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ON ALGORITHMS SOLVING THE L_∞ - APPROXIMATION IN GEOMETRIC MODELLING

by

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ABSTRACT

The progress in digital photogrammetry might also call for powerful algorithms solving problems such as system design for data preprocessing and most recently digital object reconstruction. Though the method of least-squares serves as an efficient tool for these tasks, there are some cases in which it is insufficient because of its smoothing effect. The alternative is given by a generalization of least-squares to provide for a Chebyshev formulation (L_∞-formulation) of the problem.

For that reason the method of inequality constrained least-squares is introduced. The algorithms used to solve L_∞-approximation problems are the REMEZ algorithm and the linear complementarity algorithm. Examples from digital and analytical photogrammetry demonstrate the advantages of these algorithms.

1. Introduction

Digital photogrammetry nowadays is far more than only an extension of analytical photogrammetric methods: entirely new concepts of disciplines like electronic engineering, digital signal processing, computer science and statistics are to take into account. This implicates on the one hand a re-orientation of photogrammetry, but on the other hand it serves as basis for new application fields for instance in robotics and computer vision. As far as the algorithmization is concerned, there are two premises on it: first it should be fast and secondly it should be reliable, that means to take care for exact geometric modelling and object reconstruction. A look at the data flow in digital photogrammetry (see Fig. 1) indicates, that powerful algorithms have to be used in data preprocessing and data analysis.

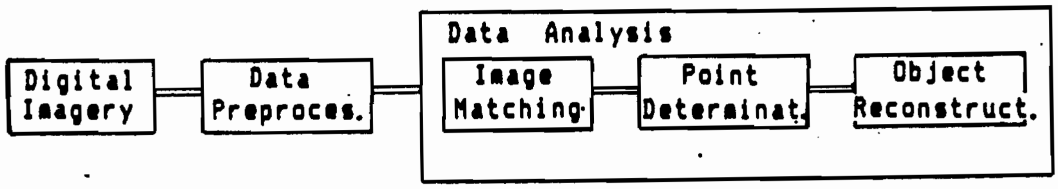

Fig. 1: Data flow in digital photogrammetry

The digital imagery process serves as digitizer of continous objects and delivers the data for the second main step in digital photogrammetry: it includes methods of noise reduction, contrast improvement, filtering, image segmentation and so on. This data preprocessing has primarily to maintain the geometry of the digital image, what is very important and demands for so-called "system tuning". That means, the system or algorithm applied to the data has to be designed properly.

The other main step of digital photogrammetry is the final data analysis, what is standing for image matching, point determination and object reconstruction. Adjustments, approximations and perhaps some statistical assertions are the tools for these tasks.

The algorithmization commonly used to solve most of the problems above is based on the method of least-squares introduced by C.F. Gauss in 1793 (H. Wussing, 1982). Most of the modern statistical inference is derived from that method and there is no reason to refuse this powerful approach. Some advantages are given in the following:

(i) there is a simple mathematical formulation behind it

(ii) efficient solution strategies such as sparse matrix algorithms and most recently multigrid methods can be used

(iii) one obtains error measures for the goodness-of-fit.

But there is also a great disadvantage of the method of least-squares: it does not provide for the worst case, but smoothes the observation errors by the minimization of

$$\min_x \sum_{i=1}^{n} p_i e_i^2 = \min_x \| e \|_P^2 =: \min_x L_2 \text{ norm} \qquad (1)$$

with regard to the unknown parameters x. For that reason it is not suited for the approximation of sharp-edged linear systems such as filters etc. as well as object reconstruction, in which it is desired to place a bound on the errors.

