DIETER FRITSCH - BURKHARD SCHAFFRIN

☆

The "Choice of Norm,, Problem for the Free Net Adjustment with Orientation Parameters

ESTRATTO DAL « *BOLLETTINO DI GEODESIA E SCIENZE AFFINI* »
RIVISTA DELL' ISTITUTO GEOGRAFICO MILITARE
ANNO XLI - N. 3 - LUGLIO - AGOSTO - SETTEMBRE 1982

ANNO XLI – BOLLETTINO DI GEODESIA E SCIENZE AFFINI – N. 3, 1982

The « Choice of Norm » Problem for the Free Net Adjustment with Orientation Parameters (*)

DIETER FRITSCH — BURKHARD SCHAFFRIN
Institute of Theoretical Geodesy, University of Bonn

Summary. — When adjusting free geodetic networks, solutions of type MINOLESS (« minimum-norm least-squares solution ») are widely used. While the (weighted) least-squares norm is chosen in such a way that estimable functions of the (non-estimable) point coordinates become BLUE (« best linear unbiased estimates »), the question remained still open of which (weighted) Euclidean norm should be applied to the minimization of the solution vector, in particular, e.g., if orientation parameters occur beside the point coordinates as it happens within triangulation networks.

For some typical choices of this norm the resulting parameter vectors are compared as well as their appropriate dispersion matrices showing the respective point accuracy. The dual problem, however, concerning solutions of type BLIMBE (« best linear minimum bias estimate ») is not yet treated. Some interpretations of the theory are given by a small example.

(*) Presented at the 8th Symposium on Mathematical Geodesy — 5th Hotine Symposium — Como, Italy, September 7 - 9, 1981.

0. — INTRODUCTION

In order to get realistic information about the actual accuracy of geodetic networks, P. Meissl (1962 ; 1965 ; 1969, p. 8-21) developed his famous « inner error theory » consisting in an instruction how to compute the suited dispersion matrix of the point coordinates. Since the contributions of E. Mittermayer (1971 ; 1972 a, b) it has become well known that identical dispersion matrices are obtained if the parameter vector containing the coordinates is of type MINOLESS (« minimum-norm least-squares solution ») ; in this case the dispersion matrix turns out to be the (unique) *pseudoinverse matrix* of the normal equations matrix, sometimes also called « stochastic ring inverse » as introduced by A. Bjerhammar (1958, p. 20) or « Helmert inverse » as proposed by H. Wolf (1972 ; 1973).

One powerful method of evaluating the pseudoinverse matrix uses the so-called « solution space » which means the « null-space of the singular normal equations matrix ». This method has been described in full length by A.J. Pope (1973) as well as by H. Pelzer (1974), but previously, e.g., by A.J. Goldman/M. Zelen (1964), P. Meissl (1969), and R.M. Pringle/A.A. Rayner (1971, p. 90-98 ; 1976) who attribute it to R.L. Plackett (1950) justly, I suppose ; a synopsis can be found in the book of K.R. Koch (1980, p. 57-60). Notice that the required basis of the « solution space » is readily obtained as far as geodetic networks are concerned, see e.g. P, Meissl (1969), A.J. Pope (1973) and K.R. Koch (1980, p. 173-175) ; a remarkable connection exists further with the S-transformations of W. Baarda (1973) as shown by J. van Mierlo (1980).

If the unknown parameters are estimated in such a way that their dispersion matrix results as that one described above, it gets the most important property, too, of having *minimal trace* under all other possible dispersion matrices, thus securing an even and, for the present, quite reasonable error situation with respect to *all* parameters. However, this fact might become less satisfactory if the parameter vector to be estimated contains not only point coordinates but also orientation unknowns, e.g., as it happens within triangulation networks. In this case, not the entire dispersion matrix is of interest, but mainly the block matrix corresponding to the point coordinates and containing their variances and covariances.

Although this problem has been touched in part by P. Meissl (1969, p. 16-18), it seems to be still far from being solved in a convincing manner, nervetheless. Hence our approach shall contribute to a deeper knowledge about the changes which the covariance matrix of the point coordinates will be subject to within triangulation networks, if the *weighted* Euclidean norm is varied with respect to which the parameter vector has to be minimized after the performance of the least-squares process. In particular, we shall investigate the following alternatives to be applied to the normal equations:

(*i*) elimination of the orientation parameters, and afterwards minimization of the coordinates in the (unweighted) l^2-norm (« classical approach ») ;

(*ii*) minimization only of the point coordinates in the (unweighted) l^2-norm, and afterwards minimization of the orientation parameters in the (unweighted) l^2-norm as well (« dual approach ») ;

(*iii*) minimization of the whole parameter vector containing both, point coordinates *and* orientation unknowns, in the (unweighted) l^2-norm (« pseudoinverse approach ») ;

(*iv*) minimization of the parameter vector in a weighted l^2-norm (if it exists) such that the covariance matrix of the point coordinates becomes the pseudoinverse only of that block of the normal equations matrix which belongs just to the point coordinates (« naive approach »).

The resulting parameter vectors as well as their appropriate dispersion matrices are presented in a form suitable for some comparisons ; moreover, a necessary and sufficient condition for existence is given in the case (*iv*). The theoretical results are further cleared up by a small example where computational aspects are treated, too,

Nevertheless, several questions remain still unsolved, particularly which norm on the parameter vector would be responsible for minimizing the trace of the covariance matrix of the point coordinates only. Furthermore, if taking in consideration the equivalence between MINOLESS and BLIMBE (« best linear minimum bias estimate »), see e.g. C.R. Rao (1971; 1972) or B. Schaffrin (1975), the same problems arise concerning the respective norms ; e.g. the requirement « best » standing for «minimal total variance » may be related only to the variances of the point coordinates as well. However, this is beyond the scope of the present paper.

1. — THE « CHOICE OF NORM » PROBLEM

Let us start with introducing some notations and describing the adjustment model. As usually, we presuppose the Gauss-Markov model defined by

$$E(\bar{y}) = \bar{A}x\,, \tag{1.1a}$$

$$D(\bar{y}) = \sigma^2 P^{-1} \tag{1.1b}$$

where $\bar{y}$ is the $n\times 1$ observation vector, x the unknown $(u+r)\times 1$ vector of fix parameters, $\bar{A}$ the $n\times(u+r)$ coefficient matrix, P the positive-definite $n\times n$ weight matrix, and σ^2 the unknown variance of unit weight (« variance component »); E denotes « expectation » and D « dispersion » of a random vector. Then by the common procedure of « homogeneization »

$$y := P^{1/2}\,\bar{y}\,, \quad A := P^{1/2}\,\bar{A}\,, \tag{1.2}$$

we arrive at the « Simple Gauss-Markov Model »

$$E(y) = Ax\,, \quad D\,(y) = \sigma^2 I \tag{1.3}$$

with I being the identity matrix, and by the least-squares process further at the « normal equations »

$$Nx = b \tag{1.4a}$$

with

$$N := A'A = \bar{A}'P\bar{A}\,, \qquad b := A'y = \bar{A}'P\bar{y}\,. \tag{1.4b}$$

Moreover, it is well known, confer e.g. K.R. Koch (1980, p. 170), that *any* solution of (1.4) yields the *unique* BLUE (« best linear unbiased estimate ») of every arbitrary estimable function, independently of the variance component σ^2 and even of the probability distribution of y at all. This fundamental theorem turns out to be a straight generalization of « Gauss' 2. Begründung » by the help of C.A. Aitken (1934) as pointed out by R.L. Plackett (1949). Therefore, in the case of a *singular* matrix N we are free, for the time, in reflecting which solution of (1.4) we should prefer, perhaps the virtual reason why it is called « *free* adjustment ».

