INTERNATIONAL SOCIETY FOR PHOTOGRAMMETRY AND REMOTE SENSING

Symposium '86 Commission III
Rovaniemi, Finland, August 19-22, 1986

THE MULTIGRID METHOD AND ITS APPLICATION IN PHOTOGRAMMETRY

Heinrich Ebner / Dieter Fritsch

Chair of Photogrammetry
Technical University Munich
Arcisstrasse 21, D-8000 Munich 2
Federal Republic of Germany

International Archives of Photogrammetry and Remote Sensing
Vol. XXVI, Part 3/3

THE MULTIGRID METHOD AND ITS APPLICATION IN PHOTOGRAMMETRY

Heinrich Ebner, Dieter Fritsch
Technical University Munich
Munich, F.R.G.

ABSTRACT

The multigrid method is a highly efficient iterative process for the solution of large sparse systems of linear equations. It has been developed during the last decade and is being more and more applied in numerical mathematics, physics and engineering sciences.

Under rather general conditions the rate of convergence of this iterative procedure is independent of the size of the problem. Consequently the computational effort for the solution increases only linearly with the number of unknowns.

This paper introduces the multigrid method and investigates its application to digital terrain model interpolation by finite elements. Further possible applications in photogrammetry are mentioned.

1. INTRODUCTION

The solution of large systems of linear equations has a long tradition in photogrammetry. Going back in history it began with the introduction of the analytical block triangulation (H.H. Schmid, 1956) and the digital terrain model (C.L. Miller / R.A. Laflamme, 1958). Since then there has been a flood of contributions dealing with both subjects.

For most of the procedures proposed the least squares method is applied to solve the adjustment in point determination as well as the approximation problem in digital terrain modelling. The resultant linear equation systems are frequently sparse and may even become banded.

Until now, the direct methods for solving the linear equation systems are used. Today, however, iterative procedures become very attractive too, especially for problems based on regular grid structures. The most advanced technique solving linear equations iteratively is the multigrid method, which in recent years has been applied for the solution of difference equations resulting from various problems in physics and engineering sciences (W. Hackbusch / U. Trottenberg, 1982). A first application of the multigrid method for the generation of digital terrain models has been given by U. Rüde (1985).

2. THE MULTIGRID METHOD - A BRIEF REVIEW

The multigrid principle for the solution of large linear equation systems for gridded unknowns is extremly simple: approximations with smooth errors are obtained efficiently by applying suitable relaxation methods. Because of the error smoothness, corrections to the approximation can be calculated on coarser grids. If this approach is used recursively employing coarser and coarser grids one obtains optimal iterative solutions, i.e. solutions for which the numerical expenditure required to achieve a fixed accuracy is proportional to the number of unknowns.

Let

$$N_h x_h = f_h \qquad (1)$$

be a linear equation system with the $h \times h$ coefficient matrix N_h, the $h \times 1$ vector of unknowns x_h and the $h \times 1$ right hand side f_h, all defined on a grid Ω_h.

Within the classical relaxation procedures the Jacobi method and the Gauss-Seidel method are of direct relevance to the multigrid method. A closer analysis of these procedures shows, that they converge fast for the high frequency part of the solution, but slowly for low frequent solution components having wavelengths much larger than the distance between two gridpoints. Because of this smoothing effect on the high frequency errors the classical relaxation procedures are called 'smoothers' speaking in the multigrid context. Let x_h^j be any approximation of the solution x_h of (1) and let us assume that x_h^j has been improved by ν_1 relaxation steps. Then the corrections of x_h^j can be denoted by

$$\Delta x_h^j := x_h - x_h^j \qquad (2)$$

and the defect (residuals) by

$$d_h^j := f_h - N_h x_h^j \qquad (3)$$

Accordingly, the equation (1) can be replaced by the defect equation

$$N_h \Delta x_h^j = d_h^j \qquad (4)$$

Because of the preceding smoothing the defect equation (4) contains mainly the low frequent solution components. Consequently, no fine grid is necessary for its approximation. If in (4) N_h is replaced by any 'simpler' operator $\hat{N}_h$ supposing that $\hat{N}_h^{-1}$ exists, the solution Δx_h^j resulting from