The way out from this weakness of least-squares is given by the Chebyshev formulation (L_∞-formulation) of the problem, which minimizes the maximum error

$$\min_x \max_{1 \le i \le n} | e_i |_{p_i} = \min_x \| e \|_P^\infty =: \min_x L_\infty \text{ norm} \qquad (2)$$

and thus takes care for the worst case. But the disadvantage of this objective function in the past was the lack of algorithms; most of the problems had to be solved by linear programming techniques. Further developments in numerical mathematics and the discipline of operation research changed the situation. Today there are powerful algorithms available for instance the REMEZ algorithm and the linear complementarity algorithm being shortly commented on in the following.

2. The REMEZ Algorithm

In one-dimensional applications such as the design of edge operators and filters, respectively, the REMEZ algorithm provides for fast and accurate computation of the impulse response. Very soon after the development of this algorithm, the IEEE society used it for the design of selective systems (filters, differentiators and Hilbert transformers) in digital speech, signal and image processing.

Thus, T.W. Parks/J.H. McClellan (1972) gave already a comprehensive program package coded in FORTRAN for the design of linear phase finite impulse response digital filters, which can also be found in L.R. Rabiner/B. Gold (1975). This program package is nowadays still the standard in digital data preprocessing; therefore digital photogrammetry should have some profit of it, too.

An extension of this package in such a way, that it includes also Wiener filtering for noise elimination is given by E.U. Fischer/H. Friedsam (1977).

Let be

$$e(x) = p(x)(f(x) - h(x)) \qquad (3)$$

a continous approximation problem with e(x) as error function, p(x) is any weight function, f(x) is the ideal to be approximated and h(x) the polynomial of the approximation; all functions are dependent of the one-dimensional variable x. The REMEZ algorithm is an iterative procedure to determine the polynomial coefficients h(x) numerically with regard to (2). A complete description of the algorithm is given by H. Rutishauser (1976), thus only its basis will be reviewed here:

(i) Let h(x) be the polynomial of the optimum approximation with degree n for the continous function f(x) defined within the interval I. Then there exist at least n+2 samples $x_0>x_1>x_2>....>x_{n+1}$, where the error function e(x) takes on its extrem values $\pm\|e\|_p^\infty$ alternately. That means there is an $e(x_k)=(-1)^k h\ \forall k=0,1,2,....,n+1$ with $h=\pm\|e\|_p^\infty$. These n+2 samples are called 'alternant' of f(x). Though p(x) is clearly given, it might exist several alternants.

(ii) The set of n+2 samples $x_0>x_1>x_2>....>x_{n+1}$ is called 'reference' and its belonging polynomial r(x) with $r(x_k)-f(x_k)=(-1)^k h$ is called 'reference polynomial'. In this context it is to be said that $|h|$ is the 'reference deviation'.

(iii) By f(x) and the reference $x_0>x_1>x_2>....>x_{n+1}$ the reference polynomial r(x) and h are clearly determined.

The algorithm starts with an arbitrary choice of the reference and computes the reference deviation with regard to the polynomial r(x), which might be a trigonometric or natural polynomial. Then the error function e(x) can be obtained, whose extrem values are generally greater than the reference deviation. Thus one has to replace old reference points x_k by new references $\bar{x}_k$, what results into an increase of $\bar{h}$ contrary to h. During this exchange it should be guaranteed, that the sign of $e(x_k)$ changes alternately. This procedure is called 'multiple REMEZ exchange algorithm'. As result of such an iterative process a reference polynomial is obtained, whose reference deviation corresponds with the maximum error of the error function.

The REMEZ algorithm is very fast and therefore well-suited to solve one-dimensional approximation and optimization problems. But its main disadvantage is, that it cannot be extented on higher dimensions.

3. Inequality Constrained Least-Squares

Another way to solve L_∞-approximations is given by the method of inequality constrained least-squares (ICLS), which has been developed within the discipline of operation research (C.K. Liew, 1976) and introduced into geodesy by B. Schaffrin (1981). The well-known equality constrained least-squares estimation is a special case of the ICLS estimation. But the ICLS method generalizes ordinary least-squares in such a way, that upper or lower bounds or both, are introduced which is quite practical. A first experience with ICLS-approximations in geometric modelling is given by D. Fritsch/ B. Schaffrin (1980), in which a comparison between the REMEZ algorithm and an ICLS algorithmization has been carried out. The generalization of L_∞-formulations on two-dimensional approximation problems is exhibited in D. Fritsch (1982), so that it will be reviewed in the following for higher dimensional applications.

