Accuracy of Ocean Topography from ERS-1 Altimetry as a Function of Ground Track Stability

K. R. Koch

D. Fritsch

Institut für Theoretische Geodäsie der Universität Bonn
Nussallee 17, 5300 Bonn 1, Federal Republic of Germany

Abstract If the ground track repeat pattern of ERS-1 cannot be exactly maintained so that the ground tracks move within a certain band, the altimeter measurements of ERS-1 have to be used to determine the unknown geoid undulations within the band and the time-varying ocean heights. Using a polynomial model, the standard deviations of the geoid undulations and the time-varying heights are computed. The results show that the standard deviations mainly depend on the length of the time span for the observations, if data are analyzed for ground tracks, which cover the band from one end to the other at least once. Nevertheless, the width of the band should be kept small, for instance 10 km to 20 km, to insure the validity of the polynomial model.

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Introduction

The European Space Agency (ESA) is planning the first ESA Remote Sensing Satellite (ERS-1) for launch in 1987; its mission is oriented toward ice and ocean monitoring. To determine significant wave heights of the oceans and major current systems, ERS-1 will carry a radar altimeter (Dow, 1982).

The surface velocity of the currents of the oceans can be computed if the heights of the ocean surfaces are given with respect to the geoid (Wunsch and Gaposchkin, 1980). These heights should be determined by the altimetry data. The geoid over the oceans obtained from the potential coefficients of the gravity field is only approximately known and it needs to be determined, since the short-wave components are still lacking in knowledge of the gravity field. After the orbit computations have been performed, the altimetry data can therefore give ocean heights only referred to an approximate geoid. Since the geoid undulations are constant with time, while the sea surface heights are variable with time, this variability will allow, to the extent of the variations, the separation of the sea-surface heights and the geoid undulations in the altimetry data (Koch and Fritsch, 1982).

The accuracy of the altimeter of ERS-1 is expected to be 5 cm to 10 cm, but the orbit computations will probably not be of an equivalent accuracy. However, the orbital errors are not as critical, since repetitive orbits are planned for ERS-1. This means that the ground tracks of ERS-1 repeat themselves after, say, three days. Because of the changing drag effects on a satellite, it is practically impossible to maintain an exact ground-track repeat pattern. The ground tracks do not coincide exactly, but follow approximately parallel to each other within a band of a certain width, for instance 5 km, in which the tracks are allowed to move once back and forth before the next orbit correction is initiated (Dow, 1981).

As already mentioned, the geoid undulations within the band have to be determined from the altimetry, in addition to the time-varying ocean heights. The accuracy of such a determination will depend on the stability of the ground tracks, that means, on the speed with which the tracks move in the band, and on the width of the band. Using simplifying assumptions that allow the separation of the geoid and the time-varying heights, the accuracy of such a determination will be investigated here.

For this investigation it is assumed that orbits have been computed for the ERS-1 so that ocean heights referred to an approximate geoid could be derived from the altimeter data and that known time variations such as waves and tides have been eliminated from the data. These ocean heights are considered as observations in this study, and their accuracy will be expressed by standard deviations.

Method of Analysis

Each ground track of ERS-1 over the oceans will give ocean heights measured by the altimeter. With a repeating ground track pattern of, say, three days, the ocean heights of any two colinear ground tracks will differ for three reasons:

1) unknown geoid undulations, since the first ground track is not identical with the second one, but runs parallel to it,
2) variations of the ocean heights with time,
3) errors in the orbit computations.

Even if we assume perfect orbit computations, the altimeter data alone cannot distinguish between a slope of the ocean heights caused either by a change in the geoid or from ocean variability. However, to simplify the problem we will assume that a slope, for instance, due to a current can be neglected, if tracks with ocean heights of only a few hundred kilometers in length are processed. A slope is then attributed to the geoid, while the changes of the ocean heights with time are assumed to be the same for the whole area where the data are analyzed. Hence, the geoid undulations in the measured ocean heights can be modeled by a polynomial as a function of the positions of the data points and the variations of the measured ocean heights with time by a polynomial as a function of the time of the data points.

As mentioned, the measured ocean heights are also affected by errors in orbit computation. If these errors do not depend on the position or the time of the data points, but on some variable orbit characteristics, they can be modeled also by a polynomial as a function of these orbit characteristics. At present the information for such a model is not available, so that errors in orbit computations are not considered in the sequel.

