A COMPUTER PROGRAMME FOR STRIP TRIANGULATION WITH INDEPENDENT MODELS

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1. INTRODUCTION

At the Institute for Photogrammetry of Stuttgart University in the summer of 1968 the development of a programme package for aerialtriangulation with independent models was started.

The first objective was the programming of a strip adjustment procedure. It was done with the intention of gaining programming experience for direct and iterative solutions of large systems of equations. Furthermore, strip triangulation still maintains a certain independent importance besides block triangulation and it is also still interesting for various scientific investigations. Thus the separate development of an efficient and optimized strip programme seemed expedient. It could be remarked that a programme for block adjustment should accommodate the strip just as a special case of a block. It is not conclusive however, to consider a strip programme superfluous as a block programme is not optimized with respect to the individual strip, and also several requirements are entirely different such as for instance those of the central core storage capacity of the computer and the use of external memory stores or search programmes (for tie-points).

The computer programme for strip adjustment which has been developed does therefore contain as a separate sub-programme that of strip formation (that is connexion of models into a strip) the transformation and the actual adjustment of the strip. The programme is available in both ALGOL and FORTRAN ASA-Norm computer languages. For the testing of the programme and for the practical applications carried out so far, the Telefunken TR 4 computer and the CDC 6600 computer of Control Data Corporation installed at Stuttgart University were used.

The programme working without external storage on the TR 4 is capable of handling, by assuming for instance 8 points per model, a maximum strip length of about 70 models. The CDC 6600 (with 128 K words

in the memory core as against 32 K words for the TR 4) allows the adjustment of very much longer strips. Thus the strip adjustment programme covers a full range of strip adjustment required for practical purposes and can be undertaken by medium capacity computers. As the computing times clearly indicate, the programme is very efficient which means that the costs for computing time are very low (see section 4).

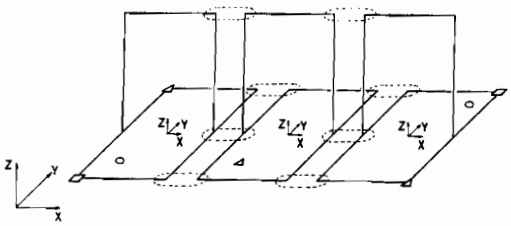


Figure 1. Scheme of strip adjustment with independent models

2. THE MATHEMATICAL PRINCIPLE OF THE STRIP ADJUSTMENT METHOD

The computer programme is based on the concept as shown in figure 1. For each model of the strip, the measured model coordinates (including both projection centres) are considered as given and these coordinates are related to an arbitrary cartesian coordinate system (X, Y, Z model system). Two requirements are made: firstly it is assumed that the z-axis points upwards, that means in approximately the same direction as the z-axis of the terrain system. In addition all coordinate systems, the terrain system included, must have the same sense of rotation. As for scale and orientation of the individual models, with respect to the terrain system, there are no restrictions. It is only necessary that the various model coordinate systems have more or less the same orientation and scale.

A spatial similarity transformation is made for each model. The transformation parameters (7 per model) of all models have to be determined simultaneously. The determination is based on terrestrial planimetric and height control points as well as the tie points with which adjacent models are interconnected. The projection centres are treated in the same manner as tie-points in the model space and their coordinates are assigned the same weight.

The conditions for control and tie-points are realized by considering the terrestrial coordinates in the transformation equations either as given (control points) or as unknowns (tie-points). By this method the adjustment problem is conveniently formulated in the form of observational equations which are suitable for automatic processing. The adjust-

ment of a strip consisting of n models with m tie-point coordinates (projection centres included,but tie-point-coordinates which are controlled excluded) has to determine in total 7n+m unknowns. From these the m unknown coordinates of the tie-points are eliminated from the normal equations and the reduced system of normal equations then to be solved contains only the 7n unknown transformation parameters. Points which have been measured once only and which do not contribute to any connection between models, are excluded from the adjustment. These points are read in together with all measuring data but then they are automatically sorted out and not carried through the adjustment. After all transformation parameters have been determined by the adjustment such points are included in the transformation of each model within each iteration step.