Let be

$$\boxed{\begin{array}{l} y + e = Ax \quad , \quad P \text{ pos. def.} \\ \{y,e\} \in \mathbb{R}^n \quad , \quad x \in \mathbb{R}^m \quad , \quad rkA = m \quad , \quad m < n \end{array}} \qquad (4)$$

a linear model, in which y is the vector of the discrete function(s) to be approximated, e its belonging error or inconsistency vector, both with regard to a given positive definite weight marix P. The right hand side is represented by the known coefficient matrix A and the vector x of unknown parameters. An ICLS formulation starts from (4) and introduces linear inequalities $Hx \geq c$, so that (4) is generalized into

$$\boxed{\begin{array}{l} y + e = Ax \quad , \quad P \text{ pos. def.} \\ \text{subject to} \\ Hx \geq c \\ \{y,e\} \in \mathbb{R}^n \quad , \quad x \in \mathbb{R}^m \quad , \quad c \in \mathbb{R}^r \quad , \quad r < m < n \\ rkA = m \; , \; rkH = r \end{array}} \qquad (5)$$

its solution can be obtained by the minimization of the objective function

$$\min_x \phi(x) = 2x'A'PAx - 2x'A'Py + y'Py$$
$$\text{subject to} \qquad (6)$$
$$Hx \geq c$$

With the introduction of the slack variables $v := |v_1, v_2, \ldots, v_r|'$ the inequalities may be transformed into equalities

$$Hx - v = c \qquad (7)$$

which will be considered by means of Lagrangian multipliers in

$$\min_{x} L(x,\lambda,v) = 2x'A'PAx-2x'A'Py+y'Py-2\lambda'(Hx-v-c) \quad (8)$$

Setting the partial derivatives of the Lagrangian fuction to zero results into

$$\frac{\partial L}{\partial x} = 2A'PA\hat{x} - 2A'Py - 2H'\hat{\lambda} = 0 \quad (9a)$$

$$\frac{\partial L}{\partial \lambda} = -2(H\hat{x} - \hat{v} - c) = 0 \quad (9b)$$

$$\frac{\partial L}{\partial v} = -4\hat{\lambda}_i \hat{v}_i = 0 \qquad \forall i=1,2,\ldots,r \quad (9c)$$

This equation system is necessary but not sufficient to solve ICLS problems. For that reason the linear inequalities have to be parameterized such that

$$Hx \geq c + ak \quad , \quad a\in \mathbb{R} \quad , \quad ak \geq 0 \quad (10)$$

which leads to further equations (D. Fritsch, 1985)

$$\left.\frac{\partial L}{\partial(ak)}\right|_{a=0} = 2\lambda_i \geq 0 \qquad \forall i=1,2,\ldots,r \quad (11)$$

System (9) and (11) correspond now with the famous Kuhn-Tucker conditions in quadratic programming. The ICLS solution x is given by solving (9)

$$\hat{x} = (A'PA)^{-1} A'Py + (A'PA)^{-1} H'\hat{\lambda} \quad (12a)$$

subject to

$$H\hat{x} - \hat{v} = c \quad , \quad \hat{v}'\hat{\lambda} = 0 \quad , \quad \hat{v},\hat{\lambda} \geq 0 \quad (12b)$$