As far as triangulation networks are concerned let us separate the parameters contained in x into the $u \times 1$ vector x_1 of point coordinates and the $r \times 1$ vector x_2 of orientation unknowns, leading to a corresponding partitioning of the normal equations

$$Nx = \begin{bmatrix} N_{11}\,, & N_{12} \\ N_{21}\,, & N_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b \tag{1.5a}$$

with

$$N_{ij} := A_i'A_j = \bar{A}_i'P\bar{A}_j\,, \qquad b_i := A_i'y = \bar{A}_i'P\bar{y} \tag{1.5b}$$

for $i, j \in \{1, 2\}$ and $A = [A_1, A_2]$. Note that a comparable partitioning of A should be possible for every other type of geodetic network, too.

Since for free networks (thus without assuming fix points) the matrix N becomes singular, in fact, we have to decide upon the choice of g-inverse N^- specifying the general solution

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} N_{11}, & N_{12} \\ N_{21}, & N_{22} \end{bmatrix}^- \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{1.6}$$

as to certain desirable properties particularly of its appropriate covariance matrix

$$D\left(\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}\right) = \sigma^2 \begin{bmatrix} N_{11}, & N_{12} \\ N_{21}, & N_{22} \end{bmatrix}^-_{rs} =: \sigma^2 \begin{bmatrix} C_{11}, & C_{12} \\ C_{21}, & C_{22} \end{bmatrix} \tag{1.7}$$

which proves to be itself a certain symmetric reflexive g-inverse of N respectively. Such a specification may be performed by introducing a distinctive (weighted) Euclidean norm with respect to which the solution vector has to be minimized; it is this what has been called the *« choice of norm » problem*.

2. — SOME ALTERNATIVE NORMS

Several alternative norms, whether composed or successive, have been collected; they are partly in use, depending on the purposes to which the adjustment is carried out, and shall be further investigated in the following:

(*i*) « classical approach »

$$[x_1', x_2'] \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2' x_2 = \|x_2\|^2 = \min \tag{2.1a}$$

$$\Rightarrow x_2 = N_{22}^{-1}(b_2 - N_{21}x_1) \tag{2.1b}$$

(*i.e.* elimination of the orientation parameters due to the regular inverse of N_{22})

$$\Rightarrow N_{11}x_1 - N_{12}N_{22}^{-1} N_{21}x_1 = b_1 - N_{12}N_{22}^{-1}b_2 , \tag{2.1c}$$

but now subject to

$$[x_1', x_2'] \begin{bmatrix} I_u & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1'x_1 = \| x_1 \|^2 = \min \tag{2.1d}$$

(*ii*) « dual approach »

$$[x_1', x_2'] \begin{bmatrix} I_u & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1'x_1 = \| x_1 \|^2 = \min \tag{2.2a}$$

$$\Rightarrow x_1 = N_{11}^{+}(b_1 - N_{12}\, x_2) \tag{2.2b}$$

(N_{11}^{+} being the pseudoinverse of N_{11})

$$\Rightarrow N_{22}x_2 - N_{21}\, N_{11}^{+}\, N_{12}x_2 = b_2 - N_{21}\, N_{11}^{+}\, b_1\,, \tag{2.2c}$$

but now subject to

$$[x_1', x_2'] \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2'x_2 = \| x_2 \|^2 = \min \tag{2.2d}$$

(*iii*) « pseudoinverse approach »

$$[x_1', x_2'] \begin{bmatrix} I_u & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1'x_1 + x_2'x_2 = \| x \|^2 = \min \tag{2.3}$$

(*iv*) « naive approach »

$$x'Rx = \| x \|_R^2 = \min \tag{2.4a}$$

where R has to be chosen such that both

$$C_{11} := \sigma^{-2}\, D(x_1) = N_{11}^{+} \tag{2.4b}$$

and

$$\operatorname{rk} C_{22} = \operatorname{rk} N_{22} \tag{2.4c}$$

hold (if possible).

3. — COMPARISON OF THE RESULTS

In order to become able to compare the different suggested possibilities, let us quote the respective solution vectors and their appropriate dispersion matrices, distinctive for each « choice of norm ».

If defining

$$S_1 := N_{11} - N_{12} N_{22}^{-1} N_{21}\,, \qquad S_2 := N_{22} - N_{21} N_{11}^{+} N_{12} \tag{3.1}$$

as (generalized) « Schur complements » with respect to the partitioned matrix N in (1.5), see e.g. R.W. Cottle (1974), we obtain for the particular cases :

(*i*) *The « classical approach »* (2.1) :

Solving

$$S_1 x_1 = b_1 - N_{12} N_{22}^{-1} b_2 \tag{3.2a}$$

subject to

$$\| x_1 \|^2 = x_1' x_1 = \min \tag{3.2b}$$

yields immediately as solution vector

$$\hat{x}_1 = S_1^{+}(b_1 - N_{12} N_{22}^{-1} b_2)\,, \tag{3.3a}$$

and by inserting this into (2.1b)

$$\hat{x}_2 = N_{22}^{-1} b_2 - N_{22}^{-1} N_{21} S_1^{+}(b_1 - N_{12} N_{22}^{-1} b_2)\,, \tag{3.3b}$$

or in comprised form

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} S_1^{+} & , & -S_1^{+} N_{12} N_{22}^{-1} \\ -N_{22}^{-1} N_{21} S_1^{+} & , & N_{22}^{-1} + N_{22}^{-1} N_{21} S_1^{+} N_{12} N_{22}^{-1} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{3.4}$$

From this representation the appropriate dispersion matrix is readily obtained by computing

$$D\left(\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}\right) = \sigma^2 \begin{bmatrix} S_1^+ , & -S_1^+ N_{12} N_{22}^{-1} \\ -N_{22}^{-1} N_{21} S_1^+ , & N_{22}^{-1} + N_{22}^{-1} N_{21} S_1^+ N_{12} N_{22}^{-1} \end{bmatrix} \tag{3.5}$$

However, since our special interest is directed to the covariance matrix of the point coordinates because of the orientation parameters being only auxiliary quantities, we try to express

$$\boxed{C_{11} = \sigma^{-2} D(\hat{x}_1) = S_1^+} \tag{3.6}$$

in terms of the pseudoinverse matrix N_{11}^+.

