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Lecture Notes on

MULTIVARIATE DATA ANALYSIS *

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Abstract

The term ' multivariate data analysis ' is mostly referred to the subject of multivariate statistical inference. In the contrary also multivariate solution strategies can be applied to be considered by this paper.

The paper starts in its first part with array algebra - a strategy known since the sixties. Although the frame, in which array algebra can be applied, is very static there are applications well suited for this solution strategy.

The second part deals with multigrid methods. This strategy is able to solve large systems of linear equations with a minimum of computational efforts in time and storage.

Some examples prove at a time the applicability of the two solution strategies given by this paper.

1. INTRODUCTION

In order to solve least squares problems geodesists and photogrammetrists are well aware in dealing with large systems of linear equations. The algorithms according to Gauss and Cholesky are well known and well researched, so that these algorithms form the basis for solution strategies which overcome thousands of unknown parameters. Many contributions have been given in the past dealing with new variants and approaches (H. Wolf, 1978, P. Meissl, 1982, K.R. Koch, 1987) to reorder and to solve linear equation systems. Most of these contributions considered problems in net adjustment which led to sparse matrices.

But as far as other applications are concerned, for instance digital image processing and object descriptions, one does not always have sparse matrices. The advantage here is that some problems have underlying regular discretizations which can be decomposed in a multivariate sense. For that reason two multivariate solution strategies are introduced in the following to show up the efficiency using elementary mathematical descriptions.

2. FUNDAMENTALS OF ARRAY ALGEBRA

The subject of array algebra has been dealt with already in the sixties, when elementary considerations on the Kronecker product were given (H. Neudecker, 1969). At that time the name ' array algebra ' was not yet known - mathematicians called the underlying solution strategy ' Kronecker decompositions '. Since the seventies U.A. Rauhala gave comprehensive considerations on Kronecker decompositions (U.A. Rauhala, 1974, 1977, 1980, 1981, 1986). He introduced a new formalism to get in touch with multivariate problems. It is this new formalism which he gave the name ' array algebra '. Because of its fascinating possibilities to solve linear equations very efficient, also other authors dealt with array algebra (G. Blaha, 1977, R.A. Snay, 1978). Even if array algebra seems to be quite elegant some criticism should be pointed out in the way, that besides some special applications it could not become as important as other classical techniques are supposed to.

But nevertheless, when working in a special hardware environment it can

* Dedicated to the 80. anniversary of Prof.Dr.mult. H. Wolf, Bonn

contribute much in some cases to

- parallelisation techniques
- solve linear equation systems
- solve fast transforms in frequency domain.

In Fig. 1 a typical hardware configuration can be found to be used in digital image processing and digital photogrammetry.

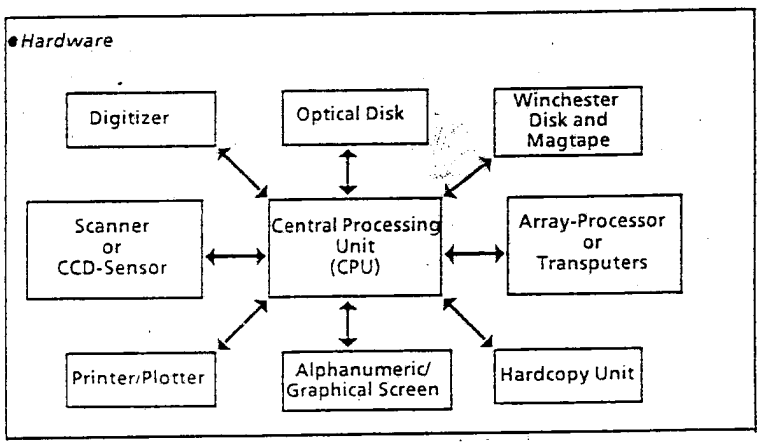


Fig. 1: Recent hardware environment in photogrammetry

Therefore, application fields of array algebra of today are

- signal processing (filter design, fast fourier transforms, convolutions)
- least squares approximations (finite element approaches, data snooping)
- digital image processing (contrast improvement, edge detection, image matching)
- robotics (pattern recognition, object reconstruction).

2.1 Mathematical approach

The application of array algebra is most efficient if gridded data have to be evaluated. The size of the grid most not necessarily be regular - in Fig. 2 three forms of gridded data can be seen.