By using the method described the models are connected to each other and simultaneously fitted onto the terrestrial control points. Thus, the method is rigorous in as much as all conditions of interconnection and of fitting to ground control are taken into account simultaneously, and the solution is direct. The term rigorous method applies of course only to the mathematical concept of independent models and so in this sense is equivalent to the rigorous bundle adjustment method of H. Schmid |1| .

Since the spatial similarity transformation is non-linear the adjustment goes through the linearization of the observational equations starting from approximate values of the unknowns and reaches the final solution in an iterative way. In general three iteration cycles are sufficient.

The description of the procedure as a direct rigorous adjustment is not affected by the fact that the computer programme provides in fact first a strip formation (rigorous adjustment without the use of control points) and then a linear planimetric transformation of the strip onto all planimetric control points. These steps serve the purpose of giving good approximations for the final adjustment and they also provide means of finding and eliminating gross errors. Throughout these steps the ordering of the measuring data in models has to be maintained which means that in contradistinction with the more conventional procedures of strip adjustment the arithmetic mean will not be taken, after strip formation, of points measured twice.

3. STRIP ADJUSTMENT PROCEDURE

3.1 Relationship between model and terrain coordinates

For the strip adjustment with independent models the central computational operation is the spatial similarity transformation, for the observational equations of which the following approach has been chosen:

$$-v_{ij} + k_i = (1 + m_j)\, R_j\, k_{ij} + k_{oj} \qquad (1)$$

Herein are

i = point number assigned to the terrain point
j = model number
k_i = vector of the terrain coordinates of point i
k_{ij} = vector of model coordinates of point i in model j
k_{oj} = vector of translation
m_j = scale correction for model j
R_j = orthogonal rotation matrix for model j
v_{ij} = vector of the corrections of the transformed model coordinates to the terrestrial coordinates of control and tie-points

The point numbers i refer to the point in the terrain. They are maintained without additional coding for the corresponding model points (ij) and thus enforce the identity of points measured twice.

For reasons of convenience the following modification of the Rodriguez-Caylay matrix is used as the rotation matrix (see |2|):

$$R_j = \frac{1}{1+\frac{1}{4}(a^2+b^2+c^2)} \begin{bmatrix} 1+\frac{1}{4}(a^2-b^2-c^2) & -c+\frac{1}{2}\,ab & b+\frac{1}{2}\,ac \\ c+\frac{1}{2}\,ab & 1+\frac{1}{4}(-a^2+b^2-c^2) & -a+\frac{1}{2}\,bc \\ -b+\frac{1}{2}\,ac & a+\frac{1}{2}\,bc & 1+\frac{1}{4}(-a^2-b^2+c^2) \end{bmatrix}_j \qquad (2$$

For small rotations the parameters a, b, c correspond to the conventional notations for the three rotations ω, ϕ, κ. The approach (1) contains for each model 7 unknown orientation parameters (m, a, b, c,

x_o, y_o, z_o) and for each tie-point coordinate (x_i, y_i, z_i) one unknown. In the case of control points one, two or all three coordinates of the vector k_i are treated as known depending on whether the control point in question is a height or planimetric control point or whether it is of the combined type.

3.2 Linearized observational equations

The expansion of relationship (1) starting from approximate values $a_j = b_j = c_j = m_j = o$ gives the linearized observational equation

$$v_{ij} = T_{ij}\, t_j + k_i - k_{ij} \qquad (3)$$

In addition to the previously defined magnitudes v_{ij}, k_i, k_{ij} which are always referred to the system in which the linearization has been carried out, the equation (3) contains the terms t_j and T_{ij} which are defined as follows:

$$t_j = \left[a_j\ b_j\ c_j\ m_j\ x_{oj}\ y_{oj}\ z_{oj}\right]^T = \text{vector of the transformation parameters}$$

T_{ij} = the coefficient matrix of the linearized similarity transformation

Each measured model point (ij) contributes to the matrix T_{ij} the following coefficients:

$$T_{ij} = \begin{bmatrix} 0 & -z_{ij} & y_{ij} & -x_{ij} & -1 & 0 & 0 \\ z_{ij} & 0 & -x_{ij} & -y_{ij} & 0 & -1 & 0 \\ -y_{ij} & x_{ij} & 0 & -z_{ij} & 0 & 0 & -1 \end{bmatrix} \qquad (4)$$

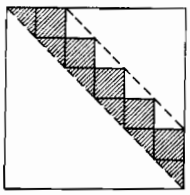
In the strip adjustment programme the weight matrix of the observations is assumed to be a unit matrix, that is, all measured model coordinates are considered as uncorrelated and are

given weight 1. The terrestrial control coordinates are considered as errorfree and thus receive no corrections. These premises are justified on the basis that for the time being they can be considered as sufficiently practicable.