At first, observe that for the rank of the normal equations matrix

$$\operatorname{rk} N = \operatorname{rk} N_{11} - f + \operatorname{rk} N_{22} \tag{3.7}$$

holds where the constant f reflects the degree of freedom lost if the orientation of the network would supposedly be fixed ; it is easily seen that in the simplest geometric cases f takes the values, e.g.,

$$f = 1 \qquad \text{for horizontal networks,} \tag{3.8a}$$

$$f = 3 \qquad \text{for 3-dimensional networks.} \tag{3.8b}$$

Thus we can conclude for the rank of the first « Schur complement »

$$\begin{aligned}\operatorname{rk} S_1 &= \operatorname{rk}\begin{bmatrix} S_1 , & 0 \\ 0 , & N_{22}\end{bmatrix} - \operatorname{rk} N_{22} = \\ &= \operatorname{rk}\begin{bmatrix} N_{11} , & N_{12} \\ N_{21} , & N_{22}\end{bmatrix} - \operatorname{rk} N_{22} = \operatorname{rk} N_{11} - f\end{aligned} \tag{3.9}$$

since the regular transformation

$$\begin{bmatrix} I , & -N_{12}N_{22}^{-1} \\ 0 , & I \end{bmatrix}\begin{bmatrix} N_{11} , & N_{12} \\ N_{21} , & N_{22}\end{bmatrix}\begin{bmatrix} I , & 0 \\ -N_{22}^{-1}N_{21} , & I\end{bmatrix} = \begin{bmatrix} S_1 , & 0 \\ 0 , & N_{22}\end{bmatrix} \tag{3.10}$$

does not influence the rank. This in mind, we can find a $u \times f$ matrix Z_1 fulfilling the conditions

$$\text{rk}\, Z_1 = f\,, \qquad\qquad i.e.\ \dim R(Z_1) = f\,, \tag{3.11a}$$

$$S_1 Z_1 = 0\,, \qquad\qquad i.e.\ R(Z_1) \subset N(S_1)\,, \tag{3.11b}$$

$$(I - N_{11}N_{11}^{+})\, Z_1 = 0\,, \quad i.e.\ R(Z_1) \subset R(N_{11})\,, \tag{3.11c}$$

which can be proved, e.g., along the lines of A.J. Goldman/M. Zelen (1964, p. 152, Lemma 1) regarding

$$R(S_1) \subset R(A_1') = R(N_{11})\,. \tag{3.12}$$

Here $N(.)$ denotes the « null space » and $R(.)$ the « range » of a matrix. Hence it follows immediately

$$N_{11}N_{11}^{+}(S_1S_1^{+} + Z_1Z_1^{+}) = N_{11}N_{11}^{+}[S_1\,,\, Z_1]\begin{bmatrix} S_1^{+} \\ Z_1^{+} \end{bmatrix} = S_1S_1^{+} + Z_1Z_1^{+} \tag{3.13a}$$

with

$$\text{rk}(S_1S_1^{+} + Z_1Z_1^{+}) = \text{rk}[S_1\,,\, Z_1] = \text{rk}\, S_1 + f = \text{rk}\, N_{11}\,, \tag{3.13b}$$

and consequently

$$N_{11}N_{11}^{+} = S_1S_1^{+} + Z_1Z_1^{+} \tag{3.14}$$

as the due generalization of A.J. Goldman/M. Zelen (1964, p. 153, Lemma 2).

Now we are in a position to apply the famous theorem of P. Meissl (1967) yielding

$$\boxed{\begin{aligned} C_{11} - N_{11}^{+} = {} & S_1^{+}N_{12}T_2^{-1}N_{21}S_1^{+} - \\ & - (I - S_1^{+}N_{12}T_2^{-1}N_{21})Z_1[Z_1'N_{12}T_2^{-1}N_{21}Z_1]^{-1}\, Z_1'(I - N_{12}T_2^{-1}N_{21}S_1^{+}) \end{aligned}} \tag{3.15a}$$

where the matrix

$$T_2 := N_{22} + N_{21}S_1^{+}N_{12} \tag{3.15b}$$

is positive-definite and thus invertible with certainty.

An alternate representation may be gained if we start from the relation

$$(S_1 + Z_1 Z_1^+)^+ = S_1^+ + Z_1 Z_1^+ \tag{3.16}$$

which can be verified by straight computation. This further leads to

$$\begin{aligned} S_1^+ &= (N_{11} - N_{12} N_{22}^{-1} N_{21} + Z_1 Z_1^+)^+ - Z_1 Z_1^+ = \\ &= [I - N_{11}^+(N_{12} N_{22}^{-1} N_{21} - Z_1 Z_1^+)]^{-1} N_{11}^+ - Z_1 Z_1^+ , \end{aligned} \tag{3.17}$$

respectively

$$\boxed{\begin{aligned} & C_{11} - N_{11}^+ = \\ & = N_{11}^+(N_{12} N_{22}^{-1} N_{21} - Z_1 Z_1^+)[I - N_{11}^+(N_{12} N_{22}^{-1} N_{21} - Z_1 Z_1^+)]^{-1} N_{11}^+ - Z_1 Z_1^+ \end{aligned}} \tag{3.18}$$

provided that the inverse matrix exists!

Notice that Z_1 shrinks to a $u \times 1$ vector in the case of horizontal networks, allowing a particularly simple calculation of

$$Z_1 Z_1^+ = Z_1 (Z_1' Z_1)^{-1} Z_1' . \tag{3.19}$$

By the way, a recently proposed procedure by K.R. Koch (1981) turns out to yield actually the solution (3.4) with dispersion matrix (3.5), too.

(*ii*) *The « dual approach »* (2.2) :

Here, solving

$$S_2 x_2 = b_2 - N_{21} N_{11}^+ b_1 \tag{3.20a}$$

subject to

$$\|x_2\|^2 = x_2' x_2 = \min \tag{3.20b}$$

yields the solution vector

$$\hat{x}_2 = S_2^+ (b_2 - N_{21} N_{11}^+ b_1) , \tag{3.21a}$$

and by inserting this into (2.2b)

$$\hat{x}_1 = N_{11}^+ b_1 - N_{11}^+ N_{12} S_2^+ (b_2 - N_{21} N_{11}^+ b_1) , \tag{3.21b}$$

thus in comprised form

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} N_{11}^+ + N_{11}^+ N_{12} S_2^+ N_{21} N_{11}^+ , & - N_{11}^+ N_{12} S_2^+ \\ - S_2^+ N_{21} N_{11}^+ , & S_2^+ \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{3.22}$$

and for the appropriate dispersion matrix

$$D\left(\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}\right) = \sigma^2 \begin{bmatrix} N_{11}^+ + N_{11}^+ N_{12} S_2^+ N_{21} N_{11}^+ , & - N_{11}^+ N_{12} S_2^+ \\ - S_2^+ N_{21} N_{11}^+ , & S_2^+ \end{bmatrix} \tag{3.23}$$

Note, in particular, that (3.23) differs from (3.5) according to G. Marsaglia/G.P.H. Styan (1974, p. 441, Corollary 2) in consideration of (3.7) with $f \neq 0$. As immediate consequence we obtain

$$\boxed{C_{11} - N_{11}^+ = N_{11}^+ N_{12} S_2^+ N_{21} N_{11}^+} \tag{3.24}$$

in this case.