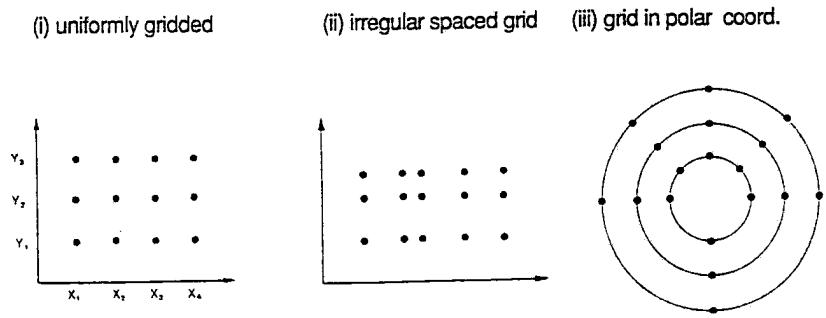


Fig. 2: Gridded data

There is no restriction using array algebra on two-dimensional grids only - the higher the grid dimension the more efficient is the approach.

Before going into details of array algebra its main advantage should be explained: The Kronecker separability. In numerical mathematics the Kronecker product ⊠ is defined as follows

$$\underset{(m,n)\ (r,s)}{G \boxtimes H} = \begin{bmatrix} g_{11}H & g_{12}H & \cdots & g_{1n}H \\ g_{21}H & g_{22}H & \cdots & g_{2n}H \\ \vdots & & & \vdots \\ g_{m1}H & g_{m2}H & \cdots & g_{mn}H \end{bmatrix} = \underset{(mr,ns)}{F} \qquad (1)$$

By the way a very simple example will illustrate the use of the Kronecker product: Let be given

$$G = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}, \qquad H = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

then G⊠H results into the matrix

$$F = \begin{bmatrix} 2 & 3 & 4 & 6 \\ 4 & 5 & 8 & 10 \\ 2 & 3 & 8 & 12 \\ 4 & 5 & 16 & 20 \end{bmatrix}$$

With the following rules for the Kronecker product in mind

$$(G \boxtimes H)' = G' \boxtimes H' \qquad \text{" transpose "} \qquad (2a)$$

$$(G_1 \boxtimes G_2)(H_1 \boxtimes H_2) = G_1H_1 \boxtimes G_2H_2 \qquad \text{" multiplication "} \qquad (2b)$$

$$(G_1 \boxtimes G_2) \boxtimes G_3 = G_1 \boxtimes (G_2 \boxtimes G_3) \qquad \text{" associative "} \qquad (2c)$$

$$(G_1 \boxtimes G_2)^{-1} = G_1^{-1} \boxtimes G_2^{-1} \qquad \text{" inverse "} \qquad (2d)$$

least squares solutions can be reformulated as demonstrated below.

Let us solve an example in approximation theory which allows Kronecker separation. The corresponding Gauss-Markov model is

$$\boxed{E(l) := l + v = Ax = (A_x \boxtimes A_y)\, x \quad , \quad D(l) := \sigma^2 P^{-1} = \sigma^2 (P_x^{-1} \boxtimes P_y^{-1})} \qquad (3)$$

it leads to the least squares estimation formula

$$\hat{x} = (A'PA)^{-1}A'Pl \qquad (4)$$

Using the rules for the Kronecker product (2) then (4) results into

$$\begin{aligned}\hat{x} &= [(A_x'\boxtimes A_y')(P_x\boxtimes P_y)(A_x\boxtimes A_y)]^{-1}(A_x'\boxtimes A_y')(P_x\boxtimes P_y)l\\ &= [(A_x'P_xA_x\boxtimes A_y'P_yA_y)]^{-1}(A_x'P_x\boxtimes A_y'P_y)l\\ &= [A_x'P_xA_x\boxtimes A_y'P_yA_y]^{-1}(A_x'P_x\boxtimes A_y'P_y)l\\ &= [(A_x'P_xA_x)^{-1}\boxtimes(A_y'P_yA_y)^{-1}](A_x'P_x\boxtimes A_y'P_y)l\\ &= [(A_x'P_xA_x)^{-1}A_x'P_x\boxtimes(A_y'P_yA_y)^{-1}A_y'P_y]l \qquad (5)\end{aligned}$$

The problem here is to avoid the computation of the Kronecker product. It is well known (H. Neudecker, 1969) that the vectorial linear equation $(A\boxtimes B)y$ can be expressed by a linear matrix equation BYA' with

$$y = \operatorname{vec} Y = \operatorname{vec}\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n}\\ y_{21} & y_{22} & \cdots & y_{2n}\\ \cdot & & & \cdot\\ \cdot & & & \cdot\\ y_{m1} & y_{m2} & \cdots & y_{mn}\end{bmatrix} = [y_{11},y_{21},\ldots,y_{m1},y_{12},\ldots]' \qquad (6)$$

