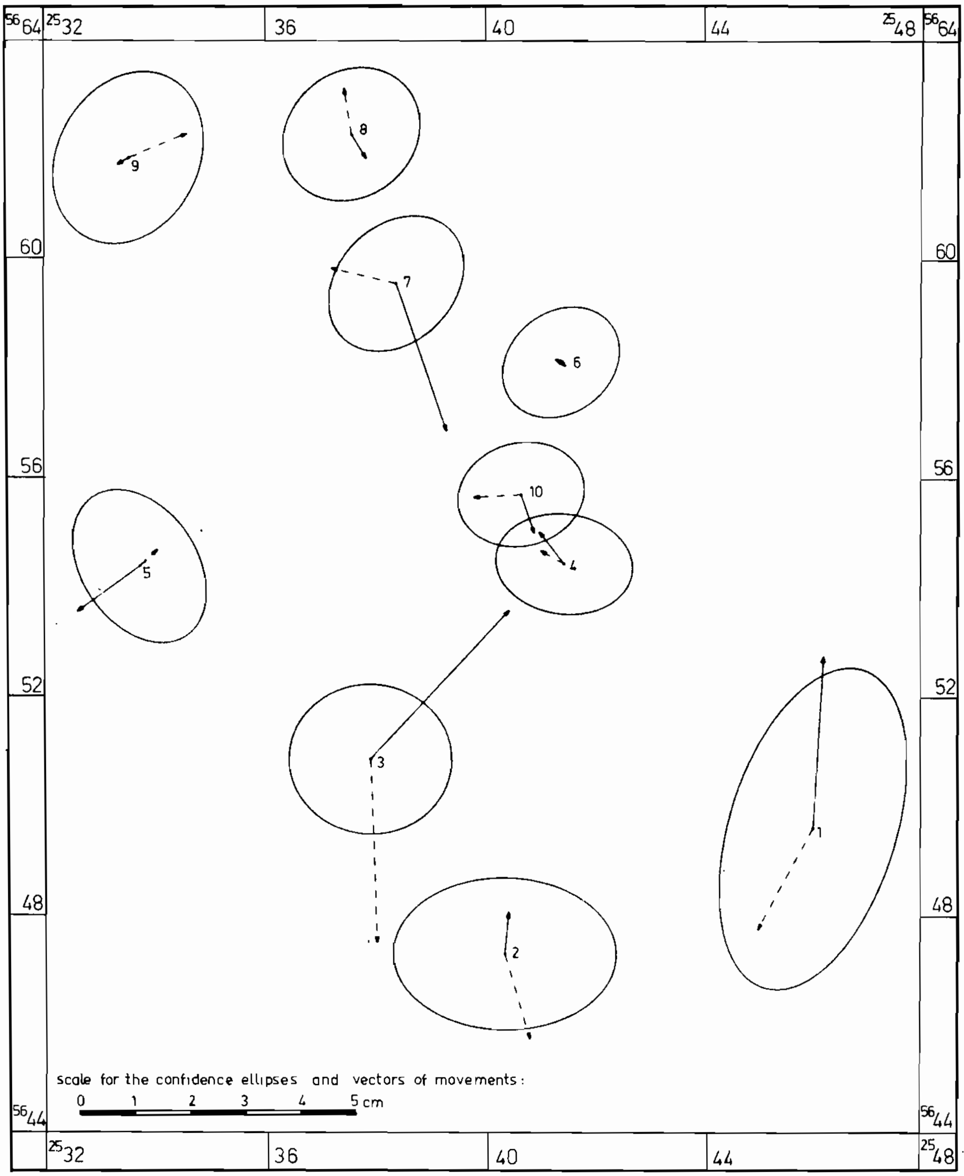

Fig. 1. Control net with 95% confidence ellipses and vectors of movements using the points 2, 4, 6, 8, 9, 10 for translation and rotation.

Using the pseudoinverse to estimate the coordinates of the control points, the hypotheses eqs. 17 and 22 that all control points are fixed points have to be rejected. The differences of the estimated coordinates indicate that points 4 and 6 did not move, so that they are used to translate and rotate the con-