(*iii*) *The « pseudoinverse approach »* (2.3) :
Since now we ask for the solution of type MINOLESS, we proceed as follows;

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} N_{11} , & N_{12} \\ N_{21} , & N_{22} \end{bmatrix}^+ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = [A_1, A_2]^+ y \tag{3.25}$$

where the pseudoinverse of the partitioned matrix can be written as

$$[A_1 , A_2]^+ = \begin{bmatrix} (I + DD')^{-1} A_1^+ (I - A_2 B^+) \\ D'(I + DD')^{-1} A_1^+ (I - A_2 B^+) + B^+ \end{bmatrix} \tag{3.26a}$$

with

$$B := (I - A_1 A_1^+) A_2 \tag{3.26b}$$

and

$$D := A_1^+ A_2 (I - B^+ B), \tag{3.26c}$$

according to S.L. Campbell/C.D. Meyer (1979, p. 58, Theorem 3.3.3). The exploitation of this formula yields

$$B = A_2 - A_1 N_{11}^+ N_{12}, \tag{3.27a}$$

$$B'B = N_{22} - N_{21} N_{11}^+ N_{12} = S_2, \tag{3.27b}$$

and thus

$$D = A_1^+ A_2 [I - (B'B)^+ B'B] = N_{11}^+ N_{12} (I - S_2^+ S_2). \tag{3.27c}$$

In full analogy to (3.9) we can conclude here, too, that for the second « Schur complement »

$$\operatorname{rk} S_2 = \operatorname{rk}\begin{bmatrix} N_{11}, & 0 \\ 0, & S_2 \end{bmatrix} - \operatorname{rk} N_{11} =$$

$$= \operatorname{rk}\begin{bmatrix} N_{11}, & N_{12} \\ N_{21}, & N_{22} \end{bmatrix} - \operatorname{rk} N_{11} = \operatorname{rk} N_{22} - f \tag{3.28}$$

holds due to the relation (3.7) and a regular transformation dual to (3.10). Therefore, we can construct a $r \times f$ matrix Z_2 now fulfilling the two conditions

$$\operatorname{rk} Z_2 = f, \qquad \textit{i.e.} \dim R(Z_2) = f, \tag{3.29a}$$

$$S_2 Z_2 = 0, \qquad \textit{i.e.} \; R(Z_2) \subset N(S_2), \tag{3.29b}$$

corresponding to (3.11a, b) ; the third condition has been omitted because of N_{22} being regular.

In the same way as (3.14), it is easily shown that

$$I - S_2^+ S_2 = I - S_2 S_2^+ = Z_2 Z_2^+ \tag{3.30}$$

holds, implying

$$D = N_{11}^+ N_{12} Z_2 Z_2^+, \tag{3.31a}$$

and consequently

$$(I + DD')^{-1} = [I + N_{11}^{+} N_{12} Z_2 (Z_2' Z_2)^{-1} Z_2' N_{21} N_{11}^{+}]^{-1} =$$

$$= I - N_{11}^{+} N_{12} Z_2 (Z_2' Z_2 + Z_2' N_{21} N_{11}^{+} N_{11}^{+} N_{12} Z_2)^{-1} Z_2' N_{21} N_{11}^{+} =$$

$$= I - G_2 (Z_2' Z_2 + G_2' G_2)^{-1} G_2' \tag{3.31b}$$

with

$$G_2 := N_{11}^{+} N_{12} Z_2 \tag{3.31c}$$

due to a well known identity concerning the inverse of a sum of matrices; refer to H.V. Henderson/S.R. Searle (1981), e.g., for a review.

Finally, by inserting (3.27) and (3.31) into (3.26a), resp. this in turn into (3.25), we obtain the solution vector

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{vmatrix} \{ I - G_2 (Z_2' Z_2 + G_2' G_2)^{-1} G_2' \} \{ N_{11}^{+} b_1 - N_{11}^{+} N_{12} S_2^{+} (b_2 - N_{21} N_{11}^{+} b_1) \} \\ S_2^{+} (b_2 - N_{21} N_{11}^{+} b_1) + \\ + Z_2 (Z_2' Z_2 + G_2' G_2)^{-1} G_2' \{ N_{11}^{+} b_1 - N_{11}^{+} N_{12} S_2^{+} (b_2 - N_{21} N_{11}^{+} b_1) \} \end{vmatrix} \tag{3.32}$$

which can be proved to fulfill the normal equations (1.5), in fact.

Note that its appropriate dispersion matrix is known to become the pseudo-inverse

$$D\left(\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \right) = \sigma^2 \begin{bmatrix} N_{11} & , & N_{12} \\ N_{21} & , & N_{22} \end{bmatrix}^{+} = \sigma^2 \begin{bmatrix} C_{11} & , & C_{12} \\ C_{21} & , & C_{22} \end{bmatrix} \tag{3.33}$$

with the upper left block matrix now readily found by

$$\boxed{C_{11} = [I - G_2 (Z_2' Z_2 + G_2' G_2)^{-1} G_2'] (N_{11}^{+} + N_{11}^{+} N_{12} S_2^{+} N_{21} N_{11}^{+}) [I - G_2 (Z_2' Z_2 + G_2' G_2)^{-1} G_2']} \tag{3.34}$$

Note the difference again between (3.34) and (3.24), resp. (3.6) established by G. Marsaglia/G.P.H. Styan (1974, p. 441, Corollary 2) if regarding formula (3.7) with $f \neq 0$ as above.

(*iv*) *The « naive approach »* (2.4) :

This case is a slightly different one in comparison with the others treated so far, since now we have to secure, in advance, the *existence* of a reflexive symmetric g-inverse

$$\begin{bmatrix} N_{11} , & N_{12} \\ N_{21} , & N_{22} \end{bmatrix}^{-}_{rs} = \begin{bmatrix} N_{11}^{+} , & C_{12} \\ C_{21} , & C_{22} \end{bmatrix} \tag{3.35}$$

with suitable block matrices $C_{12} = C'_{21}$ and $C_{22} = C'_{22}$, according to (1.7) and (2.4b, c), while observing rk $C_{22} =$ rk N_{22}. Note that this approach has some relevance to optimization problems, too, see B. Schaffrin (1981).

Suppose, for the moment, that (3.35) holds ; then we conclude the necessary conditions

$$\begin{bmatrix} N_{11} , & N_{12} \\ N_{21} , & N_{22} \end{bmatrix} \cdot 2 \begin{bmatrix} N_{11}^{+} , & C_{12} \\ C_{21} , & C_{22} \end{bmatrix} \begin{bmatrix} N_{11} , & N_{12} \\ N_{21} , & N_{22} \end{bmatrix} =$$

$$= 2 \begin{bmatrix} N_{11} , & N_{12} \\ N_{21} , & N_{22} \end{bmatrix} + \begin{bmatrix} N_{12}(C_{22} N_{21} + 2C_{21} N_{11}) , & N_{12}(C_{22} N_{22} + 2C_{21} N_{12}) \\ N_{22}(C_{22} N_{21} + 2C_{21} N_{11}) , & N_{22}(C_{22} N_{22} + 2C_{21} N_{12} - N_{22}^{-1} S_2) \end{bmatrix} +$$

$$+ \begin{bmatrix} (N_{12} C_{22} + 2N_{11} C_{12}) N_{21} , & (N_{12} C_{22} + 2N_{11} C_{12}) N_{22} \\ (N_{22} C_{22} + 2N_{21} C_{12}) N_{21} , & (N_{22} C_{22} + 2N_{21} C_{12} - S_2 N_{22}^{-1}) N_{22} \end{bmatrix} =$$

$$= 2 \begin{bmatrix} N_{11} , & N_{12} \\ N_{21} , & N_{22} \end{bmatrix} \tag{3.36}$$