Reformulating the model (3) delivers

$$\boxed{E(L) := L + V = A_yXA_x', \quad D(L) = \Sigma\boxtimes(P_x^{-1}\boxtimes P_y^{-1})} \qquad (7)$$

for which the following least squares solution is obtained

$$\hat{X} = (A_y'P_yA_y)^{-1}A_y'P_yLP_xA_x(A_x'P_xA_x)^{-1} \qquad (8)$$

The equivalence and computation of (5) and (8) is demonstrated by a further example. Let us search for an approximation

$$z = z(x,y), \quad z = \sum_{p=0}^{1}\sum_{q=0}^{1} c_{pq}x^py^q \qquad (9)$$

to be derived within a regular discretized grid. The observation equations obtained are

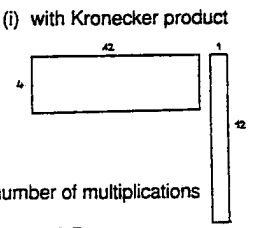
$$A := \begin{bmatrix} 1 & y_1 & x_1 & x_1y_1\\ 1 & y_2 & x_1 & x_1y_2\\ 1 & y_3 & x_1 & x_1y_3\\ 1 & y_1 & x_2 & x_2y_1\\ 1 & y_2 & x_2 & x_2y_2\\ \cdot & & & \cdot\\ \cdot & & & \cdot\\ \cdot & & & \cdot\\ 1 & y_3 & x_4 & x_4y_3\end{bmatrix}, \quad x := \begin{bmatrix} c_{oo}\\ c_{o1}\\ c_{1o}\\ c_{11}\end{bmatrix} \quad l := \begin{bmatrix} z_{11}\\ z_{12}\\ z_{13}\\ z_{21}\\ z_{22}\\ \cdot\\ \cdot\\ \cdot\\ z_{43}\end{bmatrix}, \quad v := \begin{bmatrix} v_{11}\\ v_{12}\\ v_{13}\\ v_{21}\\ v_{22}\\ \cdot\\ \cdot\\ \cdot\\ v_{43}\end{bmatrix} \qquad (10a)$$

These observation equations can be Kronecker-separated according to $(A_x \boxtimes A_y)x = l+v \Rightarrow A_y X A_x' = L+V$

$$A_x = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix}, \quad A_y = \begin{bmatrix} 1 & y_1 \\ 1 & y_2 \\ 1 & y_3 \end{bmatrix}, \quad \boxed{\text{Proof: } A = A_x \boxtimes A_y} \qquad (10b)$$

$$X = \begin{bmatrix} c_{oo} & c_{1o} \\ c_{o1} & c_{11} \end{bmatrix}, \; L = \begin{bmatrix} z_{11} & z_{21} & z_{31} & z_{41} \\ z_{12} & z_{22} & z_{32} & z_{42} \\ z_{13} & z_{23} & z_{33} & z_{43} \end{bmatrix}, \; V = \begin{bmatrix} v_{11} & \cdots & v_{41} \\ v_{12} & \cdots & v_{42} \\ v_{13} & \cdots & v_{43} \end{bmatrix}$$

The main advantage of the Kronecker separation is to avoid the Kronecker product which has to be demonstrated for the example introduced before. In Fig. 3 the computational effort is given with and without using the Kronecker product. It can be seen that avoiding this product saves computing time and space.



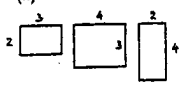
(i) with Kronecker product

- number of multiplications

48 for KP
48 for solution
---
96 in all

- storage space

48 for KP-matrix
12 for right hand side
---
60 in all



(ii) without Kronecker product

24 last products
12 first products
---
36 in all

6 + 12 + 8
---
26 in all

Fig. 3: Computational effort for the example

The term ' array algebra ' is now referred to rules when vectors and matrices are converted into arrays in the sense, that a k-array is a vector whose components are (k-1) arrays. This leads to arrays of different size, for example in

$$\begin{aligned} \mathbb{R}^1&: \text{ one - array} \quad (\text{" vector "}) \\ \mathbb{R}^2&: \text{ two - array} \quad (\text{" matrix "}) \\ \mathbb{R}^k&: \text{ k - array} \end{aligned} \qquad (11)$$

As shown before, the conversion of vectors into matrices and vice versa in $\mathbb{R}^2$ is carried out by means of the vec-operator

$$\mathbb{R}^2: \quad x = \text{vec } X \;, \quad l = \text{vec } L \;, \quad v = \text{vec } V \qquad (12)$$

But as far as higher dimensions are concerned more general rules have to be introduced. One general rule given by U.A. Rauhala is the ' R-matrix ' multiplication (R=Rauhala).