$$\hat{N}_h \Delta x_h^j = d_h^j \qquad (5)$$

provides for a new approximation

$$x_h^{j+1} = x_h^j + \Delta x_h^j \qquad (6)$$

The choice of $\hat{N}_h$, and this leads to the multigrid idea, is given by an approximation N_H of N_h on a coarser grid Ω_H. Therefore the defect equation (4) will be replaced by

$$N_H \Delta x_H^j = d_H^j \tag{7}$$

assuming that N_H^{-1} exists. As d_H^j and Δx_H^j are grid functions on a coarser grid Ω_H one needs linear transfer operators I_h^H and I_H^h between the grids Ω_h and Ω_H as follows

$$\begin{aligned} &\Omega_h \xrightarrow{I_h^H} \Omega_H \\ &\Omega_H \xrightarrow{I_H^h} \Omega_h \end{aligned} \tag{8}$$

The operator I_h^H is used to restrict the defect d_h^j to Ω_H

$$d_H^j := I_h^H \, d_h^j \tag{9}$$

and I_H^h is used to interpolate the correction Δx_H^j to Ω_h

$$\Delta x_h^j := I_H^h \, \Delta x_H^j \tag{10}$$

Because of the use of a coarse grid when solving for Δx_h^j this correction is called 'coarse grid correction' (CGC).

Summarizing the considerations above it is reasonable to combine the processes of classical relaxation and of coarse grid correction. Following this way one obtains an iterative (h,H) two-grid method, whereby each iteration step consists of smoothing and coarse grid correction (see Fig.1)

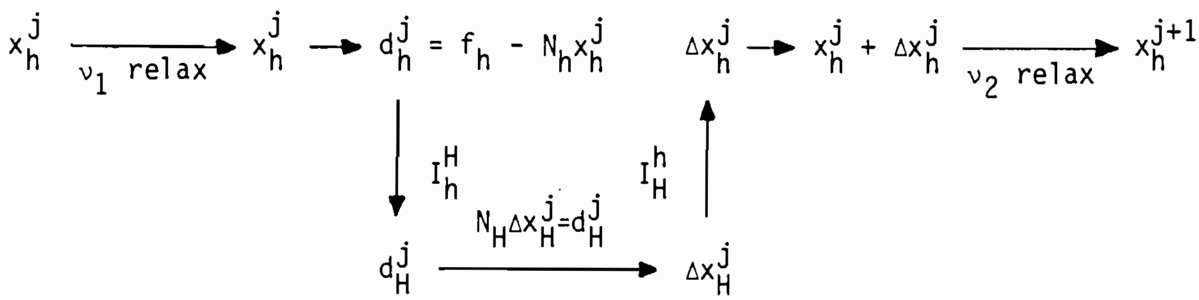

Fig.1: Structure of a two-grid method (h,H)

The generalization of the two-grid method into the multigrid principle is now reasonable: it is not necessary to solve the coarse grid defect equation (7) exactly but it may be replaced by a suitable approximation.

A quite natural way obtaining the approximation is to apply a further two-grid method to (7), where an even coarser grid than Ω_H is used. This idea can be applied recursively using coarser and coarser grids down to the coarsest grid, where the defect equation is solved directly or iteratively. Thus a hierarchy of grids is constructed. In standard applications the meshsizes Δ, 2Δ, 4Δ, 8Δ, etc. are used, so that the grids are nested. On the coarsest grid the problem has only very few unknowns; the effort for its solution by any method is therefore negligible.

An illustration of different options to build up one multigrid iteration cycle j is given in Fig. 2.