The following polynomial is applied

$$\begin{aligned} &\beta_1 + x_i\beta_2 + y_i\beta_3 + t_i\beta_4 \\ &[\,+ x_i^2\beta_5 + x_iy_i\beta_6 + y_i^2\beta_7 + x_i^3\beta_8 \\ &\qquad + x_i^2y_i\beta_9 + t_i^3\beta_{10}] \\ &= E(h_i) \quad \text{with} \quad i\in\{1,\ldots,n\} \end{aligned} \tag{1}$$

where h_i are the measured ocean heights; $E(h_i)$ their expected values; n the number of observations; β_j with $j\epsilon\{1,\ldots,10\}$ the unknown parameters of the polynomials; x_i, y_i the coordinates of the positions of the data points, which are defined in a horizontal plane with the x-axis pointing in the directions of the ground tracks and the y-axis perpendicular to it; and t_i the time of the data points. For the first data point h_1 we define $x_1=y_1=t_1=0$.

As indicated in Equation (1) by brackets, a polynomial of lower degree, which is restricted to the linear terms, and a polynomial of higher degree are chosen to represent the measured ocean heights. The polynomial of higher degree includes the linear terms and the terms of second degree for x_i and y_i but not for t_i, since y_i is a function of $t^2{}_i$, as will be shown by Equation (10), and the terms of the third degree for x_i and t_i but not for y_i, since the band with data points is long in comparison to its width. The coordinate x_i can be assumed as independent of the time t_i, because of the high velocity of a satellite. The choice of the degree of the polynomials, of course, is arbitrary. However, as will be seen by the results, polynomials of lower degree are to be preferred.

Standard Deviations for the Geoid Undulations and the Time-Variable Heights

The variances and covariances of the measured ocean heights h_i in Equation (1) have to be known in order to compute variances and covariances for quantities derived from h_i. It will be assumed that the measurements h_i have equal variances σ^2, and that they are independent from each other. Hence,

$$V(h_i) = \sigma^2 \text{ and } C(h_i, h_k) = 0 \text{ for } i \neq k \tag{2}$$

where $V(h_i)$ denotes the variance of h_i, and $C(h_i,h_k)$ the covariance of h_i and h_k. Using matrix notation we get from Equations (1) and (2) the Gauss-Markoff model for the estimation of the unknown parameters of the polynomials (Koch, 1980: 144)

$$\underline{X}\underline{\beta} = E(\underline{y}) \text{ with } D(\underline{y}) = \sigma^2 \underline{I} \tag{3}$$

and

$$\underline{\beta} = (\beta_j),\ \underline{y} = (h_i)$$

$$\underline{X} = \left| \begin{array}{c} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 1 \quad x_i \quad y_i \quad t_i \quad [x_i^2 \quad x_i y_i \quad \ldots \quad t_i^3] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array} \right|$$

D(y) is the covariance matrix of *y*, and *I* the unit matrix. Only coefficient matrices *X* of full column rank will be considered. The covariance matrix $D(\beta)$ of the estimates β with $\beta = (\beta_j)$ of the unknown parameters β is given by

$$D(\hat{\underline{\beta}}) = \sigma^2 (\underline{X}'\underline{X})^{-1} \tag{4}$$

and the variance $V(a'\beta)$ of the estimate of any linear function $a'\beta$ of the unknown parameters by (Koch, 1980: 148)

$$V(\underline{a}'\hat{\underline{\beta}}) = \sigma^2 \underline{a}'(\underline{X}'\underline{X})^{-1}\underline{a} \tag{5}$$

In Equation (5) the standard deviations of the geoid undulations and of the time-variable heights are computed. The geoid undulation $N(x_1,y_1)$ at the point with the coordinates x_1,y_1 is obtained from Equation (1) by

$$\begin{aligned} N(x_1,y_1) &= \hat{\beta}_1 + x_1\hat{\beta}_2 + y_1\hat{\beta}_3 + 0\hat{\beta}_4 \\ &\quad [+ x_1^2\hat{\beta}_5 + x_1y_1\hat{\beta}_6 + y_1^2\hat{\beta}_7 + x_1^3\hat{\beta}_8 \\ &\quad + x_1^2y_1\hat{\beta}_9 + 0\hat{\beta}_{10}] \\ &= \underline{b}_1'\hat{\underline{\beta}} \end{aligned} \tag{6}$$

and the time-variable height $h(t_m)$ at the time t_m by

$$\begin{aligned} h(t_m) &= 0\hat{\beta}_1 + 0\hat{\beta}_2 + 0\hat{\beta}_3 + t_m\hat{\beta}_4 \\ &\quad [+ 0\hat{\beta}_5 + 0\hat{\beta}_6 + 0\hat{\beta}_7 + 0\hat{\beta}_8 + 0\hat{\beta}_9 + t_m^3\hat{\beta}_{10}] \\ &= \underline{c}_m'\hat{\underline{\beta}} \end{aligned} \tag{7}$$