Let be X a k-dimensional array

$$X = \langle G_1, G_2, \ldots, G_k \rangle\ L \tag{13a}$$

with

$$X = (x_{i1,i2,\ldots,ik})$$

and the dimensions

$$\begin{aligned} o(X) &= m_1, m_2, \ldots, m_k \\ o(G_i) &= m_i, n_i \\ o(L) &= n_1, n_2, \ldots, n_k \end{aligned} \tag{13b}$$

then the R-matrix multiplication is defined as

$$x_{i1,i2,\ldots,ik} = \sum_{j1=1}^{n_1} \sum_{j2=1}^{n_2} \ldots \sum_{jk=1}^{n_k} g(1)_{i1j1} g(2)_{i2j2} \ldots g(k)_{ikjk} \cdot l_{j1j2\ldots jk}$$

$$1 \le i_r \le m_r\ ,\quad 1 \le r \le k\ ,\quad G_r = (g(r)_{ij}) \tag{14}$$

With reference to the example above in which we had

$$\hat{x} = (G_1 \otimes G_2)l \tag{15}$$

for the linear vector equation and

$$\hat{X} = G_2 L G_1' \tag{16}$$

for the linear matrix equation with substitutions $G_1 := (A_x'P_xA_x)^{-1}A_x'P_x$, $G_2 := (A_y'P_yA_y)^{-1}A_y'P_y$ the corresponding array algebra notation is

$$\hat{X} = \langle G_1, G_2 \rangle\ L \tag{17}$$

According to the rules of R-matrix multiplication one gets the following computing formula

$$\hat{X} = \langle G_1, I \rangle \{ \langle I, G_2 \rangle\ L \} \tag{18}$$

The efficiency of array algebra is proven by the following example from digital signal processing:

Design a lowpass filter from regular distributed 100x100 discretization points within the frequency domain. The number of filter coefficients should be K=L=33 which means the filter has rotational symmetry with $u_k = u_l = 17$ unknown parameters. Furthermore, the passband frequency $\omega_p = 0.07 * 2\pi$ and the stopband

frequency $\omega_s = 0.16 * 2\pi$ have to be considered. The results of the corresponding approximation problem are given in Fig. 4.

a) frequency response - linear scale -

b) approximation errors - log scale -

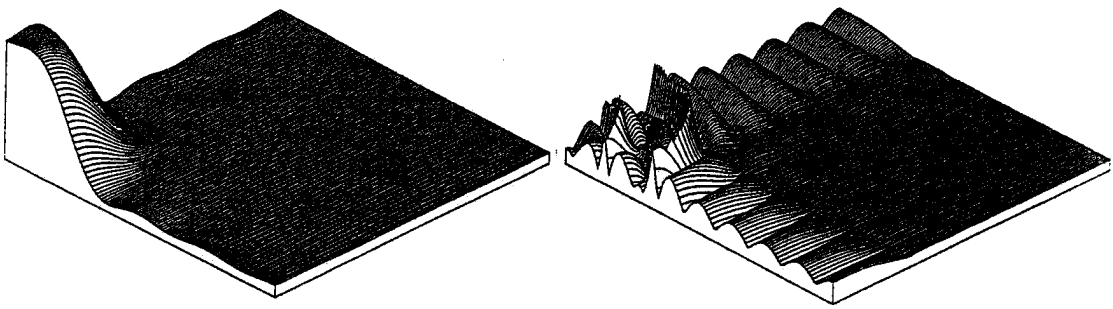


Fig. 4: Design of a lowpass filter

In order to show up the advantage of using array algebra only the normal equation system is reconsidered. Without array algebra notation one has to solve a normal equation matrix

$$\begin{array}{|l} o(A'PA) = 289 \times 289 \quad , \text{ resulting from} \\ o(A) \quad = 10\,000 \times 289 \end{array} \tag{19a}$$

Using array algebra these numbers are decreased to

$$\begin{array}{|l} o(A_x'P_xA_x) = o(A_y'P_yA_y) = 17 \times 17 \quad , \text{ resulting from} \\ o(A_x) = o(A_y) = 100 \times 17 \end{array} \tag{19b}$$

This gives clear benefits in the way that we have less computational effort in filter design and round-off errors are kept minimum.

2.2 Critics and outlook

This short excursion on array algebra showed that it can be very efficient in case that fixed discretization structures are available. If we do not have these regular discretization schemes then the computational efforts are increasing. This increase results from switching over from the array algebra structure to irregular structures and is in some applications considerable. Therefore, it is not recommended to solve nearly every problem by means of the Rauhala-approach - very often it exists alternatives leading also to less computational efforts, for instance, sparse matrix calculus and multigrid methods. But in some cases one can combine array algebra with efficient models used in economics, so that its potential may be broadened to solve approximation and statistical problems.

3. MULTIGRID METHODS

The multigrid method is a highly efficient iterative process for the solution of large sparse systems of linear equations. It has been developed during the beginning of the seventies and is being more and more applied in numerical mathematics, physics and engineering sciences.