Another formulation of (12) can be obtained by

$$\boxed{\begin{array}{l}\text{find } \hat{v},\hat{\lambda} \geq 0 \\ \text{such that } \hat{v} = M\hat{\lambda} + q \\ \text{and } \hat{v}'\hat{\lambda} = 0\end{array}} \tag{13}$$

with substitutions

$$M := H(A'PA)^{-1}H' \quad , \quad q := H\tilde{x} - c \tag{14}$$

$$\tilde{x} := (A'PA)^{-1}A'Py \quad \text{"simple least-squares solution"}$$

But how can the method of ICLS be used to solve L_∞-approximations? This question is now simply reduced to the formulation of linear inequalities. Let be ε the maximum error of the L_∞-formulation

$$\varepsilon = \min_x L_\infty \text{ norm} \tag{15}$$

thus the following inequalities will provide for the Chebyshev solution

$$\boxed{\begin{array}{l}Ax \geq y - \varepsilon \\ Ax \leq y + \varepsilon\end{array} = \begin{bmatrix} A \\ -A \end{bmatrix} x \geq \begin{bmatrix} y - \varepsilon \\ -y - \varepsilon \end{bmatrix}} \tag{16}$$

That means, the solution space has been reduced to a band with bandwidth ε (see Fig. 2). Naturally, there is no a priori knowledge on the worst case ε, thus it has to be found out during the solution process. This implicates once more iterations, because also the ICLS solution is based on an iterative procedure.

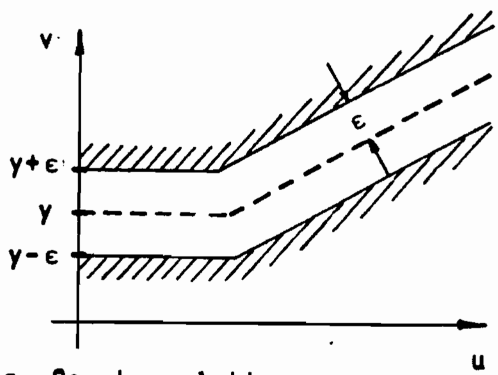

Fig. 2: L_∞-solution space

A reasonable approach to get L_∞-approximations by means of the method of ICLS starts with a first Chebyshev number (initial value) ε_1

$$\boxed{\begin{array}{l} \text{1st step:} \quad \varepsilon_1 = \alpha\, e_0, \quad e_0 = \max\limits_{1 \le i \le n} |\tilde{e}_i| \\ \text{subject to} \\ 0.6 \le \alpha \le 0.9 \end{array}} \qquad (17)$$

in which e_0 is the maximum least-squares error or residual. The second step considers the initial value ε_1 and the first maximum ICLS residual $e_1 = \max\limits_{1 \le i \le n} |\hat{e}_i|$

$$\text{2nd step:} \quad \varepsilon_2 = \frac{1}{2}\,(\varepsilon_1 + e_1) \qquad (18)$$

so that the i-th step looks as follows

$$\boxed{\begin{array}{lll} \text{compute} & \varepsilon_i = \frac{1}{2}\,(\varepsilon_{i-1} + e_{i-1}) & \\ \text{if} & \varepsilon_i = \varepsilon_{i-1} + \delta \;, & \delta \ll 1 \\ \multicolumn{3}{l}{\text{then the iterations are finished}} \end{array}} \qquad (19)$$

with δ as inconsistency number depending on the accuracy level desired.

4. The Linear Complementarity Algorithm

The algorithms used to solve ICLS approximation and estimation problems are based on linear complementary pivot theory (C.Z. Lemke, 1968, R.W. Cottle / G.B. Dantzig, 1968). A detailed description of the algorithm according to Lemke is given by D. Fritsch (1985), but there is also a master thesis on a comparison of the Lemke and the Cottle-Dantzig algorithm (Ch. Heipke, 1986). Both algorithms solve (13) by a finite number of iterations, if the problem has a complementary solution. Because of the complementarity of the slack variables v_i and the Lagrangian multipliers λ_i $\forall i=1,2,\dots,r$, (13) is called 'linear complementarity problem (LCP)'.

