INTERNATIONAL SOCIETY FOR PHOTOGRAMMETRY AND REMOTE SENSING
WORKING GROUP III/3 - DIGITAL TERRAIN MODELS

PROCEEDINGS OF
THE INTERNATIONAL COLLOQUIUM

PROGRESS IN TERRAIN MODELLING

TECHNICAL UNIVERSITY OF DENMARK

20 - 22 MAY 1987

Edited by

O.JACOBI
P.FREDERIKSEN

ON THE USE OF CURVATURE MEASURES IN DIGITAL TERRAIN MODELLING

by

Dieter Fritsch, Gabriele Düsedau

Chair of Photogrammetry
Technical University Munich
Arcisstraße 21
D-8000 Munich 2
Fed. Rep. of Germany

Abstract

During the last decade different procedures in digital terrain modelling have been proposed using an elastic grid for the description of the terrain surface. The determination of the elasticity is performed by means of the minimization of curvature in least squares approximations.

For that reason the paper reviews the methods employed and introduces different curvature measures. Also the problem of correct weighting of hybrid norms will be treated. Examples demonstrate the goodness-of-fit of the approximations, if the curvature measure varies in the objective function of the least squares approach.

1. Introduction

Modelling a digital terrain is subject of many contributions during its three decades of history (H. Ebner / D. Fritsch, 1986). But following the trend it seems to be obvious using local procedures for an improved presentation of the surface such as by local polynomials and triangles, respectively, or a combination of both.

A quite elegant method to solve this task is given by the method of finite elements (FEM): in the meantime a powerful tool to model complex structures in all engineering sciences. The basic idea of the FEM uses elastic elements for the description of surface data, whereby elasticity is determined by means of the minimization of curvature measures in least-squares approximations. Following this idea, the contributions of K. Kubik (1971), G. de Masson d'Autume (1976, 1979), H. Ebner / P. Reiß (1978), H. Ebner (1979, 1983), G. Melykuti (1982) and P. Reiß (1985) can be classified into this concept, in which an elastic grid is used to model the terrain surface. The grid can be composed of bilinear or bicubic elements; its elasticity may be obtained by the minimization of a hybrid L_2 norm of residuals of observed points and of additional curvature equations. But there are two main problems in this approach

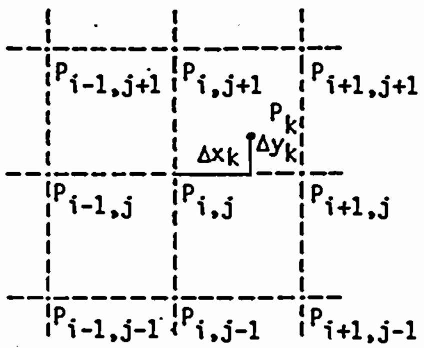
(i) which is the best curvature measure with regard to an optimum local approximation?

(ii) what about the influence or smoothing factor(s) of the curvature equations?

In the following different curvature measures will be introduced, furthermore the determination of optimal smoothness factors (global ones) is formulated as variance component estimation (VCE) problem. Local smoothness factors can be obtained by a simple trial-and-error method; some practical examples prove the efficiency of this procedure.

2. Terrain Modelling by Means of Simple Finite Elements

The finite-element-approach in digital terrain modelling is demonstrated by the simple least-squares approximation problem (see Fig.1): estimate the unknown grid height $h_{i,j}$ of grid point $P_{i,j}$ $\forall$ $i=1,2,\dots,m$, $j=1,2,\dots,n$ by means of the observed height h_k of arbitrarily distributed points P_k $\forall$ $k=1,2,\dots,s$ and additional equations of a curvature measure or approximations on it.



$$\Delta x_k = (x_k - x_i)/d, \quad 0 \le \Delta x_k \le 1$$
$$\Delta y_k = (y_k - y_j)/d, \quad 0 \le \Delta y_k \le 1$$
$$d = x_{i+1} - x_i = y_{j+1} - y_j$$

Fig. 1: Definition of observables and unknown parameters

As curvature measures may be used:

(i) the "simple" formulation K_1, which is not invariant against rotations

$$K_1^2 = \left(\frac{d^2h}{dx^2}\right)^2 + \left(\frac{d^2h}{dy^2}\right)^2 \qquad (1)$$

(ii) the "Laplacian" curvature measure

$$K_2^2 = \left(\frac{d^2h}{dx^2} + \frac{d^2h}{dy^2}\right)^2 \qquad (2)$$

(iii) the "total" curvature

$$K_3{}^2 = \left(\frac{d^2h}{dx^2} + 2\,\frac{d^2h}{dxdy} + \frac{d^2h}{dy^2}\right)^2 \qquad (3)$$