The rate of convergence of this iterative procedure is independent of the size of the problem. Consequently the computational effort for the solution increases only linearly with the number of unknowns. A comprehensive review on multigrid methods can be found in W. Hackbusch/U. Trottenberg (1982) - application in DTM-generation has been given by H. Ebner/D. Fritsch (1986).

The main approach of multigrid methods is, that it iterates linear equation systems on grids of different grid size (see Fig. 5). It uses the fact that iterative procedures converge fast for the high frequency part of the solution but converge slow for the lower frequencies. For that reason they are called ' smoother '. If the lower frequencies of the dense grid are mapped onto a coarser grid we will have the high frequencies of the coarse grid and so on. This mapping is demonstrated by Fig. 5, in which four grids can be seen. The mapping operators are symbolically written as I_h^H or its inverse I_H^h if only two grids will be considered.

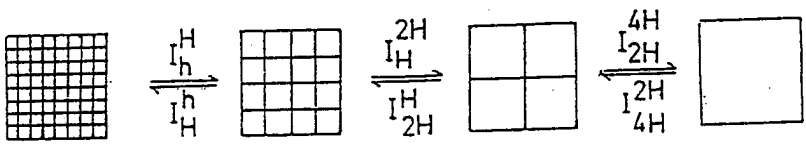


Fig. 5: Multigrid mappings

Multigrid methods can be found within solution strategies to solve difference equations, for instance in physics, engineering sciences and meteorology. In geodesy this approach is highly effective for ' banded ' normal equation systems.

3.1 Multigrid procedure

The multigrid principle for the solution of large linear equation systems for gridded unknowns is extremely simple: Approximations with smoothed errors are obtained by applying suitable relaxation methods, for instance Gauss-Seidel and Jacobi, respectively. Because of the error smoothness, corrections to the iterates have to be calculated on coarser grids. If this approach is used recursively employing coarser and coarser grids we obtain optimal iterative solutions.

Let be given the linear equation

$$N_h x_h = f_h \tag{20}$$

defined on a dense grid Ω_h with grid size h. Using the method of Gauss-Seidel or Jacobi as iterative (relaxational) solver, this can be described by

$$x_h^{(j)} = B_h x_h^{(j-1)} + f_h \quad , \quad B_h := I_h - N_h \tag{21}$$

The corresponding residuals or corrections contain dominating lower fre-

quencies because of the smoothing behaviour of the relaxational procedures

$$\Delta x_h^{(j)} := x_h - x_h^{(j)} \tag{22}$$

The multiplication with N_h of both sides of (22) leads to the ' defect ' equation system

$$N_h \Delta x_n^{(j)} = f_h - N_h x_h^{(j)} =: d_h^{(j)} \tag{23}$$

If this defect equation system can be approximated by

$$\hat{N}_h \Delta x_h^{(j)} = d_h^{(j)} \tag{24}$$

we arrive at a new iteration

$$x_h^{(j+1)} = x_h^{(j)} + \Delta x_h^{(j)} \tag{25}$$

The choice of $\hat{N}_h$, and this leads to the multigrid idea, is given by an approximation N_H of N_h on a coarser grid Ω_H. Therefore (24) will be replaced by

$$N_H \Delta x_H^{(j)} = d_H^{(j)} \quad , \quad N_H \text{ regular} \tag{26}$$

Because $d_H^{(j)}$ and $\Delta x_H^{(j)}$ are grid functions defined on a coarser grid one needs mapping operators between the grids Ω_h and Ω_H.

$$\Omega_h \xrightarrow{I_h^H} \Omega_H \quad , \quad \Omega_H \xrightarrow{I_H^h} \Omega h \tag{27}$$

The operator I_h^H restricts the defect $d_h^{(j)}$ to Ω_H whereas I_H^h interpolates $x_H^{(j)}$ to Ω_h.

$$d_H^{(j)} = I_h^H d_h^{(j)} \quad , \quad x_h^{(j)} = I_H^h \Delta x_H^{(j)} \tag{28}$$

The correction $\Delta x_h^{(j)}$, which results from a coarse grid, is also called ' coarse grid correction '. With the above equations in mind we are able to combine relaxation techniques with coarse grid corrections leading to a two-grid method, whereby each iteration step consists of smoothings and coarse grid corrections (see Fig. 6).

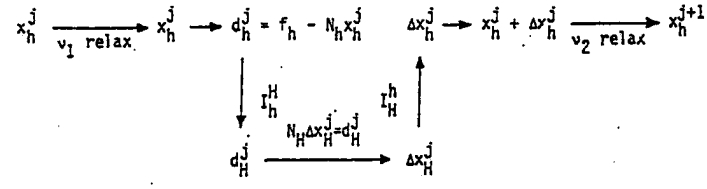


Fig. 6: Structure of a two-grid method

Standard mapping operators of multigrid methods are

$$I_h^H = \frac{1}{16}\begin{bmatrix}1 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 1\end{bmatrix}, \qquad I_H^h = \frac{1}{4}\begin{bmatrix}1 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 1\end{bmatrix} \tag{29}$$

refined operators can be obtained by linear system design to be derived by given frequency responses (see Fig. 7).