in order to be a g-inverse, as well as

$$\begin{bmatrix} N_{11}^{+} , & C_{12} \\ C_{21} , & C_{22} \end{bmatrix} \begin{bmatrix} N_{11} , & N_{12} \\ N_{21} , & N_{22} \end{bmatrix} \cdot 2 \begin{bmatrix} N_{11}^{+} , & C_{12} \\ C_{21} , & C_{22} \end{bmatrix} =$$

$$= 2 \begin{bmatrix} N_{11}^{+} , & N_{11}^{+} N_{11} C_{12} \\ C_{21} N_{11} N_{11}^{+} , & C_{22} \end{bmatrix} + \begin{bmatrix} C_{12}(N_{22} C_{21} + 2N_{21} N_{11}^{+}), & C_{12}(N_{22} C_{22} + 2N_{21} C_{12}) \\ C_{22}(N_{22} C_{21} + 2N_{21} N_{11}^{+}), & C_{22}(N_{22} C_{22} + 2N_{21} C_{12}) - K_2 \end{bmatrix} +$$

$$+ \begin{bmatrix} (C_{12} N_{22} + 2N_{11}^{+} N_{12}) C_{21} , & (C_{12} N_{22} + 2N_{11}^{+} N_{12}) C_{22} \\ (C_{22} N_{22} + 2C_{21} N_{12}) C_{21} , & (C_{22} N_{22} + 2C_{21} N_{12}) C_{22} - K_2 \end{bmatrix} =$$

$$= 2 \begin{bmatrix} N_{11}^{+} , & C_{12} \\ C_{21} , & C_{22} \end{bmatrix} \tag{3.37}$$

with

$$K_2 := C_{22} - C_{21} N_{11} C_{12} \tag{3.38}$$

in order to be reflexive. Then (3.36) implies the equalities

$$N_{12} C_{22} N_{21} + N_{12} C_{21} N_{11} + N_{11} C_{12} N_{21} = 0 \,, \tag{3.39a}$$

$$N_{12} C_{22} N_{22} + N_{12} C_{21} N_{12} + N_{11} C_{12} N_{22} = 0 \,, \tag{3.39b}$$

$$N_{22} C_{22} N_{22} + N_{22} C_{21} N_{12} + N_{21} C_{12} N_{22} - S_2 = 0 \,, \tag{3.39c}$$

and hence, for reasons of symmetry,

$$\begin{aligned} N_{12} C_{22} N_{21} &= - N_{12} C_{21} N_{11} - N_{11} C_{12} N_{21} = \\ &= - N_{12} C_{21} N_{12} N_{22}^{-1} N_{21} - N_{11} C_{12} N_{21} = \\ &= - N_{12} C_{21} N_{11} - N_{12} N_{22}^{-1} N_{21} C_{12} N_{21} = \\ &= N_{12} N_{22}^{-1} S_2 N_{22}^{-1} N_{21} - N_{12} C_{21} N_{12} N_{22}^{-1} N_{21} - N_{12} N_{22}^{-1} N_{21} C_{12} N_{21} \end{aligned} \tag{3.40}$$

By putting these equalities skillfully together we obtain the relations

$$N_{12} C_{21} S_1 = 0 = S_1 C_{12} N_{21} \,, \tag{3.41}$$

and altogether finally

$$0 = N_{12} N_{22}^{-1} S_2 N_{22}^{-1} N_{21} = N_{12} N_{22}^{-1} N_{21} - (N_{12} N_{22}^{-1} N_{21}) \, N_{11}^{+} (N_{12} N_{22}^{-1} N_{21}) \tag{3.42a}$$

or even equivalently, see e.g. K.R. Koch (1980, p. 50, formula (153.6)),

$$\boxed{N_{12} N_{22}^{-1} = N_{12} N_{22}^{-1} (N_{21} N_{11}^{+}) \, N_{12} N_{22}^{-1}} \tag{3.42b}$$

as the *fundamental* (*necessary*) *condition* for the existence of a g-inverse of type (3.35) ; it means that $N_{21} N_{11}^{+}$ *must* turn out to be a g-inverse of $N_{12} N_{22}^{-1}$ in order that (3.35) applies!

On the other hand, the g-inverse in (3.35) has to become necessarily positive-semidefinite, and hence there exists a suitable $r \times n$ matrix F with

$$\begin{bmatrix} A_1^+ \\ F \end{bmatrix} \begin{bmatrix} A_1^+ \\ F \end{bmatrix}' = \begin{bmatrix} A_1^+(A_1^+)' & , & A_1^+F' \\ F(A_1^+)' & , & FF' \end{bmatrix} = \begin{bmatrix} N_{11}^+ & , & C_{12} \\ C_{21} & , & C_{22} \end{bmatrix} \tag{3.43a}$$

yielding immediately

$$N_{11}^+ N_{11} C_{12} = N_{11}^+ N_{11} A_1^+ F' = A_1^+ F' = C_{12} \tag{3.43b}$$

and consequently with (2.4c) and (3.7)

$$\begin{aligned} \text{rk } K_2 &= \text{rk} \begin{bmatrix} N_{11}^+ & , & C_{12} \\ C_{21} & , & C_{22} \end{bmatrix} - \text{rk } N_{11}^+ = \\ &= \text{rk} \begin{bmatrix} N_{11} & , & N_{12} \\ N_{21} & , & N_{22} \end{bmatrix} - \text{rk } N_{11} = \text{rk } C_{22} - f, \end{aligned} \tag{3.43c}$$

according to D. Carlson et al. (1974, p. 170, Theorem 1), since K_2 is yet the second « Schur complement » of the g-inverse in (3.35).

Moreover, due to (2.4c) the matrix C_{22} proves to be invertible ; thus (3.37), if observing (3.43b), gives the particular equalities

$$C_{12} N_{22} C_{21} + C_{12} N_{21} N_{11}^+ + N_{11}^+ N_{12} C_{21} = 0 , \tag{3.44a}$$

$$C_{12} N_{22} C_{22} + C_{12} N_{21} C_{12} + N_{11}^+ N_{12} C_{22} = C_{12} - N_{11}^+ N_{11} C_{12} = 0 , \tag{3.44b}$$

$$C_{22} N_{22} C_{22} + C_{22} N_{21} C_{12} + C_{21} N_{12} C_{22} + C_{21} N_{11} C_{12} = C_{22} , \tag{3.44c}$$

from which we conclude by applying (3.44) to (3.41)

$$S_1 C_{12} = - S_1 N_{11}^+ N_{12} N_{22}^{-1} = N_{12} N_{22}^{-1} N_{21} N_{11}^+ N_{12} N_{22}^{-1} - N_{12} N_{22}^{-1} = 0 \tag{3.45}$$

because of (3.42b), respectively with (3.43b)

$$\boxed{C_{12} = N_{11}^+ N_{11} C_{12} = N_{11}^+ N_{12} N_{22}^{-1} N_{21} C_{12}} \tag{3.46a}$$

confining the block matrix $C_{12} = A_1^+ F' = C_{21}'$. From (3.44b) directly, we get the restraint

$$\boxed{C_{12} = -N_{11}^{+} N_{12} N_{22}^{-1} - C_{12} N_{21} C_{12} C_{22}^{-1} N_{22}^{-1}} \tag{3.46b}$$

and from (3.44a)

$$\boxed{C_{12} N_{22} C_{21} = -C_{12} N_{21} N_{11}^{+} - N_{11}^{+} N_{12} C_{21} = N_{11}^{+} N_{12} C_{22} N_{21} N_{11}^{+}} \tag{3.46c}$$

with the aid of (3.39a) and (3.43b) whereas (3.39c) yields immediately

$$\boxed{C_{22} = N_{22}^{-1} - N_{22}^{-1} N_{21} N_{11}^{+} N_{12} N_{22}^{-1} - C_{21} N_{12} N_{22}^{-1} - N_{22}^{-1} N_{21} C_{12}} \tag{3.46d}$$

