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A universal computer program for analytical aerotriangulation

by H. Meixner, Stuttgart

Herbert Wichmann Verlag GmbH, 75 Karlsruhe 21, Rheinstraße 122

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1. Introduction

The development of computer programs for aerial triangulation at the Photogrammetric Institute of Stuttgart University started with the strip- and blockadjustment for independent models (PAT–M, [1]). At present the system of programs is being completed by a program for fully analytical aerial triangulation, also known as strip- and blockadjustment by the bundle method. The program is called PAT–B (B for bundle adjustment). Based on the same philosophy as the program for independent models (see [1]) also this program is intended to be very generally applicable with as little restrictions and limitations as possible. In particular there should be virtually no restrictions concerning block-size and number of points. In addition the handling of the program should be as comfortable as possible.

The general scope and specifications of triangulation programs as given in [1] are not repeated here. This paper rather intends to present the mathematical system and some features of the program, and to report empirical results concerning convergence and computing time.

2. Mathematical model

The mathematical model is based on the perspective relationship between terrain points and image points, also referred to as collinearity condition.

The basic relationship in linearized form is given by the observational equations (1).

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix}_{i,j} = f_j \cdot \begin{bmatrix} \bar{X}/\bar{Z} \\ \bar{Y}/\bar{Z} \end{bmatrix}_{i,j} - \begin{bmatrix} x \\ y \end{bmatrix}_{i,j} \tag{1}$$

$$\begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix}_{i,j} = R_j \cdot \begin{bmatrix} X_i - X_{oj} \\ Y_i - Y_{oj} \\ Z_i - Z_{oj} \end{bmatrix} \tag{1a}$$

$[x, y]^t_{i,j}$	Vector of image-coordinates of the point i in photo j.
$[v_x, v_y]^t_{i,j}$	Vector of corrections to image coordinates.
f_j	principal distance of photo j.
$[X_i, Y_i, Z_i]^t$	Vector of ground coordinates of point i.
$[X_{oj}, Y_{oj}, Z_{oj}]^t$	Vector of ground coordinates of the projection centre j.
R_j	Orthogonal Rodriguez-Matrix for photo j.

$$R_j = \frac{1}{k_j} \cdot \begin{bmatrix} 1 + (a_j^2 + b_j^2 - c_j^2)/4 & -c_j + a_j b_j/2 & b_j + a_j c_j/2 \\ c_j + a_j b_j/2 & 1 + (-a_j^2 + b_j^2 - c_j^2)/4 & -a_j + b_j c_j/2 \\ -b_j + a_j c_j/2 & a_j + b_j c_j/2 & 1 + (-a_j^2 - b_j^2 + c_j^2)/4 \end{bmatrix} \tag{2}$$

$$k_j = 1 + (a_j^2 + b_j^2 + c_j^2)/4 \tag{2a}$$

The Rodriguez-Matrix is, for electronical computation, advantageous in comparison to other orthogonal matrices, because it does not include trigonometric functions. For small tilts the parameters a, b, c match with conventional ω, φ, $\varkappa$.

The approach (1) contains as unknowns corrections to both the 6 elements of exterior orientation and the pass point ground coordinates. Linearization requires initial values, the determination of which will be discussed in section 3. Image coordinates are given weight 1. For theoretical and practical reasons (easier error finding) it is desirable to treat also the terrestrial coordinates of ground control points as observations to be corrected by the adjustment. The ground control points can be weighted individually, allowing per point a full 3×3 weight coefficient matrix. Considering plate coordinates x, y and ground control as observations gives finally the complete system of observational equations (3):

$$\begin{aligned} v_{i,j} &= K_{i,j} \cdot dk_j + L_{i,j} \cdot dt_j - f_{i,j} \\ v_{o,j} &= dk_j - f_{o,j} \end{aligned} \tag{3}$$

$v_{o,j}$ = Vector of residuals of terrestrial ground coordinates.
$f_{o,j}$ = Vector of constant terms.
$K_{i,j}$ = Coefficient matrix (4) of partial derivatives of the three unknown corrections of the object point coordinates (dk_j).
$L_{i,j}$ = Coefficient matrix (5) of partial derivatives of the six unknown corrections of exterior orientation (dt_j).
$f_{i,j}$ = Vector of constant terms of measured image point coordinates.
$v_{i,j}$ = Residuals of the measured image point coordinates (point j in photo i).