The sum of the geoid undulation and the time-variable height at x_1,y_1,t_m is given by

$$\begin{aligned} N(x_1,y_1) + h(t_m) &= \hat{\beta}_1 + x_1\hat{\beta}_2 + y_1\hat{\beta}_3 \\ &\quad + t_m\hat{\beta}_4 \\ [+ x_1^2\hat{\beta}_5 + x_1y_1\hat{\beta}_6 &+ y_1^2\hat{\beta}_7 + x_1^3\hat{\beta}_8 + x_1^2y_1\hat{\beta}_9 \\ &\quad + t_m^3\hat{\beta}_{10}] \\ &= \underline{d}'\underline{\beta} \end{aligned} \tag{8}$$

In Equation (5) we get the variances of these quantities

$$V(N(x_1,y_1)) = \sigma^2 \underline{b}_1' (\underline{X}'\underline{X})^{-1} \underline{b}_1 \tag{9}$$

$$V(h(t_m)) = \sigma^2 \underline{c}_m' (\underline{X}'\underline{X})^{-1} \underline{c}_m$$

$$V(N(x_1,y_1) + h(t_m)) = \sigma^2 \underline{d}' (\underline{X}'\underline{X})^{-1} \underline{d}$$

and the standard deviations by taking positive square roots.

Coordinates of the Data Points

To compute the standard deviations in Equation (9), the coordinates x_i,y_i,t_i of the data points are needed. They are therefore generated. For the altimeter of ERS-1 a data rate of one point per second can be assumed so that the observations of one ground track follow parallel to the x-axis with a distance of 6.8 km, if a period of revolution of ERS-1 of 98.18 min, a constant velocity, and a mean earth radius of 6,367 km are adopted.

The distance between the ground tracks of repetitive orbits depends on the width of the band in which the ground tracks are allowed to move and on the time between the orbit maneuvers that are necessary to maintain the ground-track repeat pattern. The time is a function of solar activity. With y_b being the bandwidth and T_m the time betwen maneuvers, the y-coordinate of a repetitive ground track is obtained for the time t counted from the end of the previous maneuver by (Dow, 1981)

$$y = y_b - c(t-T_m/2)^2$$
$$\text{with } c = y_b/(T_m/2)^2 \tag{10}$$

Hence, the ground tracks move from one end of the band toward the other end and then return. The movements are fast at the side of the band where they start and end; they are slow at the other side of the band where the direction of the movement is reversed.

To avoid a regular pattern of data points generated within the band, which means the same x-values for the data points of all ground tracks, the x-coordinates of the data points of one ground track are shifted by 0.68 km with respect to the data points of the preceding ground track.

By assuming in Equations (2) and (9) a variance of $\sigma^2 = 25\ cm^2$, which is equivalent to a standard deviation of $\sigma = 5$ cm, for the measured ocean height h_i, the standard deviations for the geoid undulations are computed by Equation (9) in a regular grid of nine points that lie at the borders and in the middle of the rectangle formed by the width y_b of the band in which the ground tracks move and the length x_g of the ground tracks chosen for the analysis (see Figure 1). The standard deviations of the time-variable heights are determined by Equation (9) for three different times: for one third of the time span for the observations chosen for the analysis, for two thirds and for the end of the time for the observations (see Figure 1).

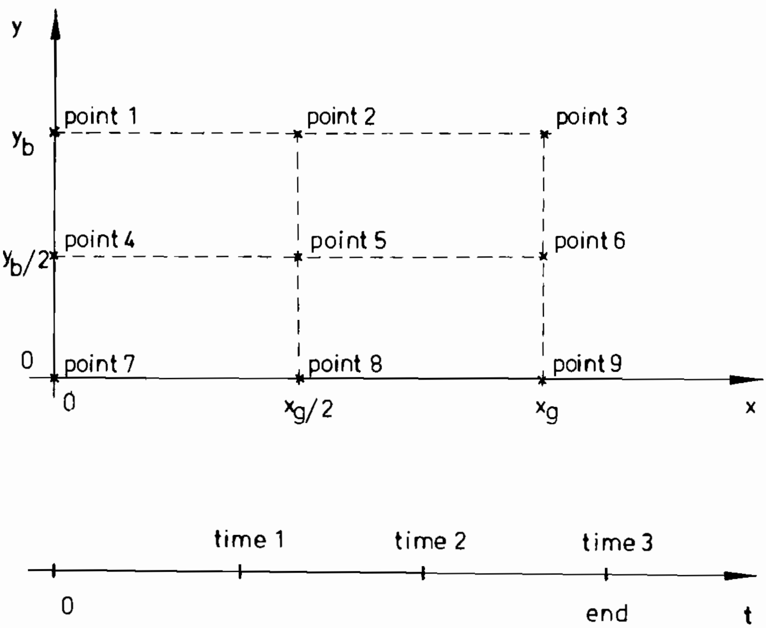

Figure 1. Distribution of the points in position and in time for which standard deviations are computed.