Now we are in a position to show also that (3.42b) is *sufficient* for the existence of a g-inverse of type (3.35), namely by constructing a particular one. For if the restraints (3.46a, b, c, d) are closely scrutinized we may ascertain that they are fulfilled by setting

$$F := -N_{22}^{-1}(A_2' + N_{21} N_{11}^{+} A_1') \tag{3.47}$$

which gives in turn

$$C_{12} = F(A_1^{+})' = -2N_{22}^{-1} N_{21} N_{11}^{+} \tag{3.48a}$$

as well as

$$C_{22} = FF' = N_{22}^{-1} + 3N_{22}^{-1} N_{21} N_{11}^{+} N_{12} N_{22}^{-1}\,; \tag{3.48b}$$

note that here (3.42b) enters quite crucially. Now it is no more difficult to establish the matrix

$$\begin{bmatrix} N_{11}^{+} & , & -2N_{11}^{+}N_{12}N_{22}^{-1} \\ -2N_{22}^{-1} N_{21} N_{11}^{+} & , & N_{22}^{-1} + 3N_{22}^{-1} N_{21} N_{11}^{+} N_{12} N_{22}^{-1} \end{bmatrix} = \begin{bmatrix} N_{11} & , & N_{12} \\ N_{21} & , & N_{22} \end{bmatrix}_{rs}^{-} \tag{3.49}$$

to be a reflexive symmetric g-inverse of the normal equations matrix, in fact, belonging to the solution vector

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} N_{11}^+ & , & -2N_{11}^+ N_{12} N_{22}^{-1} \\ -2N_{22}^{-1} N_{21} N_{11}^+ & , & N_{22}^{-1} + 3N_{22}^{-1} N_{21} N_{11}^+ N_{12} N_{22}^{-1} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{3.50}$$

To sum up thus we may state the following

THEOREM: The partitioned normal equations matrix $N = [N_{ij}]$ has a reflexive symmetric g-inverse of type

$$\begin{bmatrix} N_{11} & , & N_{12} \\ N_{21} & , & N_{22} \end{bmatrix}_{rs}^{-} = \begin{bmatrix} N_{11}^+ & , & C_{12} \\ C_{21} & , & C_{22} \end{bmatrix}$$

with rk C_{22} = rk N_{22} *if and only if*

$$\boxed{N_{12} N_{22}^{-1} = N_{12} N_{22}^{-1} (N_{21} N_{11}^+) N_{12} N_{22}^{-1}}$$

holds, in which case (3.49) is one choice of it belonging to the solution vector (3.50)

4. — AN IMPORTANT INEQUALITY

Now we want to compare the various covariance matrices obtained by the respective approaches. For this purpose define the matrices

$$C_{11}(i) := S_1^+ = (N_{11} - N_{12} N_{22}^{-1} N_{21})^+ \tag{4.1}$$

as in (3.6), (3.15a), or (3.18),

$$C_{11}(ii) := N_{11}^+ + N_{11}^+ N_{12} S_2^+ N_{21} N_{11}^+ \tag{4.2}$$

as in (3.24),

$$\begin{aligned} C_{11}(iii) := {} & [I - G_2(Z_2' Z_2 + G_2' G_2)^{-1} G_2'](N_{11}^+ + N_{11}^+ N_{12} S_2^+ N_{21} N_{11}^+) \cdot \\ & \cdot [I - G_2(Z_2' Z_2 + G_2' G_2)^{-1} G_2'] \end{aligned} \tag{4.3}$$

as in (3.34), and

$$C_{11}(iv) := N_{11}^{+}, \quad \text{provided that} \quad N_{12} N_{22}^{-1} = N_{12} N_{22}^{-1} (N_{21} N_{11}^{+}) N_{12} N_{22}^{-1}, \tag{4.4}$$

as in (3.35) in connection with (3.42b).

Then we can establish by turns

$$S_1 C_{11}(i) S_1 = S_1 S_1^{+} S_1 = S_1, \tag{4.5a}$$

$$\begin{aligned} S_1 C_{11}(ii) S_1 &= S_1 N_{11}^{+}(N_{11} - N_{12} N_{22}^{-1} N_{21}) + S_1 N_{11}^{+} N_{12} S_2^{+} N_{21} N_{11}^{+} S_1 = \\ &= S_1 - N_{12} N_{22}^{-1} S_2 N_{22}^{-1} N_{21} + N_{12} N_{22}^{-1} S_2 S_2^{+} S_2 N_{22}^{-1} N_{21} = S_1 \end{aligned} \tag{4.5b}$$

because of

$$S_1 N_{11}^{+} N_{12} = N_{12} N_{22}^{-1} S_2, \tag{4.6}$$

further with (4.5b)

$$S_1 C_{11}(iii) S_1 = S_1(N_{11}^{+} + N_{11}^{+} N_{12} S_2^{+} N_{21} N_{11}^{+}) S_1 = S_1 \tag{4.5c}$$

since

$$\begin{aligned} S_1[I - G_2(Z_2' Z_2 + G_2' G_2)^{-1} G_2'] &= S_1 - S_1 N_{11}^{+} N_{12} Z_2(Z_2' Z_2 + G_2' G_2)^{-1} G_2' = \\ &= S_1 - N_{12} N_{22}^{-1} S_2 Z_2(Z_2' Z_2 + G_2' G_2)^{-1} G_2' = S_1 \end{aligned} \tag{4.7}$$

holds due to (3.29b), and finally

$$\begin{aligned} S_1 C_{11}(iv) S_1 &= (N_{11} - N_{12} N_{22}^{-1} N_{21}) N_{11}^{+} S_1 = \\ &= S_1 - [N_{12} N_{22}^{-1} - N_{12} N_{22}^{-1} (N_{21} N_{11}^{+}) N_{12} N_{22}^{-1}] N_{21} = S_1 \end{aligned} \tag{4.5d}$$

thus guaranteeing C_{11} (α) to be a (*positive-semidefinite*) *g-inverse* of the first « Schur complement » for every $\alpha \in \{i, ii, iii, iv\}$.

Unfortunately, at least C_{11} (ii) and C_{11} (iv) prove to be non-reflexive in consideration of (3.9) for $f \neq 0$, and hence the classical theorem concerning the trace of the pseudoinverse within the class of reflexive symmetric g-inverses, see e.g. K.R. Koch (1980, p. 61, Satz (156.1)), is *not* applicable.