(iv) the "total-like" curvature

$$K_4{}^2 = \left(\frac{d^2h}{dx^2}\right)^2 + A\left(\frac{d^2h}{dxdy}\right)^2 + \left(\frac{d^2h}{dy^2}\right)^2 \qquad (4)$$

$A \in \mathbb{N}$

(v) the "combined" curvature measure

$$K_5{}^2 = K_4{}^2 + \frac{1}{2}\,K_2{}^2 \,. \qquad (5)$$

(vi) the "energy" measure of an infinitesimal thin plate with Poisson number v

$$K(v) = K_4{}^2 + 2v\left(\frac{d^2h}{dx^2}\,\frac{d^2h}{dy^2} - \left(\frac{d^2h}{dxdy}\right)^2\right) \qquad (6)$$

in which the Gaussian curvature is to be weighted with v.

The curvature measures (2)-(6) are invariant against translations and rotations of the coordinate system and therefore well-suited to describe terrain surfaces. In discrete applications the second derivatives have to be approximated by difference formulas (see Fig.2, J. Dankert, 1977)

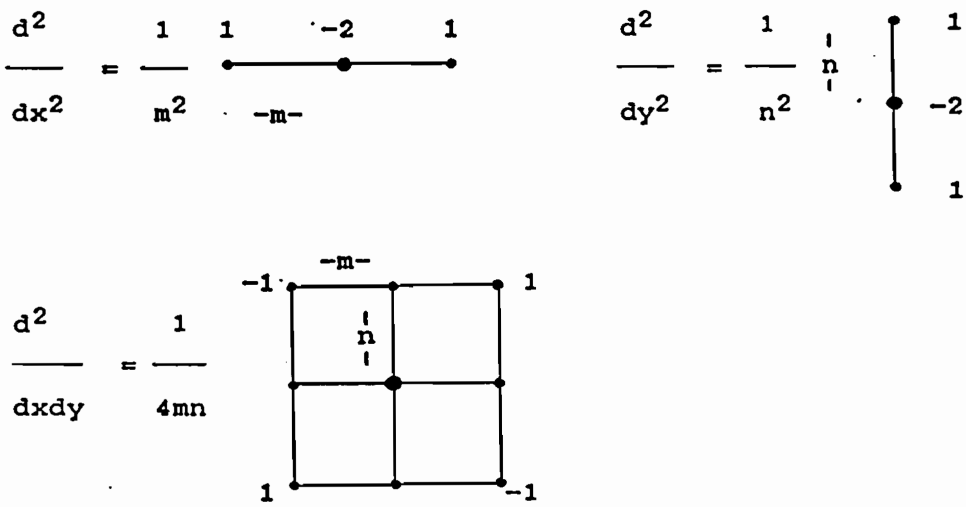


Fig.2: Difference formulas to be used in curvature measures.

Most of the contributions above deal with the curvature measure K_1, which is approximated by its second differences in x- and y-direction, and is only invariant against translations. The observation equations of the resulting least-squares approach may look as follows, if bilinear finite elements are used

$$h_k + v_k = a_{i,j}h_{i,j} + a_{i+1,j}h_{i+1,j} + a_{i,j+1}h_{i,j+1} + a_{i+1,j+1}h_{i+1,j+1} \qquad \forall\ k=1,2,\ldots,s \qquad (7a)$$

as well as

$$\left(\frac{d^2h}{dx^2}\right)_{i,j} + v_{xx,i,j} = h_{i-1,j} - 2h_{i,j} + h_{i+1,j}$$

$$\forall \; i=2,3,\ldots,m-1 \; , \quad j=1,2,\ldots n \qquad (7b)$$

$$\left(\frac{d^2h}{dy^2}\right)_{i,j} + v_{yy,i,j} = h_{i,j-1} - 2h_{i,j} + h_{i,j+1}$$

$$\forall \; i=1,2,\ldots,m \; , \quad j=2,3,\ldots,n-1 \qquad (7c)$$

whereby the second derivatives (second differences) are considered as fictitious observables with observation values set to zero.