3.3 Direct formation of a reduced system of normal equations

According to the standard rules of adjustment by indirect observations from linearized observational equations (3) the normal equation can be formulated, they can also clearly be classified according to the two groups of unknowns t_j and k_i. It has been previously demonstrated [3] that this special structure of the normal equation makes it possible to eliminate one of the two groups of unknowns and consequently directly set-up a reduced system of normal equations containing only one group of unknowns.

In general the number of transformation parameters t_j is smaller than the number of unknown coordinates of tie-points ($7\ n < m$). Therefore it is preferable to eliminate the coordinates k_i of the unknown tie-points from the normal equations. It is also useful to conveniently provide in the computer programme for what is known as format reservations for the reduced system, the number of unknown parameters t_j being just 7x the number of models of the strip, that means not dependent from the number of points measured.



7x7 submatrices,
band width 14

Figure 2. Scheme of reduced normal equations

Thus the computer programme omits the formation of the observational and normal equations and goes directly to calculate the non-zero coefficients and non-zero absolute terms of the reduced normal equations of the parameters t_j. This system of coefficients forms a simple and favourably structured band matrix (see figur 2). It is composed of square sub-matrices of 7 x 7 elements. In a strip a model is only connected with the 2 models adjoining it. Therefore the band width of the coefficient matrix (counted from the main diagonal) extends only over two sub-matrices to give a band width of 14 elements which is consi-

dered very narrow. No attempt will be made here to describe in detail the matrix operations to get the coefficients of the reduced normal equations system. It is necessary that attention is paid to distinguishing clearly the three types of control points. The model coordinates are introduced in units of kilometers in order to give the various coefficients of the reduced normal equations (for instance proportional to x^2, to x and scalar) approximately the same magnitude and thereby improve the conditioning of the equation system.

3.4 Solution of the reduced normal equations and transformation of individual models

The reduced system of normal equations which for n models contains 7n unknown parameters is directly solved according to the Cholesky method. This requires that the coefficient matrix of the system is positive definite. The mathematical approach to the adjustment problem as chosen here ensures that this is so. The Cholesky solution is equivalent to a direct Gauss solution as far as the numerical operations are concerned (see |4|). This method is even somewhat better with regard to the numerical acuity, particularly with very large systems of equations. The storage and the solution of the reduced normal equations is done entirely in the computer's central core without the need to employ external storage. The programme occupies about 10 000 words, in addition to that $100n_M + 4n_p$ words are required as working storage (where n_M = the number of models and n_p = the number of points measured). For instance with 20 points measured per model the TR 4 computer can handle strips with up to 50 models whilst with the CDC 6600 computer it would be possible to handle even more. As a consequence any strip adjustment which may occur in practice is adequately covered by the use of this programme and computers of this type.

After the numerical solution of the transformation parameters the transformed coordinates of the model points and the finally adjusted coordinates of all tie-points have to be calculated. For this one goes back to the original formulation (1) of the spatial similarity transformation and carries out the transformation model by model. Then by taking the mean of all points appearing twice is yielded the final adjusted coordinates of those points. It can be shown that thus taking the mean is equivalent to and gives the same results as a direct least squares solution.