However, according to A.J. Goldman/M. Zelen (1964, p. 152, Lemma 1), we can find a suitable matrix Z of full row rank fulfilling the conditions

$$\operatorname{rk} Z = u - \operatorname{rk} S_1, \qquad i.e.\ \dim R(Z) = \dim N(S_1), \tag{4.8a}$$

$$S_1 Z = 0 = S_1^{+} Z, \qquad i.e.\ R(Z) \subset N(S_1), \tag{4.8b}$$

and consequently

$$S_1^+ S_1 = S_1 S_1^+ = I - Z Z^+ = I - Z(Z' Z)^{-1} Z' \tag{4.9}$$

in full analogy to (3.29a, b), resp. (3.30). Therefore we conclude *for any positive-semidefinite g-inverse* S_1^-

$$S_1^+ = (S_1^+ S_1) S_1^- (S_1 S_1^+) = [I - Z(Z' Z)^{-1} Z'] S_1^- [I - Z(Z' Z)^{-1} Z'] , \tag{4.10a}$$

implying for the trace

$$\text{tr } S_1^+ = \text{tr } S_1^-[I - Z(Z' Z)^{-1} Z'] = \text{tr } S_1^- - \text{tr } ZZ^+ S_1^- ZZ^+ \leqq \text{tr } S_1^- \tag{4.10b}$$

since $ZZ^+ S_1^- ZZ^+$ becomes positive-semidefinite, too. Now direct application of (4.10b) leads to the *important inequality*

$$\text{tr } C_{11}(i) \leqq \text{tr } C_{11}(\alpha) \quad \text{for every} \quad \alpha \in \{i, ii, iii, iv\} . \tag{4.11}$$

On the other side, it is still an open question whether $C_{11}(i)$ has minimal trace also in the class of all block matrices C_{11} arising in (1.7), which would be true if these block matrices C_{11} turn out to be g-inverses of S_1, in any case. This may be conjectured!

5. — AN EXAMPLE

Let us consider a very simple horizontal triangulation network, namely just a triangle figured below. The observations are obtained from Table 1 with the standard deviations σ_{dir} in [mgon] and σ_{dis} in [mm]. The approximate coordinates are given by Table 2.

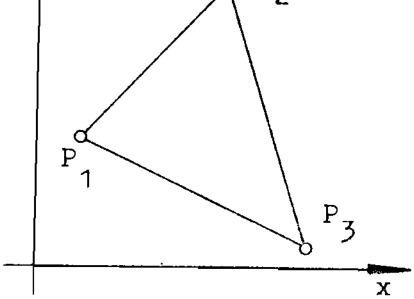

Fig. 1: A simple triangulation network

point	x [m]	y
1	30.00	40.00
2	70.00	80.00
3	90.00	10.00

TABLE 2: Approximate coordinates

from	to	directions [gon]	σ_{dir}	distances [m]	σ_{dis}
1	2	50.0010	1.0	56.576	7
1	3	129.5155	1.2	67.077	5
2	1	200.0005	0.5	56.566	3
2	3	132.2820	0.9	72.806	5
3	1	329.5175	0.8	67.087	5
3	2	382.2830	0.8	72.795	6

TABLE 1 : Observations

For the evaluation of the fomulas derived in chapter 3 let us introduce the $(u+r) \times (u+r-q)$ matrix E' with $\mathrm{rk}(E) = u+r-q = 3$, partioned for the unknown vector $x = [x_1', x_2']'$, such that

$$E = [E_1\,, E_2] := \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -y_1 & x_1 & -y_2 & x_2 & -y_3 & x_3 & \rho & \rho & \rho \end{bmatrix} \tag{5.1}$$

because the network may undergo two differential translations and one differential rotation. The third row of E contains the approximate coordinates related to their « center of gravity » and a factor of dimension for the direction measurements, respectively. With $AE' = 0$ one gets the conditions

$$S_1 E_1' = N_{12}(N_{22}^{-1} N_{22} - I)\, E_2' = 0 \tag{5.2a}$$

$$S_2 E_2' = N_{21}^{\bullet}\, (N_{11}^{+} N_{11} - I)\, E_1' = 0 \tag{5.2b}$$

From these it follows for $E_1 := [U_1', z_1']'$ and $E_2 := [U_2', z_2']'$

$$\mathrm{rk}(z_1) = \mathrm{rk}(z_2) = f = 1$$

as well as

$$S_1 z_1 = 0, \quad i.e.\ R(z_1) \subset N(S_1)\,, \tag{5.3a}$$

$$S_2 z_2 = 0, \quad i.e.\ R(z_2) \subset N(S_2)\,, \tag{5.3b}$$

hence Eq. (3.1a, b) and Eq. (3.29) are fulfilled. Further we have to prove Eq. (3.11c), which can easily derived from $\sum_i y_i = \sum_i x_i = 0$, because in our case

$$U_1' z_1 = 0 \tag{5.4}$$

is valid. With Eq. (5.4) and $N_{11} U_1 = 0$ it follows

$$z_1 \in R(U_1)^{\perp} = N(N_{11})^{\perp} = R(N_{11}) \tag{5.5}$$

and for $z_1 = N_{11} w$ with a certain vector w condition (3.11c) is also fulfilled.

With the previous considerations in mind the solutions for the different « free net adjustments » can be taken from Table 3 and Table 4. For the solution (*iv*) Eq. (3.42b) was not fulfilled, because

$$\| N_{12} N_{22}^{-1} (I - (N_{21} N_{11}^{+}) N_{12} N_{22}^{-1}) \|^2 = 0.365973 \ ,$$

and thus there exists no g-inverse as introduced in Eq. (3.35). As one can see, the solutions for the unknown vector and the appropriate covariance matrices are influenced by the choice of norm. The important feature is the trace of the covariance matrix for coordinate parameters only, also calculated in Table 4 for this example.

The computational aspects of the formulas derived in chapter 3 might be of more or less importance to some readers, and so we will also discuss them briefly. A main solution method for the computation of a pseudoinverse is the « Singular-Value-Decomposition » (SVD), however, for larger networks the run-time on a computer will increase quickly. Thus the use of the matrix E, perhaps partitioned for parts of the normal equations matrix, is recommended. Formulas for the computation of pseudoinverses by means of the matrix E are collected by K.R. Koch (1980, p. 56, 170), e.g.; from our formulas the exploitation of Eq. (3.15) cannot be recommended because of its sensitivity against roundoff errors.