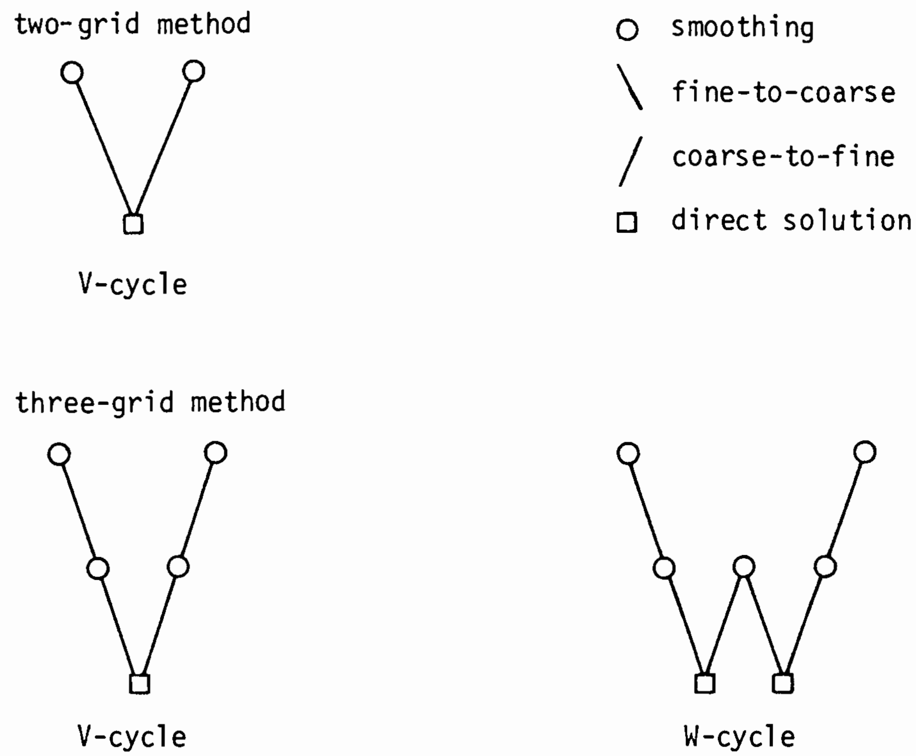

Fig. 2: Structure of multigrid cycles for two and three grids

As operators I_h^H for fine-to-coarse transfer of the defect and I_H^h for coarse-to-fine transfer of the correction the fullweighting (FW) operator

$$I_h^H := \frac{1}{16} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}_h^H \qquad (11)$$

and the bilinear interpolation (BI) operator

$$I_H^h := \frac{1}{4} \left] \begin{matrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{matrix} \right[_H^h \qquad (12)$$

are mainly used. Other operators are given by K. Stüben / U. Trottenberg (1982).

3. MULTIGRID APPLICATIONS IN DIGITAL TERRAIN MODELLING

For demonstration of the multigrid method the following example is treated: interpolate a 33x33 gridded digital terrain model by the finite element approach (H. Ebner, 1983) from a synthetic data set consisting of 9x9 reference points (see Fig. 3a)