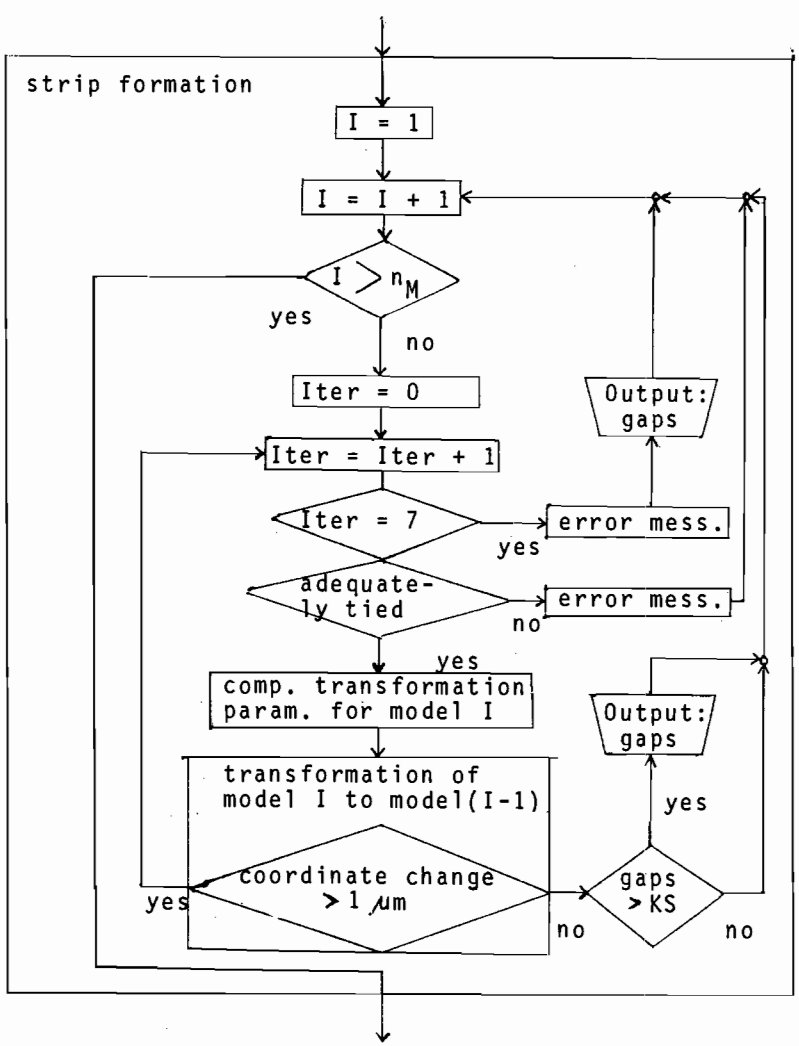
By comparison of the transformed model point coordinates with those of the terrestrial control points or adjusted tie-points one can obtain

the final corrections v_{ij} from which a number of statistical values are derived; namely, the standard error of unit weight and mean square values for various groups of the residuals.

3.5 The iteration process

Because of the approximate nature of the values to which the linearized observational equations (3) refer, the solution of the normal equations based thereon does not yet represent the final solution. The procedure has to be repeated with the improved approximations for the unknowns. There are several ways to do this. Our computer programme calculates using the improved transformation parameters the new model coordinates k_{ij}, by applying rigorously the spatial similarity transformation. Starting from the newly calculated model coordinates, the total process of adjustment is repeated. This means the observational equations are again linearized according to equation (3) and a reduced system of normal equations is formed and solved (see figure 3). The model coordinates k_{ij} which enter into the new iteration cycle are provided as a result of the previous iteration. This procedure corresponds to an iteration process of second order of the Newton type, which is distinguished by its very good convergency properties.

The iteration cycles are repeated until it is indicated by meeting the requirements of a precision criterium that further iterations would not yield substantial change anymore. The problem of which of the simple precision criteria for stopping further iterations will be reliable also with poorly conditioned systems has not been completely solved. Poor conditioning prevails in strip adjustment problems when only a few control points are given and in particular, when long strips are controlled only at the beginning and at the end. In such cases the expressions such as $[vv]$ or $[vv]_{k-1}/[vv]_k$ are not reliable. In our strip adjustment programme the iterations are interrupted and brought to an end if the maximum absolute value amongst all coordinate increments compared with the previous iteration amounts to less than FAK·1 µm in image scale. The factor FAK is read in with the measuring data and is in general equal to 1. According to the experience gained so far this criterium