For the application of LCP algorithms (13) is rewritten to

$$\boxed{\begin{array}{l} \text{find } \hat{v}, \hat{\lambda} \geq 0 \\ \text{such that } \hat{v} = M\hat{\lambda} + ez_0 + q \\ \text{and } \hat{v}'\hat{\lambda} = 0 \,, \quad z_0 \in R \,, \quad z_0 \geq 0 \end{array}} \qquad (20)$$

whereby the artificial variable z_0 has to be minimized; the vector e is defined by $e := [\ 1,1,\ldots,1\ ]'$. The algorithms prove at first whether the vector $q \geq 0$, because this is the trivial solution. Otherwise it continues with a modified Simplex method and results in the complementarity vectors v and λ, if there is a complementary solution.

A first comparison of the Lemke and the Cottle-Dantzig algorithm (Ch. Heipke, 1986) showed, that the former procedure is more appropriate than the latter one with regard to the following features

(i) indication of the existence of a complementary solution

(ii) robustness against roundoff errors

(iii) computation time

(iv) handling of the algorithm

FORTRAN-coded versions of the Lemke algorithm are given by A. Ravindran (1972), supplemented by L.G. Proll (1974), as well as a slightly changed version coded by C.K. Liew/J.K. Shim (1978).

5. Applications

A first application of the algorithms above in digital photogrammetry has been carried out in a comparison of classical edge operators (Sobel etc.) and differentiators designed by the REMEZ algorithm. Starting with the simplest differentiator templates (see Fig. 3)

$$\begin{bmatrix} 0 & -1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Fig. 3: Simple differentiator templates

the aim was to study edge detection of noisy digital images. Although this differentiator and also the Sobel operator worked very well in a synthetic unblurred image, they did not in the blurred case. For that reason differentiators of length 5 and 7 have been designed (see Fig. 4), which gave a better image gradient for further processing.

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                        FINITE IMPULSE RESPONSE (FIR)
                        LINEAR PHASE DIGITAL FILTER DESIGN
                        REMEZ EXCHANGE ALGORITHM

                        DIFFERENTIATOR

              FILTER LENGTH =    5

              *****IMPULSE RESPONSE*****
                   H(  1) =   -.10420300E+00 = -H(   5)
                   H(  2) =    .22122930E+00 = -H(   4)
                   H(  3) = 0.0

                        BAND  1          BAND
 LOWER BAND EDGE      .000000000
 UPPER BAND EDGE      .500000000
 DESIRED SLOPE       1.000000000
 WEIGTHING           1.000000000
 DEVIATION            .826525893

 EXTREMAL FREQUENCIES
     .0156250     .2812500     .4843750

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Fig. 4: Impulse response of a differentiator of length 5

A second application is dealing with two-dimensional system design, which might also be used in the preprocessing step of digital photogrammetry: design a lowpass filter of circular symmetry and of length 15x15 for the passband frequency $w_p = 0.15\pi$ as well as the stopband frequency $w_s = 0.32\pi$. The formulation of the approximation problem is given by D. Fritsch (1982, 1983); in the meantime a comprehensive program package has been developed for the design of digital amplifiers and filters to improve the contrast of the digital image or to remove nuisance frequencies. In this program package the Lemke algorithm serves as "system tuner" that means to provide for the minimum maximum error.

A simple least-squares approximation using 100x100 samples within the frequency domain results into a maximum deviation of $e_0 = 0.07$ at the edges of the passband and stopband, respectively (see Fig. 5). This means for the convolution, that every pixel will be falsified about 7%, a number which cannot be accepted in precise photogrammetric applications. The addition of 74 inequalities formulated at the pass- and stopband edges only reduces this number to $\varepsilon_p = \varepsilon_s = 0.04$ and leads to an optimum error behaviour in the pass- as well as a suboptimal one in the stopband (see Fig. 5).