Discussions of the Results and Conclusions

In order to study for the nine points given in Figure 1, the standard deviations of the geoid undulations, and the time-varying heights as a function of the stability of the repetitive ground tracks, the following input parameters have to be kept variable: the length of the ground tracks, the width of the band in which the ground tracks are allowed to move, the degree for the polynomials, the time between maneuvers to maintain the ground track repeat pattern, and the time span during which the observations are collected.

As mentioned in "Method of Analysis," only the data for short ground tracks shall be analyzed, so that the length x_g of the ground tracks is restricted now to $x_g = 100$ km. Computations have shown that an increase in x_g is connected with a decrease in the standard deviations, so that the results for $x_g = 100$ km are conservative.

Varying the width of the band for the ground tracks can be avoided, since according to Equation (10) y is a linear function of y_b, if the polynomials are normalized, for instance, between -1 and $+1$, as has been applied here. Hence, the computed standard deviations are independent of the width of the band. This does not mean that the band can be made arbitrarily wide. If the bandwidth exceeds 10 km to 20 km, the model (1) will not be valid anymore because of the few unknown parameters.

All the computed results show that the polynomials of lower degree give smaller standard deviations than the ones for higher degree. The polynomials of lower degree should therefore be preferred. However, one has to keep in mind, that depending on the "real" data, these polynomials might not be sufficient to model geoid undulations and time-varying heights.

To get standard deviations for the points of Figure 1 less than the standard deviations of the measured ocean heights, i.e., less than 5 cm, the computed results show that data must be available for ground tracks covering the band from one end to the other in case of the lower-degree polynomials. Applying higher-degree polynomials, data of additional ground tracks covering half the band are necessary to obtain standard deviations less than 5 cm. If exactly these data are given, the geoid undulations are still weakly determined. This can be seen in Table 2, where for the higher-degree polynomials $V(N) > V(N+h)$.

Assuming that enough data are available, the influence of the time between maneuvers and the time span for the collection of observations remains to be investigated. As demonstrated by the results of Tables 1 and 2 and confirmed by additional computa-

Table 1

Point	Polynomial Lower Degree	Polynomial Higher Degree
1	1.09	2.74
2	0.71	1.79
3	1.10	2.73
4	0.97	2.25
5	0.51	1.53
6	0.98	2.36
7	1.25	3.02
8	0.94	1.90
9	1.25	3.27

Standard Deviations (cm) for Time-Variable Heights

Time	Polynomial Lower Degree	Polynomial Higher Degree
1	0.26	0.97
2	0.26	0.97
3	0.78	1.85

Standard Deviations (cm) for Geoid Undulations and Time-Variable Heights

Time 3 Point	Polynomial Lower Degree	Polynomial Higher Degree
4	1.30	3.44
5	0.98	2.99
6	1.26	3.46

tions not shown here, with different times between maneuvers very similar results are obtained if identical time spans for the observations are analyzed. If the time spans are increased, the standard deviations decrease. Hence, the standard deviations of the geoid undulations and the time-varying heights are mainly dependent on the length of the time span for the observations, provided the data for ground tracks covering at least the whole band are available.

Table 2

Point	*Polynomial* *Lower Degree*	*Higher Degree*
1	1.19	2.82
2	0.86	1.89
3	1.20	2.76
4	1.05	2.88
5	0.65	2.35
6	1.06	3.09
7	1.82	3.86
8	1.62	2.82
9	1.83	4.47

Standard Deviations (cm) for Time-Variable Heights

Time	*Polynomial* *Lower Degree*	*Higher Degree*
1	0.37	1.55
2	0.37	1.55
3	1.12	2.03

Standard Deviations (cm) for Geoid Undulations and Time-Variable Heights

Time 3 *Point*	*Polynomial* *Lower Degree*	*Higher Degree*
4	1.77	2.52
5	1.54	1.84
6	1.74	2.68

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