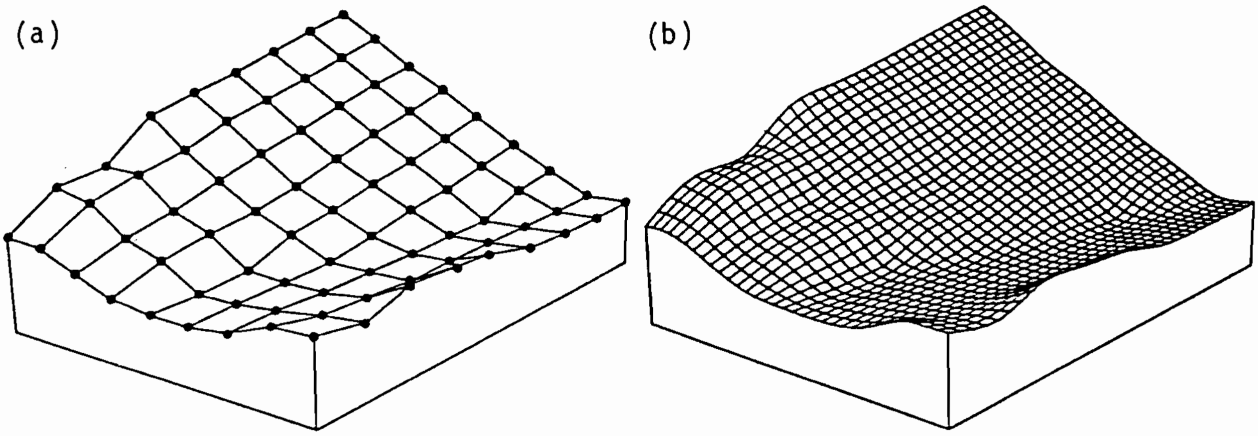

Fig.3: The data set (a) and the interpolated digital terrain model (b)

For DTM interpolation the objective function

$$p \sum_{k=1}^{81} \hat{v}_k^2 + \sum_{i=2}^{32} \sum_{j=1}^{33} \hat{v}_{xx,i,j}^2 + \sum_{i=1}^{33} \sum_{j=2}^{32} \hat{v}_{yy,i,j}^2 = \min \qquad (13)$$

with

$$\hat{v}_k = \hat{z}_{i,j} - z_k \qquad (14)$$

$$\hat{v}_{xx,i,j} = \hat{z}_{i-1,j} - 2\hat{z}_{i,j} + \hat{z}_{i+1,j} \qquad (15)$$

$$\hat{v}_{yy,i,j} = \hat{z}_{i,j-1} - 2\hat{z}_{i,j} + \hat{z}_{i,j+1} \qquad (16)$$

is used. The $\hat{v}_k$ are residuals at the 81 reference points, defined as unknown heights $\hat{z}_{i,j}$ of grid points $P_{i,j}$ minus observed heights z_k.

The $\hat{v}_{xx,i,j}$ and $\hat{v}_{yy,i,j}$ are second differences of the unknown heights of adjacent grid points in x and y direction.

In our example constant weight p=100 was assumed for all reference heights z_k. Equations (13) to (16) lead to a normal equation system (1) with h=33x33=1089 unknowns which has been solved directly (see Fig. 3b) for comparison with the following multigrid solution.

The multigrid method is demonstrated via V-cycles (see Fig. 2) with initial values resulting from bilinear interpolation of the reference points. The grids used were a 33x33, 17x17, 9x9, 5x5 and last a 3x3 grid where the normal equations have been solved directly.

Within one V-cycle the following steps were performed:

(i) smoothing on grid 33x33 with 3 Gauss-Seidel relaxations

(ii) coarse grid correction: recursive restriction of the defect and 3 Gauss-Seidel relaxations at each grid, direct solution of the normal equations at the 3x3 grid, recursive interpolation of the correction and 1 Gauss-Seidel relaxation at each grid

(iii) smoothing on grid 33x33 with 1 Gauss-Seidel relaxation

Table 1 describes the convergence of the multigrid solution. For all V-cycles and their individual steps the maximum value Δmax and the RMS value Δmean of all 33x33 deviations between the direct solution and the respective multigrid solution are listed.

Table 1: Convergence of the multigrid solution (Dim.:[m])

V-cycle	Operation	Δmax	Δmean
0	bilinear Interpol.	3,082	0,660
1	RELAX 3	2,240	0,491
	CGC	0,672	0,140
	RELAX 1	0,353	0,103
2	RELAX 3	0,241	0,073
	CGC	0,096	0,024
	RELAX 1	0,082	0,019
3	RELAX 3	0,043	0,014
	CGC	0,022	0,005
	RELAX 1	0,018	0,004
4	RELAX 3	0,010	0,003
	CGC	0,002	0,001
	RELAX 1	0,001	0,001

Fig. 4 - 9 demonstrate the co-operation of smoothing and coarse grid correction by means of the deviations between the direct solution and the multigrid solutions.