3. Problems with Curvature Measures in Least-Squares Approximations

Considering the observation equations (7) in a Gauss-Markov model leads to

$$E(l) := E\left(\begin{bmatrix} l_1 \\ 0 \end{bmatrix}\right) := \begin{bmatrix} l_1 + v_1 \\ 0 + v_2 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} x \qquad (8a)$$

and

$$D(l) := D\left(\begin{bmatrix} l_1 \\ 0 \end{bmatrix}\right) = \sigma^2 \begin{bmatrix} P_1^{-1} & 0 \\ 0 & P_2^{-1} \end{bmatrix} \qquad (8b)$$

with the s*1 observation vector l_1 and its corresponding residual

vector v_1, the r*1 residual vector v_2 of the fictitious curvature observations, the s*mn matrix A of coefficients for the bilinear interpolation, the 2(mn-m-n)*mn coefficient matrix B of difference equations as well as the mn*1 vector x of the unknown grid heights; E is the expectation operator. Its dispersion D is represented by the variance σ^2 of unit weight and the weight matrices P_1 and P_2 of the observations.
This model serves as starting point for a hybrid least-squares approach with objective function

$$\|v_1\|^2_{P_1} + \|v_2\|^2_{P_2} = \min \qquad (9)$$

from which the following normal equations can be derived

$$(A'P_1A + B'P_2B)\,\hat{x} = A'P_1 l_1 \qquad (10)$$

But the main problem in this procedure is the proper choice of the weight matrix P_2. There are different ways to come to an estimation of this matrix depending on global or local approaches.

A global procedure is the introduction of a smoothness factor γ, so that (9) and (10) can be rewritten to

$$\|v_1\|^2_{P_1} + \gamma\,\|v_2\|^2_{\bar{P}_2} = \min \qquad (11)$$

as well as

$$(A'P_1A + \gamma\, B'\bar{P}_2B)\,\hat{x} = A'P_1 l_1 \qquad (12)$$

whereby $\bar{P}_2 = I$ may be assumed in some applications. This factor is a measure for the smoothness of terrain and might also be used for terrain classifications. Its determination can be formulated as variance component estimation (VCE) problem, though there are also trial-and-error methods which will lead to an optimum value.

A local approach can be found in H. Ebner (1983), who defines variable weights simply by quadratic inverses of the second differences, for instance

$$p_{xx} = (p_{i,j})_{xx} = \frac{1}{(h_{i-1,j} - 2h_{i,j} + h_{i+1,j})^2}$$

$$p_{yy} = (p_{i,j})_{yy} = \frac{1}{(h_{i,j-1} - 2h_{i,j} + h_{i,j+1})^2} \qquad (13)$$

if the curvature measure K_1 is used to model the terrain surface. The advantage of this formulation is that the roughness of terrain is taken into account. This strategy of variable weighting seems to be the right way considering inhomogeneity of terrain, but moreover its smoothing effect has to be investigated in detail.

3.1 Estimation of the Smoothness Factor

A powerful tool to solve this task is given by the method of VCE. This procedure has been developed in statistics and is more and more used for the improvement of dispersion in geodetic least-squares problems.

Let the model (8) be reformulated to

$$E(l) = \tilde{A}x \qquad (14a)$$

and

$$V(l) = \sigma^2_1 Q_1 + \sigma^2_2 Q_2 := C \tag{14b}$$

with $\tilde{A} := [A', B']'$; its dispersion has been generalized into a two-component variance model in which the σ^2_1 and σ^2_2 are unknown and the $(s+2(mn-m-n))*(s+2(mn-m-n))$ matrices $Q_i \; \forall \; i=1,2$ are given by approximate values σ^2_{i0} as well as the weight matrices P_i

$$Q_1 := \begin{bmatrix} \sigma^2_{10} P_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_2 := \begin{bmatrix} 0 & 0 \\ 0 & \sigma^2_{20} P_2^{-1} \end{bmatrix} \tag{15}$$

An estimation of the unknown variance components may be obtained by the following equation system (W. Förstner, 1985, K. R. Koch, 1987)

$$S \hat{\Phi} = w \tag{16}$$

whereby $\hat{\Phi} := [\hat{\sigma}^2_1, \hat{\sigma}^2_2]'$; the "quasi-coefficient" 2*2 matrix S and its corresponding 2*1 right-hand side w are defined by

$$S = (s_{i,j}) = (\operatorname{tr} [C^{-1}DQ_iC^{-1}DQ_j]) \tag{17a}$$

$$w = (w_i) = (\operatorname{tr} [l'C^{-1}DQ_iC^{-1}Dl]) \tag{17b}$$

$$D = I - \tilde{A} (\tilde{A}'C^{-1}\tilde{A})^{-1} \tilde{A}'C^{-1} \tag{17c}$$