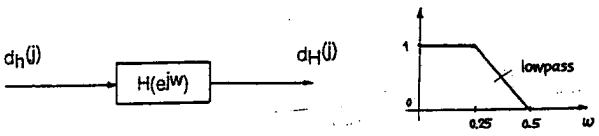


Fig. 7: Linear system design

3.2 Numerical examples

Before demonstrating a two-grid method iterative solution strategies must numerically be proven. Experience has shown, that the method of Gauss-Seidel converges fast, thus it is applied in the following.

3.2.1 Gauss-Seidel solver

Using (21) for an iterative description it can be expressed more ecplicitely

$$x^{(j)} = Lx^{(j)} + Rx^{(j-1)} + f \tag{30a}$$

with

$$x_i^{(j)} = \sum_{k=1}^{i-1} b_{ik}x_k^{(j)} + \sum_{k=i}^{n} b_{ik}x_k^{(j-1)} + f_i \tag{30b}$$

Let be given the linear equation system

$$\begin{aligned}
0.78x_1 - 0.02x_2 - 0.12x_3 - 0.14x_4 &= 0.76\\
-0.02x_1 + 0.86x_2 - 0.04x_3 + 0.06x_4 &= 0.08\\
-0.12x_1 - 0.04x_2 + 0.72x_3 - 0.08x_4 &= 1.12\\
-0.14x_1 + 0.06x_2 - 0.08x_3 + 0.74x_4 &= 0.68
\end{aligned}$$

which can be rewritten according to (30b) into

$$\begin{aligned}
x_1 &= 0.22x_1 + 0.02x_2 + 0.12x_3 + 0.14x_4 + 0.76\\
x_2 &= 0.02x_1 + 0.14x_2 + 0.04x_3 - 0.06x_4 + 0.08\\
x_3 &= -0.12x_1 - 0.04x_2 + 0.28x_3 - 0.08x_4 + 1.12\\
x_4 &= -0.14x_1 + 0.06x_2 - 0.08x_3 + 0.26x_4 + 0.68
\end{aligned}$$

In Table 1 the relaxations are given for different steps - also a comparison can be made with the exact solution obtained by a direct method.

Table 1: Relaxations according to Gauss-Seidel

	x_1	x_2	x_3	x_4
$x^{(o)}$	0.76	0.08	1.12	0.68
$x^{(1)}$	1.1584	0.1184	1.6317	1.1424
$x^{(2)}$	1.3730	0.1208	1.8379	1.3090
$x^{(3)}$	1.4683	0.1212	1.9204	1.3723
$x^{(4)}$	1.5080	0.1216	1.9533	1.3969
$x^{(5)}$	1.5242	0.1218	1.9664	1,4066
exact solut.	1.5350	0.1220	1.9752	1.4130

3.2.2 Two-grid method

In order to demonstrate the multigrid method a simple two-grid approach is carried out. The example uses digital terrain modelling to set up a linear equation system.

The following task has to be performed: Interpolate the heights of a grid of 3x3 by means of a least squares approximation with bilinear finite elements. Its reference information is given by Fig. 8 and Table 2, in which 5 reference points can be found.

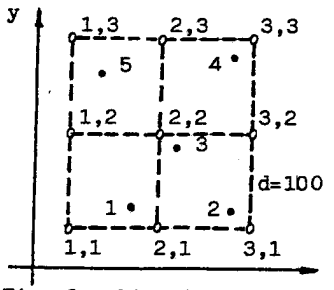


Fig. 8: Distribution of reference points

Table 2: Reference information

point	x	y	z
1	181.25	57.25	11.31
2	276.85	61.72	15.40
3	225.26	134.95	17.82
4	295.31	234.28	15.00
5	126.75	224.31	14.10

Setting up the observation equations of the finite element approach (H. Ebner et al., 1980) leads to the functional description