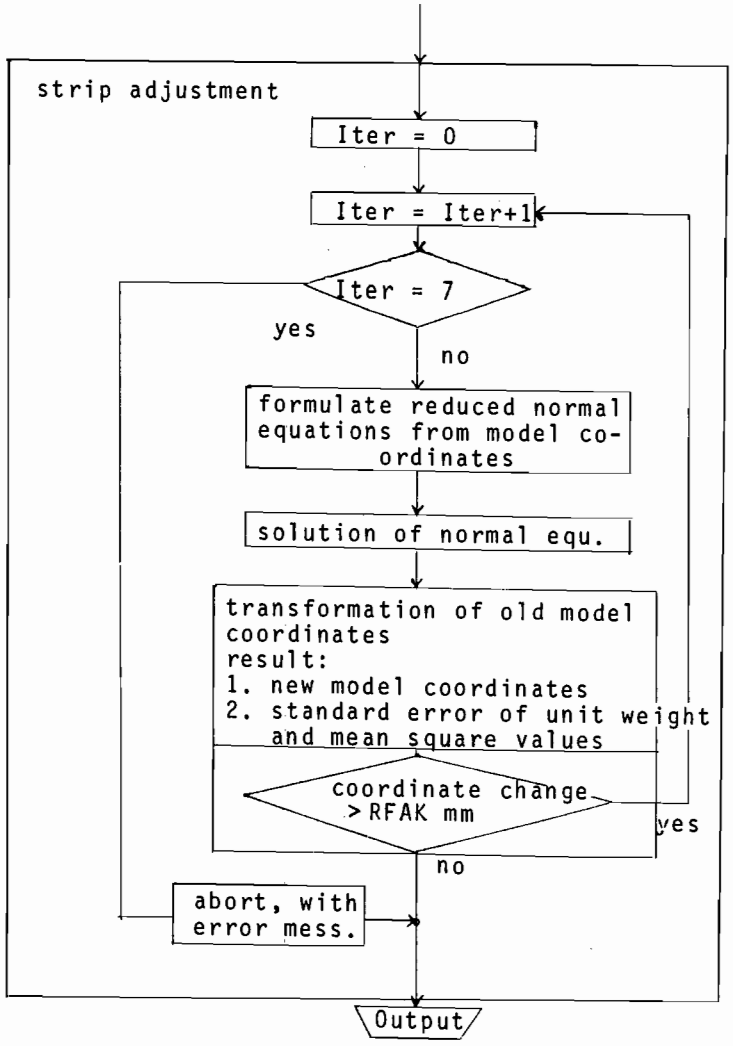


Figure 3: Flow diagram for stripformation and stripadjustment

has proven satisfactory. In general three iterations are sufficient to complete the strip adjustment.

4. Comparisons between two methods of solving the normal equations

It was our intention to gain experience from this strip adjustment programme, in particular with respect to the numerical solution of large systems of equations. Furthermore it was hoped to be able to extrapolate from such experience to the problem of solving even larger systems of equations as might occur in block triangulation. Two methods for solving reduced normal equations have been programmed: First the Cholesky method for a direct solution and second the method of conjugate gradients which provides an iterative solution (see |5|).

Both methods have been programmed with comparable degrees of optimization. Although the comparisons are based on hypothetical photogrammetric strips they can be considered realistic enough also for practical cases.

4.1 Theoretical estimation of the required computing operations

It is possible to theoretically estimate the number of numerical operations required for both methods. If n_M is the number of models in a strip and if the numerical commitment is gaged by the number of multiplications required (neglecting some other operations) all the following affords are a useful metre for comparison. Thus, for the solution of the reduced normal equations by the

- conjugate gradients method:

  $\sim 150\ n_M$ multiplications per iteration are required and for the

- Cholesky method:

  $\sim 640\ n_M$ multiplications are required.

4.2 Empirical comparison

With the Telefunken TR 4 computer the following computing times for solving the reduced normal equations have been obtained with the ALGOL programme:

- Conjugate gradients

10 models	10 iterations	6 sec	= 0,060 sec/iteration and model
	26	14	0,054
30 models	15	32	0,071
	50	102	0,068
	68	154	0,075
	112	226	0,067
	143	289	0,067

average computing time about $0.07 \cdot n_M$ sec/Iteration
computing time for one multiplication t_{Mult} = 0,0005 sec. (according to 4.1)

- Cholesky

30 models 5.5 sec = 0.18 sec/model
50 models 9 sec = 0.18 sec/model

average computing time $0.18 \cdot n_M$ sec
computing time for one multiplication t_{Mult} = 0,0003 sec. (according to 4.1)

(The shorter time for multiplication is to be expected here as the Cholesky method works in loops with linear progression).