After the estimation of the variance components the smoothness factor γ is obtained

$$\gamma = \hat{\sigma}^2_1 \, / \, \hat{\sigma}^2_2 \qquad (18)$$

4. Experience with Curvature measures

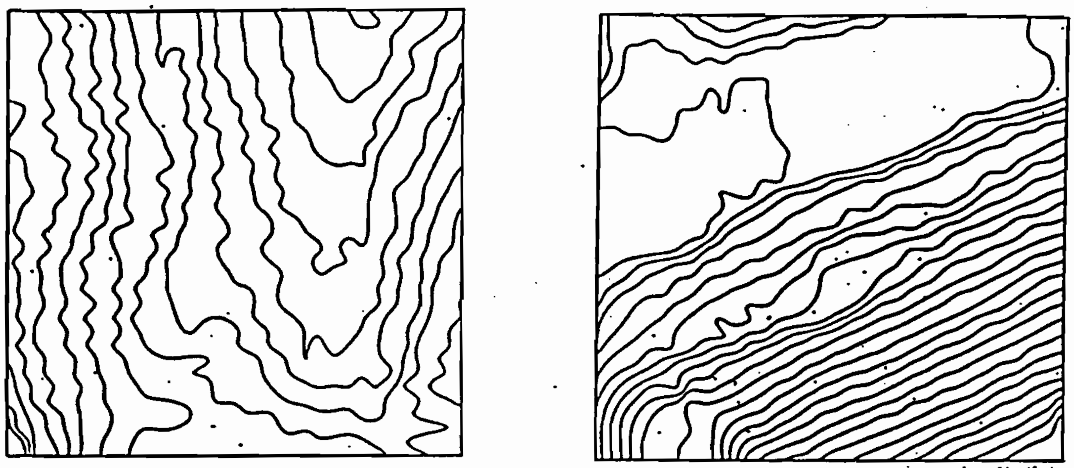
For demonstration of the use of curvature measures the following two examples (see Fig. 3) are dealing with flat and rough terrain. Example (1) represents a flat terrain surface, whereas (2) describes a rough terrain area.

Starting with (1) the upper figure contains the 1(m) contour lines derived from the original data. Because of the data acquisition by profiles the aim is here to eliminate the data capturing effect.

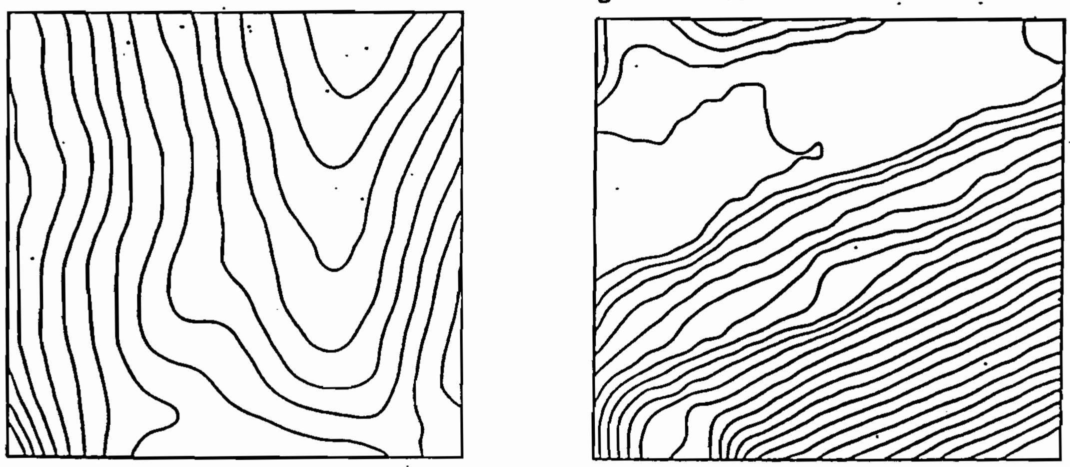
A first smoothing with the difference equations of K_1 is given by Fig. (b). In this application constant weights $p_1=p_2=1$ have been used with roughness factor γ =1.0 . The smoothing below (see Fig. c) is derived by the curvature measure K_4 with constant weights p_1 for the observed points and variable weights p_2 according to (13) for the difference equations; its belonging smoothness factor is γ=1.0. A comparison of the contour lines of Fig. (b) and (c) shows congruity, what means, there is no significant sign to switch over to more complex curvature measures as well as to variable weighting. This is not in contradiction with the expectations, because the second differences do not vary very much.

Example (2) starts with 5(m) contour lines of the original data. The first smoothing by means of the difference equations of K_1 and constant weights $p_1=p_2=1$ as well as a smoothness factor γ =0.1 is represented by Fig.(b). There is a "swing" effect within the flat area, whereas the rough terrain has been smoothed too much. A second smoothing with the difference equations of K_4 and its corresponding variable weighting is given by Fig.(c), in which constant weights for the observed weights and a smoothness factor γ =1.0 was assumed. The "swing" effect in the flat area is eliminated, whereas in the rough terrain characteristic features have been maintained. This means, that roughness adapted smoothing should be the right way to go.