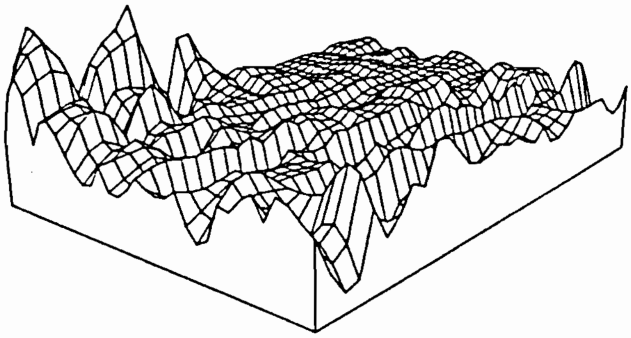

Fig. 4: Deviations between the direct solution and the initial values

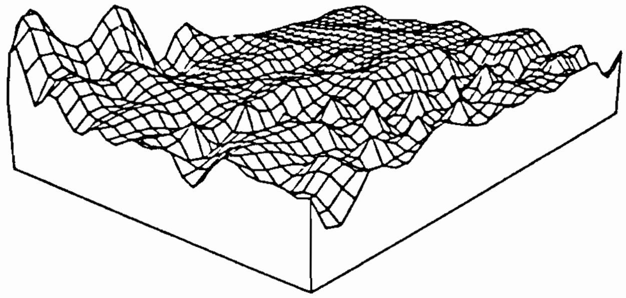

Fig. 5: Deviations after three relaxations

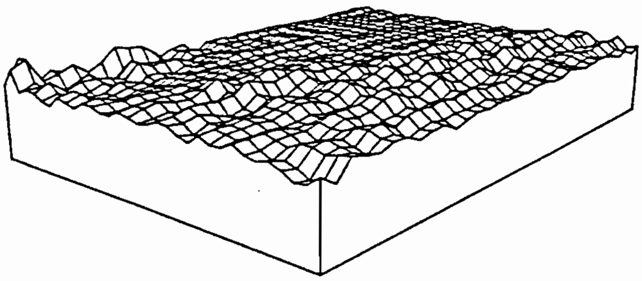

Fig. 6: Deviations after the first coarse grid correction

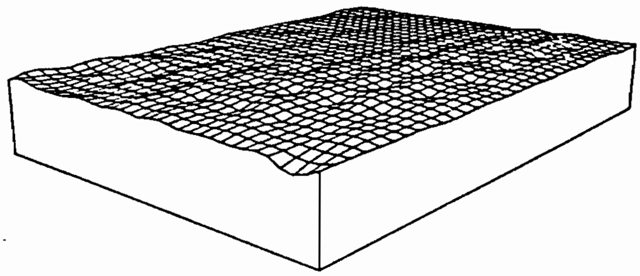

Fig. 7: Deviations after the first V-cycle and three relaxations

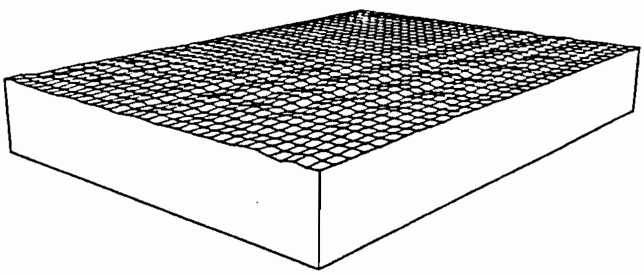

Fig. 8: Deviations after the second coarse grid correction

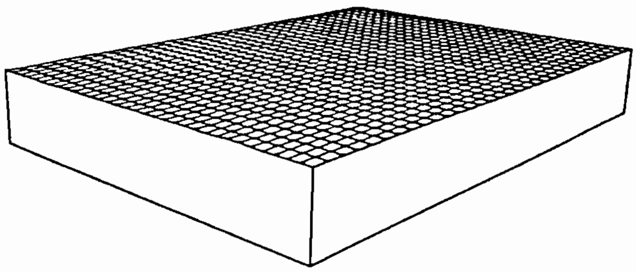

Fig. 9: Deviations after the second V-cycle and three relaxations

About 318 000 multiplications have to be executed for set-up and processing of the linear equation systems within all four V-cycles. This means about 290 multiplications per grid point contrary to about 2 200 operations in the direct solution by band algorithms. The demonstrated high efficiency of the multigrid method in terms of reduction of the computational work gives cause for some optimism for its broadening in photogrammetry.

4. CONCLUSIONS

It has been shown that iterative solution strategies nowadays become very attractive especially for large linear systems based on grid structures. At first view the multigrid method is associated with applications in digital terrain modelling and digital image processing. A closer analysis of the solution behaviour by means of local fourier analysis and two-dimensional system design (D. Fritsch, 1984, 1985) gives reason to believe in even more general applications.

Future investigations have to be made on the design of restrictors and interpolators for the improvement of coarse grid corrections. Another point of interest is the improvement of smoothing and the transfer of coefficient matrices between different grids.