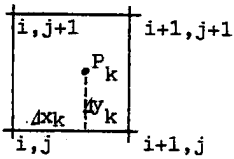
$$v_k = (1-\Delta x_k)(1-\Delta y_k)z_{ij} + \Delta x_k(1-\Delta y_k)z_{i+1,j} + (1-\Delta x_k)\Delta y_k z_{i,j+1} + \Delta x_k \Delta y_k z_{i+1,j+1} - z_k \qquad (31)$$

$$v_{xx,i,j} = z_{i-1,j} - 2z_{ij} + z_{i+1,j} - 0$$

$$v_{yy,i,j} = z_{i,j-1} - 2z_{ij} + z_{i,j+1} - 0$$

$$\Delta x_k := (x_k - x_i)/d$$

$$\Delta y_k := (y_k - y_i)/d$$



Finally, the method of least squares approximation results into (4). The numerical treatment of the observation equation is as follows:

$$v_1 = 0.1875*0.9275z_{11}+0.8125*0.9275z_{21}+0.1875*0.0725z_{12}+0.8125*0.0725z_{22}-11.31$$
$$v_2 = 0.2315*0.8828z_{21}+0.7685*0.8828z_{31}+0.2315*0.1172z_{22}+0.7685*0.1172z_{32}-15.40$$
$$v_3 = 0.7474*0.1505z_{21}+0.2526*0.1505z_{31}+0.7474*0.8495z_{22}+0.2526*0.8495z_{32}-17.82$$
$$v_4 = 0.0469*0.1572z_{22}+0.9531*0.1572z_{32}+0.0469*0.8428z_{23}+0.9531*0.8428z_{33}-15.00$$
$$v_5 = 0.7325*0.2569z_{12}+0.2675*0.2569z_{22}+0.7325*0.7431z_{13}+0.2675*0.7431z_{23}-14.10$$
$$v_6 = z_{11}-2z_{21}+z_{31}$$
$$v_7 = z_{12}-2z_{22}+z_{32}$$
$$v_8 = z_{13}-2z_{23}+z_{33}$$
$$v_9 = z_{11}-2z_{12}+z_{13}$$
$$v_{10} = z_{21}-2z_{22}+z_{23}$$
$$v_{11} = z_{31}-2z_{32}+z_{13}$$

For the setup of the normal equations these observation equations are weighted with $p_i=1$,$\forall i=1,2,3,4,5$, and $p_i=0.1,\forall i=6,7,\ldots,11$. The normal equations follow

z_{11}	z_{21}	z_{31}	z_{12}	z_{22}	z_{23}	z_{13}	z_{23}	z_{33}	
0.230243	-0.06895	0.1	-0.197636	0.010244	0	0.1	0	0	
..	1.12232	-0.005708	0.010244	-0.078646	0.04254	0	0.1	0	
..	..	0.661714	0	0.042535	-0.130741	0	0	0.1	
..	..	..	0.535596	-0.186267	0.1	-0.09757	0.037406	0	
..	..	..	..	1.212101	-0.060209	0.03741	-0.186048	0.005922	
..	..	..	..	..	0.576607	0	0.00592	-0.07695	
..	..	..	..	..	..	0.496285	-0.091800	0.1	
..	..	..	..	..	..	..	0.541076	-0.168249	
..	..	..	..	..	..	..	..	0.845248	
1.967	13.674	11.125	2.807	13.477	7.458	7.674	3.395	12.049	= 1'

Its parameter estimation obtained by a direct solution gives

$$\hat{z}_{11} = 7.979 \quad \hat{z}_{21} = 12.479 \quad \hat{z}_{31} = 16.807$$
$$\hat{z}_{12} = 11.816 \quad \hat{z}_{22} = 15.817 \quad \hat{z}_{23} = 17.359$$
$$\hat{z}_{13} = 14.826 \quad \hat{z}_{23} = 15.627 \quad \hat{z}_{33} = 15.148$$

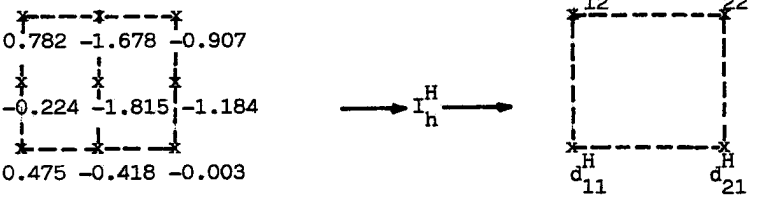
Now the two-grid cycle starts in the way that at first the Gauss-Seidel solver is used, which is the first step in a V-cycle

$x^{(o)}$	1.9669	13.6749	11.1250	2.8071	13.4778	7.4583	7.6749	3.3957	12.0491
$x^{(1)}$	2.9604	13.2156	14.5436	6.9469	13.0259	12.9845	10.5422	8.7142	13.8283
$x^{(2)}$	3.8878	12.6859	16.1409	8.5021	14.4968	15.5989	12.3008	11.8688	14.4984
$x^{(3)}$	4.5218	12.5586	16.8229	9.4171	14.9843	16.7729	13.4169	13.5939	14.8031