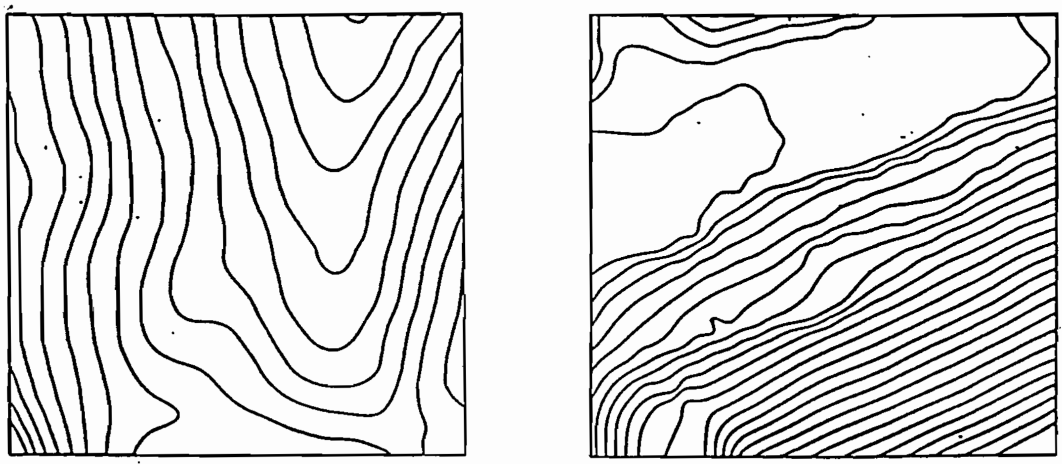
(1) flat terrain (2) rough terrain



(a) contour lines derived from the original data



(b) contour lines derived from the model with the difference equations of K_1 (constant weighting)



(c) contour lines derived from the model with the difference equations of K_4 (variable weighting)

Fig. 3: Demonstration of the influence of different curvature measures

5. Conclusions

This paper presented some theoretical considerations on curvature measures and the estimation on global roughness factors. As far as curvatures are concerned in terrain modelling, one should take care of translation and rotation invariant measures. Furthermore, the estimation of roughness factors should be realized also with regard to the classification of terrain.

Reconsidering the variable weighting approach it seems to be an efficient tool for maintaining terrain details. But also here further investigations have to be made to combine optimum smoothing with optimum detail representation.

References

Dankert, J. (1977): Numerische Methoden der Mechanik. Springer, Wien - New York, 318p.

Ebner, H. (1979): Zwei neue Interpolationsverfahren und Beispiele für ihre Anwendung. Bildmess., Luftbildwes., 47: pp.15-27.

Ebner, H. (1983): Berücksichtigung der lokalen Geländeform bei der Höheninterpolation mit finiten Elementen. Bildmess., Luftbildwes., 51: pp. 3-9.

Ebner, H. / D. Fritsch (1986): High Fidelity Digital Elevation Models - Elements of Land Information Systems. Proceed. XVIII . Int. Congr. Survey. (FIG) , 5: pp. 155-170, Toronto.

Ebner, H. / P. Reiß (1978): Height Interpolation by the Method of Finite Elements. Proc. DTM Symp. : pp. 241-254, St. Louis.

Förstner, W. (1985): Determination of the Additive Noise Variance in Observed Autoregressive Processes Using Variance Component Estimation Technique. Statistics & Decisions, Supplement Issue, 2: pp. 263-274.

Koch, K. R. (1987): Parameterschätzung und Hypothesentests. Dümmler, Bonn, 374p.

Kubik, K. (1971): The Application of Piecewise Polynomials to Problems of Curve and Surface Approximation. Rijkswaterstaat Communicat., 12, The Hague.

Masson D'Autume, G. de (1976): Interpolation by a Flexible Rod (Spline): Applications to Digital Photogrammetry. XIII . Int. Congr. ISPRS , Arch. 4: pp. 3 39 045 / 1-19 /, Helsinki.

Masson D'Autume, G. de (1979): Surface Modelling by means of an Elastic Grid. Photogrammetria, 35: pp. 65-74.

Melykuti, G. (1982): Flächenbestimmung mit der Methode der Finiten Elemente. Bildmess., Luftbildwes., 50: pp. 49-58.

Reiß, P. (1985): Aufbau digitaler Höhenmodelle auf der Grundlage einfacher finiter Elemente. Deutsche Geod. Komm., Reihe C, Nr. 315, München.