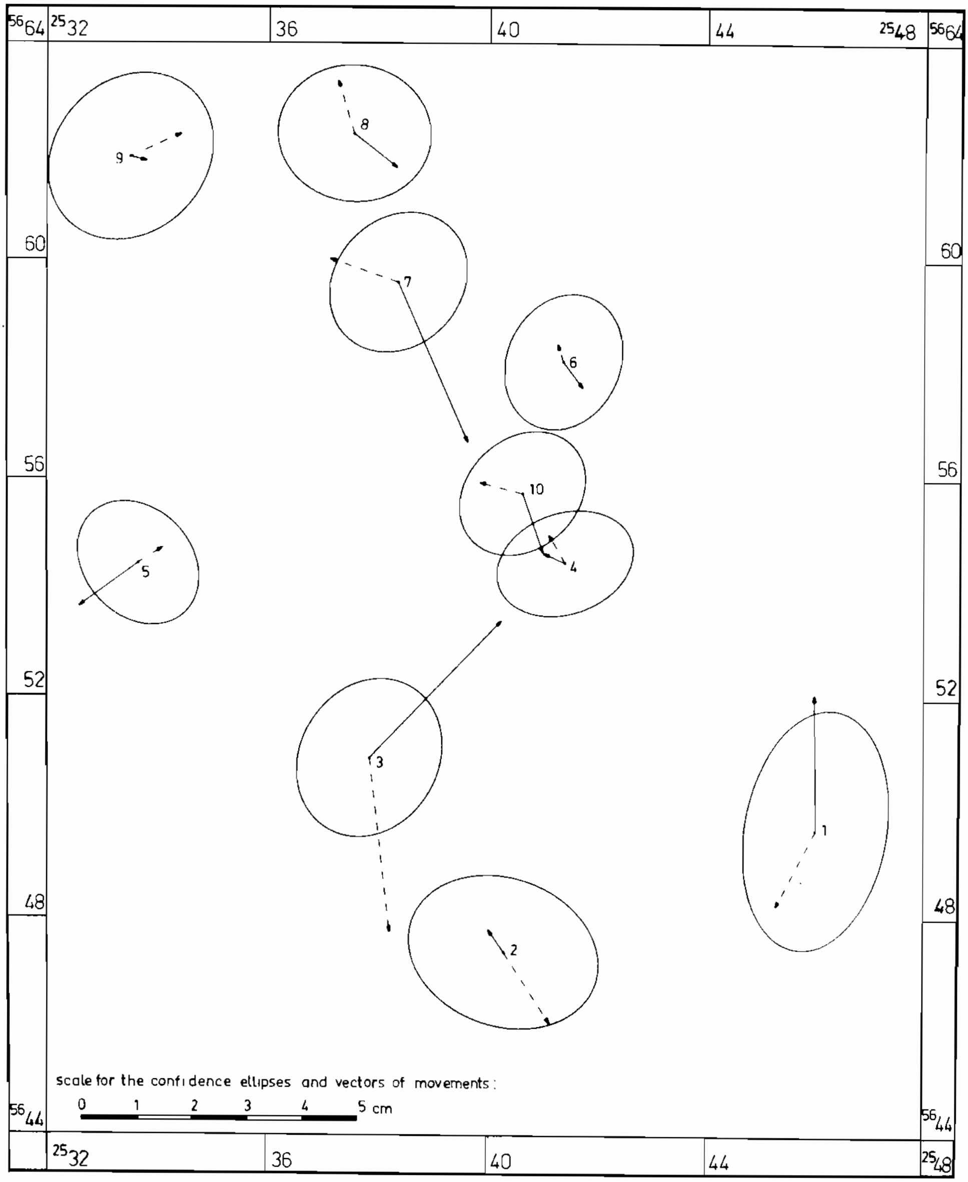

Fig. 2. Control net with 95% confidence ellipses and vectors of movements using the points 1, 2, 4, 5, 6, 8, 9, 10 for translation and rotation.

trol net, i.e. to form the matrix $\boldsymbol{B}$ in eq. 14. The tests of eqs. 17 and 22 confirm 4 and 6 as fixed points and indicate that in addition the points 2, 4, 6, 8, 9 and 10 did not move. These points are therefore used for the translation and rotation and the coordinates of the control points are estimated. The

TABLE II

Test results when using the points 1, 2, 4, 5, 6, 8, 9, 10 for the translation and rotation

Additional fixed points	Test of eq. 17				Test of eq. 20			
	Λ	1%p	T^2	1%p	T_1	T_2	T_3	1%p
—	0.40	0.40	1.16	1.10	1.73	1.49	2.39	2.44
7	0.22	0.37	2.34	1.24	3.34	3.32	2.93	2.34
3, 7	0.09	0.34	4.78	1.57	6.15	6.21	4.84	2.28
	Test of eq. 22				Test of eq. 23			
	Λ	5%p	T^2	5%p	T_1	T_2	5%p	
—	0.71	0.57	0.38	0.66	0.88	1.09	1.89	
7	0.53	0.54	0.76	0.74	1.97	1.34	1.83	
3, 7	0.32	0.50	1.55	0.83	3.69	2.22	1.80	

vectors of movements and the 95% confidence ellipses for the (x, y)-coordinates of the control points are shown in Fig. 1. By only judging from the amount of movements, the points 1, 3, and 7 could have moved. But the size of the 95% confidence ellipse for the point 1 indicates, that the movements of 1 might have been caused by the variances of the measurements. Indeed, despite the small amount of movements, which were introduced, the tests of eqs. 17 and 22 together with the univariate tests of eqs. 20 and 23 clearly indicate, that the points 3 and 7 are variable points. The test results are collected in Table I where the lower 1% points for eq. 18 and the upper 1% points for eqs. 19 and 21 when testing eqs. 17 and 20 and the lower 5% points for eq. 18 and the upper 5% points for eqs. 19 and 21 when testing eqs. 22 and 23 are given. The percentage points are taken from Kres (1975).

Finally the fixed points 1, 2, 4, 5, 6, 8, 9, 10 are used for translation and orientation and the tests for fixed points are applied. The results are given in Table II. They confirm, that the points 3 and 7 are variable points. Their movements between the two time epochs together with the movements of the fixed points are given in Fig. 2, where again the movements between epoch 2 and 3 are indicated by dashed lines. The movements shown in Fig. 2 are smaller than the ones in Fig. 1 which demonstrates that all fixed points have to be used to translate and rotate the control net.

CONCLUSION

The developed method of analysis uses univariate and multivariate tests to detect successively fixed points in a set of control points, which are then used to improve the translation, the rotation or the scale of the control net, so that the coordinates estimated for the fixed points become independent of the movements which have occurred. The successive procedure can be combined in one analysis of the data, and it is very well suited, as shown by the example, to detect horizontal movements in the order of magnitude of the lengths of the 95% confidence ellipses.

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