Practical computing times equate oneCholesky solution of the normal equations with about 2.5 iterations for the method of conjugate gradients. If one considers the different times for multiplication, the theoretical estimation is therefore confirmed.

The solution of the normal equations represents only a part of the total process of strip adjustment, in particular, according to 3.5, after the solution of the normal equations a new linearization has to take place. Both methods investigated have to undertake such iterations, these for the sake of clearity are denoted as first adjustment, second adjustment and so on. The first practical calculations had already shown that the Cholesky method which involves the direct solution of normal equations would require only two, at most three adjustments whilst the method of conjugate gradients with the same number of ad-

justments would require in total several hundred iteration cycles. We did in fact dive deeper into the empirical investigations because the method of conjugate gradients which had been programmed first showed unexpectedly long computing times.

4.3 Comparison for the total adjustment

For the investigation of the convergence properties of the two methods it was necessary to provide somewhat more realistic data. For this purpose the theoretically ideal model coordinates which were introduced originally were transformed by random transformation parameters. In addition to that random observational errors were superimposed on the model coordinates. These random figures were assigned the following standard deviations: $\sigma_a = \sigma_b = 0{,}05$; $\sigma_c = \sigma_m = \sigma_{xo} = \sigma_{yo} = \sigma_{zo} = 0{,}1$; $\sigma_x = \sigma_y = \sigma_z = 20$ µm in the image scale.

The repetition of the adjusting process was terminated if the incremental changes to the transformed model coordinates from the successive adjustments were smaller than 0,1 µm in the image scale or, if the alteration of the standard error of unit weight remained below 10^{-4} ($(\sigma_o^{k-1} - \sigma_o^{k})/\sigma_o^{k-1} < 10^{-4}$): For the strip adjustment the control points were always assumed to be located in the four corners of the strip.

Cholesky

With the solution of the normal equations according to Cholesky which was terminated always after two iterations, the standard error of unit weight $\sigma_o = \sqrt{(|vv|/n-u)}$ showed the following behaviour:

- strip of 30 models, 210 unknowns:
  - after strip formation σ_o = 39,84074 µm in image scale
  - after 1st adjustment σ_o = 22,74914 µm
  - after 2nd adjustment σ_o = 22,74911 µm
- strip of 50 models, 350 unknowns:
  - after strip formation σ_o = 29,30562 µm in image scale
  - after 1st adjustment σ_o = 20,66543 µm
  - after 2nd adjustment σ_o = 20,66542 µm

Conjugate gradients

For this method a strip with 30 models was used (the same input data as above). After three adjustment cycles (within which the reduced normal equations were always formed independently) and a total of 270 iterations σ_0 gave a value 22.75000 µm.

This solution was still not as good as the one obtained after the first Cholesky adjustment. Furthermore it was observed that the adjusted coordinates in the middle of the strip differred by 30 µm in the nagative scale compared with results of the Cholesky adjustment. This means that the solution had not yet iterated through although the second of the computational cut-off criteria was fulfilled.

4.4 Conclusions

The results of the empirical investigation have shown that for the numerical solution of the reduced normal equations about 2,5 iterations of the conjugate gradient method are equivalent to a single direct solution according to Cholesky. For the total adjustment the Cholesky method requires 2 to 3 adjustments independent of the strip length. With the same numerical commitment only about 5 to 8 iteration cycles of the conjugate gradient method could be carried out. But in fact with three linearization steps in total, several hundert iteration cycles were required and this number increases with increasing strip length.

The tests have confirmed that the conjugate gradient method converges sufficiently fast only if very good approximate values are introduced. Such approximate values are usually not directly available. Even if this problem could be solved, there would still remain the problem of where to terminate the iteration cycles in the case of the poor condition of the equation system. Having duely considered this we made a clear decision to discard the conjugate gradient method and apply the Cholesky method for the direct solution of the reduced normal equations. The appropriateness of this decision has in the meantime been borne out by additional experience with iteration methods.

5. Additional details

In the following a number of details about the various sub-programmes are collected.