Digital terrain modelling seems to be a broad application field for the multigrid method. On the one hand in data processing the areas of interest have no longer to be segmented. In that way, consistency problems and redundant processing are avoided. On the other hand it gives promise to very fast DTM interpolations. This will be needed in association with on-line DTM generation and verification (W. Reinhardt, 1986).

5. ACKNOWLEDGEMENTS

The authors are indebted to Prof. Ch. Zenger and U. Rüde for helpful hints and discussions on multigrid methods. Further thanks are given to R. Hössler and G. Düsedau for their assistance.

REFERENCES

EBNER, H. (1983): Berücksichtigung der lokalen Geländeform bei der Höheninterpolation mit finiten Elementen. Bildmess., Luftbildwes., 51: pp. 3-9.

FRITSCH, D. (1984): Two-Dimensional Finite Impulse Response (FIR) Linear Systems in Digital Photogrammetry. Arch. A3a, XV. Int. Congr. Photogr. Rem. Sens., Rio de Janeiro.

FRITSCH, D. (1985): Some Remarks on Duality between Fourier Series and Discrete Fourier Transforms. Allgem. Verm. Nachr., Int. Supplement, 2: pp. 41-47.

HACKBUSCH, W. / U. TROTTENBERG (1982): Multigrid Methods. Springer, Berlin, 650 p.

MILLER, C.L. / R.A. LAFLAMME (1958): The Digital Terrain Model - Theory and Application. Phot. Eng. Rem. Sens., 24: pp. 433-442.

REINHARDT, W. (1986): Optical Superimposition of Stereomodel and Graphical Information as a Tool for DEM Quality Control. Proceed. ISPRS Com. IV, pp. 207-215, Edinburgh.

RÜDE, U. (1985): Anwendungen der Mehrgittermethode zur Berechnung von digitalen Höhenmodellen in der Photogrammetrie. Inst. Informatik, Techn. Univ. München, TUM-I 8525.

SCHMID, H.H. (1956): An Analytical Treatment of the Problem of Triangulation by Stereophotogrammetry. Photogrammetria, 13: pp. 67-77, 91-116.

STÜBEN, K. / U. TROTTENBERG (1982): Multigrid Methods: Fundamental Algorithms, Model Problem Analysis and Applications. In: Multigrid Methods, Ed.W. Hackbusch / U. Trottenberg, Springer, Berlin, 650 p.