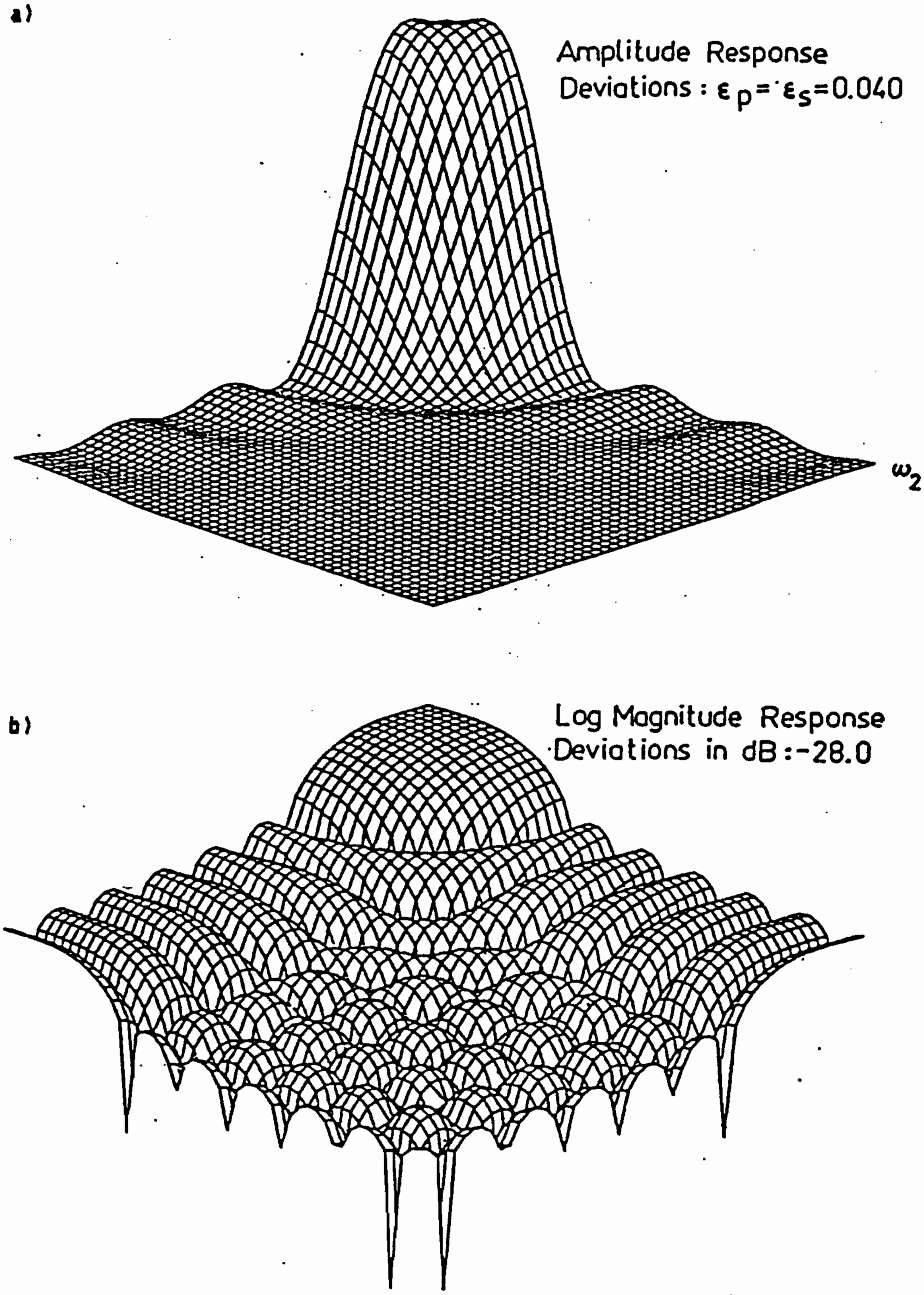

Fig. 5: Frequency response of the "tuned" lowpass filter
a) linear scale b) log scale