TABLE 3 : Solutions for the unknown vector x

vector $\hat{x}$		solution (*i*)	solution (*ii*)	solution (*iii*)
Dim.: [decameter]	$\delta\hat{x}_1$	—0.00003225	—0.00003149	—0.00003169
	$\delta\hat{y}_1$	0.00004083	0.00003326	0.00003525
	$\delta\hat{x}_2$	—0.00000014	—0.00000847	—0.00000628
	$\delta\hat{y}_2$	—0.00007362	—0.00007210	—0.00007250
	$\delta\hat{x}_3$	0.00003239	0.00003996	0.00003797
	$\delta\hat{y}_3$	0.00003279	0.00003885	0.00003726
Dim.: [decagon]	$\delta\hat{o}_1$	—0.00004035	—0.00002589	—0.00002969
	$\delta\hat{o}_2$	—0.00003978	—0.00002532	—0.00002912
	$\delta\hat{o}_3$	0.00003676	0.00005122	0.00004742

TABLE 4 : Covariance matrix for the coordinate parameters only (without σ^2)

solution	C_{11}					
(i)	1.07333855	0.10216959	—0.26499315	—0.98904076	—0.80834540	0.88687116
	0.10216959	0.14620378	—0.12601521	—0.15081253	0.02384562	0.00460875
	—0.26499315	—0.12601521	0.16685746	0.19153428	0.09813569	—0.06551907
	—0.98904076	—0.15081253	0.19153428	1.26562897	0.79750648	—1.11481644
	—0.80834540	0.02384562	0.09813569	0.79750648	0.71020971	—0.82135209
	0.88687116	0.00460875	—0.06551907	—1.11481644	—0.82135209	1.11020769
tr $C_{11}(i)$	4.47244616					
(ii)	1.06882987	0.12424915	—0.24207987	—0.99581247	—0.82675000	0.87156333
	0.12424915	0.15548016	—0.10206897	—0.12910974	—0.02218018	—0.02637042
	—0.24207987	—0.10206897	0.20831478	0.21265890	0.03376509	—0.11058994
	—0.99581247	—0.12910974	0.21265890	1.25657680	0.78315357	—1.12746706
	—0.82675000	—0.02218018	0.03376509	0.78315357	0.79298492	—0.76097339
	0.87156333	—0.02637042	—0.11058994	—1.12746706	—0.76097339	1.15383748
tr $C_{11}(ii)$	4.63602401					
(iii)	1.06993329	0.11926016	—0.24720667	—0.99419567	—0.82272662	0.87493551
	0.11926016	0.14491774	—0.11729848	—0.13318731	—0.00196167	—0.01173043
	—0.24720667	—0.11729848	0.18759037	0.20889574	0.05961630	—0.09159726
	—0.99419567	—0.13318731	0.20889574	1.25863033	0.78529993	—1.12544301
	—0.82272662	—0.00196167	0.05961630	0.78529993	0.76311032	—0.78333825
	0.87493551	—0.01173043	—0.09159726	—1.12544301	—0.78333825	1.13717344
tr $C_{11}(iii)$	4.56135549					

REFERENCES

A.C. AITKEN (1934), *On least squares and linear combination of observations*. Proc. Roy. Soc. Edinb., A-**55** (1934), 42-47.

W. BAARDA (1973), *S-transformations and criterion matrices*. Neth. Geod. Comm., Publ. on Geodesy, New Series **5**, No. 1, Delft 1973.

A. BJERHAMMAR (1958), *A generalized matrix algebra*. Kungl. Tekn. Högskolan, Div. of Geodesy, Bull. No. 9, Stockholm 1958.

S.L. CAMPBELL and C.D. MEYER (1979), *Generalized Inverses of Linear Transformations*. London/San Francisco/Melbourne 1979.

D. CARLSON, E. HAYNSWORTH, T. MARKHAM (1974), *A generalization of the Schur complement by means of the Moore-Penrose inverse*. SIAM J. Appl. Math. **26** (1974), 169-175.

R.W. COTTLE (1974), *Manifestations of the Schur complement*. Lin Alg. Appl. **8** (1974), 189-211.

A.J. GOLDMAN and M. ZELEN (1964), *Weak generalized inverses and minimum variance linear unbiased estimation*. J. Res. Nat. Bur. Stand. B-**68** (1964), 151-172.

H.V. HENDERSON and S.R. SEARLE (1981), *On deriving the inverse of a sum of matrices*, SIAM Rev. **23** (1981), 53-60.

K.R. KOCH (1980), *Parameterschätzung und Hypothesentests in linearen Modellen*, Bonn 1980.

K.R. KOCH (1981), *Private Communication*.

G. MARSAGLIA and G.P.H. STYAN (1974), *Rank conditions for generalized inverses of partitioned matrices*. Sankhya A-**36** (1974), 437-442.

P. MEISSL (1962), *Die innere Genauigkeit eines Punkthaufens*, ÖZfV **50** (1962), 159-165 + 186-194.

P. MEISSL (1965), *Über die innere Genauigkeit dreidimensionaler Punkthaufen*. ZfV **90** (1965), 109-118.

P. MEISSL (1967), *Die verallgemeinerte Inverse einer modifizierten Matrix*. ZAMM **47** (1967), T66-T67.

P. MEISSL (1969), *Zusammenfassung und Ausbau der inneren Fehlertheorie eines Punkthaufens*, in: K. Rinner, K. Killian and P. Meissl, Beiträge zur Theorie der geodätischen Netze im Raum, DGK-Publ. A-**61**, München 1969, 8-21.

J. VAN MIERLO (1980), *Free network adjustment and S-transformations*, in: Beiträge aus der BR Deutschland zur Vorlage bei der XVII. Generalversammlung der IUGG vom 2.-14, Dez. 1979 in Canberra/Australien, DGK-Publ. B-**252**, München 1980, 41-54.

E. MITTERMAYER (1971), *Eine Verallgemeinerung der Methode der kleinsten Quadrate zur Ausgleichung freier Netze*. ZfV **96** (1971), 401-410.

E. MITTERMAYER (1972a), *Zur Ausgleichung freier Netze*. ZfV **97** (1972), 481-489.

E. MITTERMAYER (1972b), *A generalization of the least-squares method for the adjustment of free networks*. Bull. Géod. **46** (1972), 139-157.

H. PELZER (1974), *Zur Behandlung singulärer Ausgleichungsaufgaben*. ZfV **99** (1974), 181-194, 479-488.

R.L. PLACKETT (1949), *A historical note on the method of least ssquares*, Biometrika **36** (1949), 458-460.

R.L. PLACKETT (1950), *Some theorems in least squares*. Biometrika **37** (1950), 149-157.

A.J. POPE (1973), *The use of the « solution space » in the analysis of geodetic network adjustment*. Pap. pres. to the IAG-Symp. on Computat. Methods in Geometric Geodesy, Oxford/U.K., 2.-8. Sept. 1973.

R.M. PRINGLE and A.A. RAYNER (1971), *Generalized Inverse Matrices with Applications to Statistics*, New York 1971.

R.M. PRINGLE and A.A. RAYNER (1976), *Some aspects of the solution of singular normal equations with the use of linear restrictions*. SIAM J. Appl. Math. **31** (1976), 449-460.

C.R. RAO (1971), *Unified theory of linear estimation*, Sankhya A-**33** (1971), 371-394, + Corrigendum, Sankhya A-**34** (1972), 194.

C.R. RAO (1972), *Unified theory of least squares*. Communic. Statist. **1** (1972), 1-8.

B. SCHAFFRIN (1975), *Zur Verzerrtheit von Ausgleichungsergebnissen*. Mitt. Inst. Theoret. Geod. Univ., No. 39, Bonn 1975.

B. SCHAFFRIN (1981), *Some considerations on the optimal design of geodetic networks*, Pap. pres. to the IAG-Symp. on Geodetic Networks and Comput., München, 31. Aug.-5. Sept. 1981.

H. WOLF (1972), *Helmerts Lösung zum Problem der freien Netze mit singulärer Normalgleichungsmatrix*, ZfV **97** (1972), 189-192.

H. WOLF (1973), *Die Helmert-Inverse bei freien geodätischen Netzen*. ZfV **98** (1973), 396-398.