5.1 Data input

The data input is subject to only very few formal conditions. Point numbering is arbitrary, but it must be referred to the terrain points (including the ground control points). Data must be arranged by models (as they are immediately obtained in the measuring of independent models). Each model number is followed by the listing, in arbitrary sequence, of the points measured in that model. Every point must be accompanied by: number of the associated terrain point, model coordinates x, y, z in hundredths of millimeters. Only the projection centres are in a sense excepted, as they must always appear as the first two points in the list of points of a model. The model end is signalized by the separation code -99. The control points with their ground coordinates (in metres) are grouped separatedly according to horizontal and vertical control points in two so-called zero models, both of which have the model number 0. The end symbol -999 signifies the end of the block of data.

The input appears in the print-out in the form of a complete list of read-in data.

5.2 Ordering of data

In this sub-programme the data are checked and models and points, respectively, are ordered into the best sequence for the strip formation and the strip adjustment.

5.3 Strip formation

In this sub-routine the strip is formed by successive spatial similarity transformation of each model to the preceding-one. The first model is kept fixed, no control points are used (see fig. 3):

- For each model connexion it is checked whether there are at least 3 tie-points available.
- The transformation of model j+1 to model j is iterated until all coordinate increments in model j+1 amount to less then 1 um in model scale. The number of iterations is printed out and is usually 2 to 4.
- After each model connexion the coordinate discrepensies at the tie-points are compared with a tolerance KS. If the x y z discrepancies are larger than KS, the coordinates of all the tie-points of this model connexion are printed out with their discrepancies. Strip formation is continued.
- The transformed model points are printed out after the strip formation with a list of the corrections. Each point gets a label: EP = single point, VP = tie-point, LH = control point in x y z, HO = height control point, LA = planimetric control point.

5.4 Preliminary transformation of the strip onto control points

In this sub-programme the strip is transformed onto all planimetric control points by means of a planimetric similarity transformation. By this large amounts of swing and large scale differences between strip and terrain system are corrected for.

5.5 Strip adjustment

The sub-programme for strip adjustment is composed of the direct formation of the reduced normal equations, and their solution, according to the Cholesky method, and the calculation of the transformed model coordinates. The strip adjustment is an iterative process which is continued until all coordinate increments between two successive iteration

cycles are smaller than RFAK mm in the terrain (see fig. 3). After each cycle the print-out shows σ_0 and mean square values of various groups of the corrections. The strip formation and the preliminary strip transformation counts as iteration cycle number 0. After completion of iteration cycles, the final terrain coordinates are calculated.

5.6 Print-out

The final print-out contains two lists of results:

- The final transformed points of the individual models with a list of the corrections and the labelling of the points according to the type of points: EP, VP, LH, HO, LA.
- The adjusted terrain coordinates of all points including the unaltered control points ordered according to increasing point numbers.

5.7 Correction for earth curvature and refraction

If required, a sub-programme to correct earth curvature and refraction can be called on. The correction is applied to the model coordinates after the first strip adjustment.

5.8 Computer storage requirements and computing times

For the three versions of the programme the following storage capacity is required:

TR 4 ALGOL programme	10 186 words
TR 4 FORTRAN programme	8 005 words
CDC 6 600 FORTRAN programme	12 633 words.

The approximate working storage space required is as follows:

number of words used = $100\ n_M + 4\ n_p$

where n_M = the number of models

n_p = the number of points measured in all models.

For the working storage the TR 4 computer has still about 9 000 words available. The computing time for the strip formation and the strip adjustment is virtually proportional to the number of models. For strips with not too many points per model we obtained the following values for the total computing time:

TR 4 FORTRAN $\sim$ 10 sec/model
CDC 6600 FORTRAN $\sim$ 0,4 sec/model.

SUMMARY

The principles and the performance of a computer programme which is available in ALGOL and FORTRAN ASA-Norm for strip adjustment by the method of independent models is described. It carries out strip formation, strip transformation and a rigorous least squares adjustment by a simultaneous solution for all transformation parameters by the Cholesky method. The solution usually requires three iteration cycles. The comparison of a direct (Cholesky) solution with an iterative solution by the method of conjugate gradients of the reduced normal equations is made with the results in favour of the direct solution. On the Telefunken TR 4 computer (32 K words) strips of up to 70 models can be treated. Using the Control Data computer CDC 6600 (128 K words) much longer strips may be adjusted. The computing time is about 10 sec. per a model on the TR 4 and 0,4 seconds on the CDC 6600.

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