The last example applies the Lemke algorithm in highly precise quality control. As object serves a subreflector, which is arranged at the focus of the main parabolic antenna to achieve parallel electronic waves. Because the antennas are nowadays working within the [GHz]-band, the shape of a subreflector has to be very smooth with deviations of about 0.1-0.3 (mm). The subreflector looks as follows (see Fig. 6): it has a diameter of about 2 (m)

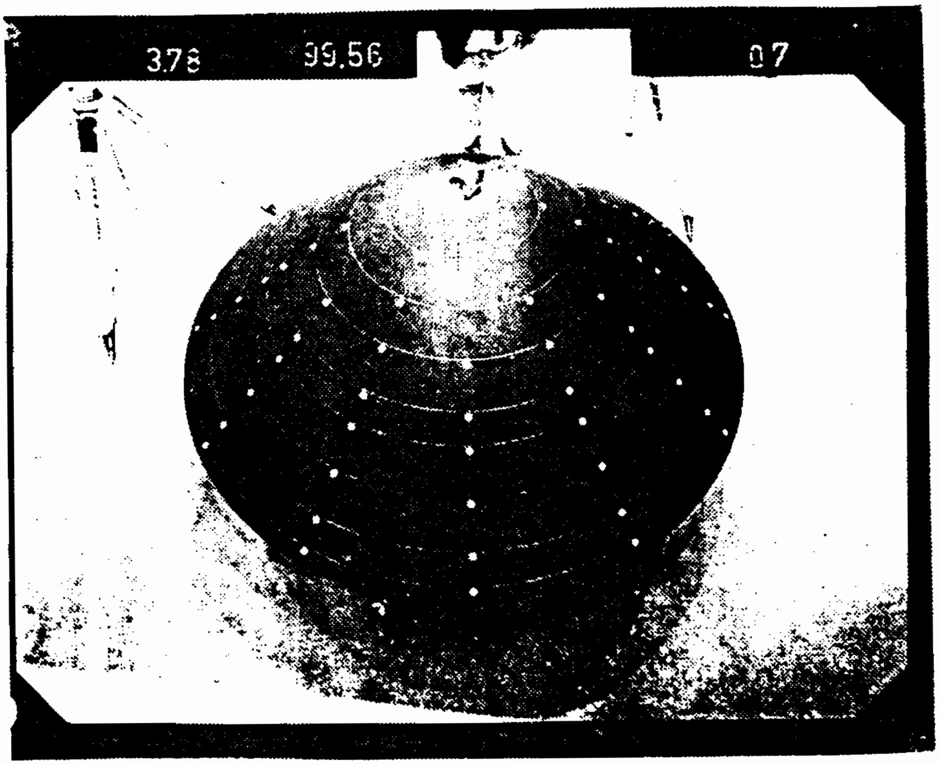

Fig. 6: Subreflector of a parabolic antenna

and a height of about 0.65 (m). 90 targets represent the samples of its surface, which have been measured geodetically and photogrammetrically (D. Fritsch et al., 1984). As results of the bundle block adjustments were obtained $\hat{\sigma}_x$ =0.053 (mm), $\hat{\sigma}_y$ =0.054 (mm) and $\hat{\sigma}_z$ =0.057 (mm), this accuracy was also in accordance with the coordinates obtained by geodetic methods.

For the reconstruction of the subreflector a shaped circular symmetry paraboloid has been used for a best-fit computation. The maximum deviation after the least-squares fit was e_0 =0.35 (mm), which was not suited for the specifications. A change of the objective function by means of 24 inequalities for the outer ring reduced this number to ϵ =0.25 (mm). By the way, providing for the worst case is the optimum evaluation for the manufacturer.

6. Conclusions

The use of algorithms to solve L_∞-approximations is not only justified in geodetic applications but also in photogrammetry. Especially the method of inequality constrained least-squares is a powerful tool in geometric modelling. As far as digital photogrammetry is concerned, it might be used in data preprocessing and data analysis. But it can also solve photogrammetric optimization problems and further applications, which could not be discussed here.

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