The defect results from $d_h^{(3)} = f_h - N_h x_h^{(3)}$

$d_h^{(3)}$ 0.4753 -0.4180 -0.0030 -0.2241 -1.8146 -1.1839 0.7824 -1.6776 -0.9070

it has to be mapped onto the coarsest grid



(1) mapping by means of operators

$$d_{11}^H = \frac{1}{9} |4*0.4753-2*0.4180-2*0.2241-1*1.8146| = -0.1331$$

$$d_{21}^H = \frac{1}{9} |4*-0.0030-2*0.4180-1*1.8146-2*1.1839| = -0.5589$$

$$d_{12}^H = \frac{1}{9} |4*0.7824-2*1.6776-2*0.2241-1*1.8146| = -0.2765$$

$$d_{22}^H = \frac{1}{9} |4*-0.907-2*1.6776-2*1.1839-1*1.8146| = -1.2406$$

(2) simple by reduction of the points in between

$$d_{11}^H = 0.4753 \quad , \quad d_{21}^H = -0.0030$$
$$d_{12}^H = 0.7824 \quad , \quad d_{22}^H = -0.9070$$

The mapping of the normal equation matrix N_h (approximation) is found by the definition of observation equations in the coarsest grid which leads to

$$\Delta v_1 = 0.5938*0.9638\Delta x_{11}^H + 0.4062*0.9638\Delta x_{21}^H + 0.5938*0.0312\Delta x_{12}^H + 0.4062*0.0312\Delta x_{22}^H$$
$$\Delta v_2 = 0.1158*0.9414\Delta x_{11}^H + 0.8842*0.9414\Delta x_{21}^H + 0.1158*0.0586\Delta x_{12}^H + 0.8842*0.0586\Delta x_{22}^H$$
$$\Delta v_3 = 0.3737*0.5752\Delta x_{11}^H + 0.6263*0.5752\Delta x_{21}^H + 0.3737*0.4248\Delta x_{12}^H + 0.6263*0.4248\Delta x_{22}^H$$
$$\Delta v_4 = 0.0234*0.0786\Delta x_{11}^H + 0.9766*0.0786\Delta x_{21}^H + 0.0234*0.9214\Delta x_{12}^H + 0.9766*0.9214\Delta x_{22}^H$$
$$\Delta v_5 = 0.8662*0.1284\Delta x_{11}^H + 0.1338*0.1284\Delta x_{21}^H + 0.8662*0.8716\Delta x_{12}^H + 0.1338*0.8716\Delta x_{22}^H$$

and furthermore to

Δx_{11}^H	Δx_{21}^H	Δx_{12}^H	Δx_{22}^H	d_H
0.3980	0.3943	0.1294	0.0847	0.4753
	.. 0.9821	0.0847	0.2150	-0.0030
		.. 0.5961	0.1503	0.7824
			.. 0.8969	-0.9070

The following solution vector is obtained

$$\Delta x_H = +1.4272 \quad -0.4154 \quad 1.3833 \quad -1.2782$$

which is mapped by bilinear interpolation onto the dense grid h

$$\Delta x^h_{11} = 1.4272 \quad \Delta x^h_{12} = 0.5059 \quad \Delta x^h_{13} = -0.4154$$
$$\Delta x^h_{21} = 1.4052 \quad \Delta x^h_{22} = 0.2792 \quad \Delta x^h_{23} = -0.8468$$
$$\Delta x^h_{31} = 1.3833 \quad \Delta x^h_{32} = 0.0526 \quad \Delta x^h_{33} = -1.2782$$

Therefore, the first coarse grid correction can be carried out leading to the solution

$$\hat{z}^{(1)}_{11} = 5.9490 \quad \hat{z}^{(1)}_{21} = 13.0645 \quad \hat{z}^{(1)}_{31} = 16.4075$$
$$\hat{z}^{(1)}_{12} = 10.8223 \quad \hat{z}^{(1)}_{22} = 15.2635 \quad \hat{z}^{(1)}_{32} = 15.9261$$
$$\hat{z}^{(1)}_{13} = 14.8002 \quad \hat{z}^{(1)}_{23} = 13.6465 \quad \hat{z}^{(1)}_{33} = 13.5249$$

Comment: For reasons of simplicity a simple bilinear interpolation was used to transport the coarse grid correction onto the dense grid. This can be the starting point for using standard multigrid operators. To complete the multigrid cycle further relaxations (1 or 2) should be performed.

4. CONCLUSIONS

It has been shown that Kronecker separations as well as iterative solution strategies nowadays become very attractive especially for large linear systems based on regular grids. As far as multigrid methods are concerned some investigations must be made in particular on the choice of mapping operators. But the idea is really fascinating - the experience up to now gives optimism to use it in general applications of digital image processing and also in the adjustment of geodetic networks.

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