$$K_{i,j} = \frac{f_j}{\overline{Z}_{i,j}{}^0} \begin{bmatrix} 1 + (a^{02} - b^{02} - c^{02})/4 & -c^0 + a^0 b^0/2 & b^0 + a^0 c^0/2 \\ c^0 + a^0 b^0/2 & 1 + (-a^{02} + b^{02} - c^{02})/4 & -a^0 + b^0 c^0/2 \end{bmatrix}_j \tag{4}$$

$$L_{i,j} = \frac{f_j}{2\,\overline{Z}_{i,j}{}^0} \cdot \overline{L}_{i,j}(m, n) \qquad \begin{matrix} 1 \le m \le 2 \\ 1 \le n \le 6 \end{matrix} \tag{5}$$

$$\overline{L}_{i,j}(1,1) = a^0 (X_j{}^0 - X^0{}_{oj}) + b^0 (Y_i{}^0 - Y_{oj}{}^0) + c^0 (Z_i{}^0 - Z_{oj}{}^0) - \\ - \frac{\overline{X}^0{}_{i,j}}{\overline{Z}^0{}_{i,j}} [c^0 (X_i{}^0 - X^0{}_{oj}) + 2 (Y_i{}^0 - Y_{oj}{}^0) - a^0 (Z_i{}^0 - Z_{oj}{}^0)]$$

$$\overline{L}_{i,j}(1,2) = -b^0 (X_i{}^0 - X_{oj}{}^0) + a^0 (Y_i{}^0 - Y_{oj}{}^0) + 2 (Z_i{}^0 - Z_{oj}{}^0) - \\ - \frac{\overline{X}_{i,j}{}^0}{\overline{Z}_{i,j}{}^0} [-2 (X_i{}^0 - X_{oj}{}^0) + c^0 (Y_i{}^0 - Y_{oj}{}^0) - b^0 (Z_i{}^0 - Z_{oj}{}^0)]$$

$$\overline{L}_{i,j}(1,3) = -c^0 (X_i{}^0 - X_{oj}{}^0) + 2 (Y_i{}^0 - Y_{oj}{}^0) + a^0 (Z_i{}^0 - Z_{oj}{}^0) - \\ - \frac{\overline{X}_{i,j}{}^0}{\overline{Z}_{i,j}{}^0} [a^0 (X_i{}^0 - X_{oj}{}^0) + b^0 (Y_i{}^0 - Y_{oj}{}^0) + c^0 (Z_i{}^0 - Z_{oj}{}^0)]$$

$$\overline{L}_{i,j}(1,4) = -2 (1 + (a^{02} - b^{02} - c^{02})/4) + \frac{\overline{X}_{i,j}{}^0}{\overline{Z}_{i,j}{}^0} (-2 b^0 + a^0 c^0)$$

$$\overline{L}_{i,j}(1,5) = -(-2 c^0 + a^0 b^0) + \frac{\overline{X}_{i,j}{}^0}{\overline{Z}_{i,j}{}^0} \cdot (2 a^0 + b^0 c^0)$$

$$\overline{L}_{i,j}(1,6) = -(2 b^0 + a^0 c^0) + \frac{2\,\overline{X}_{i,j}{}^0}{\overline{Z}_{i,j}{}^0} \cdot (1 + (-a^{02} - b^{02} + c^{02})/4)$$

$$\overline{L}_{i,j}(2,1) = b^0 (X_i{}^0 - X_{oj}{}^0) - a^0 (Y_i{}^0 - Y_{oj}{}^0) - 2 (Z_i{}^0 - Z_{oj}{}^0) - \\ - \frac{\overline{Y}_{i,j}{}^0}{\overline{Z}_{i,j}{}^0} \cdot [c^0 (X_i{}^0 - X_{oj}{}^0) + 2 (Y_i{}^0 - Y_{oj}{}^0) - a^0 (Z_i{}^0 - Z_{oj}{}^0)]$$

$$\overline{L}_{i,j}(2,2) = a^0 (X_i{}^0 - X_{oj}{}^0) + b^0 (Y_i{}^0 - Y_{oj}{}^0) + c^0 (Z_i{}^0 - Z_{oj}{}^0) - \\ - \frac{\overline{Y}_{i,j}{}^0}{\overline{Z}_{i,j}{}^0} [-2 (X_i{}^0 - X_{oj}{}^0) + c^0 (Y_i{}^0 - Y_{oj}{}^0) - b^0 (Z_i{}^0 - Z_{oj}{}^0)]$$

$$\overline{L}_{i,j}\ (2, 3) = 2\,(X_i{}^0 - X_{oj}{}^0) - c^0\,(Y_i{}^0 - Y_{oj}{}^0) + b^0\,(Z_i{}^0 - Z_{oj}{}^0) - \frac{\overline{Y}_{i,j}{}^0}{\overline{Z}_{i,j}{}^0}\,[a^0\,(X_i{}^0 - X_{oj}{}^0) + b^0\,(Y_i{}^0 - Y_{oj}{}^0) + c^0\,(Z_i{}^0 - Z_{oj}{}^0)]$$

$$\overline{L}_{i,j}\ (2, 4) = -\,(2\,c^0 + a^0\,b^0) + \frac{\overline{Y}_{i,j}{}^0}{\overline{Z}_{i,j}{}^0}\,(-\,2\,b^0 + a^0\,c^0)$$

$$\overline{L}_{i,j}\ (2, 5) = -\,2\,(1 + (-\,a^{02} + b^{02} - c^{02})/4) + \frac{\overline{Y}_{i,j}{}^0}{\overline{Z}_{i,j}{}^0}\,(2\,a^0 + b^0\,c^0)$$

$$\overline{L}_{i,j}\ (2, 6) = -\,(-\,2\,a^0 + b^0\,c^0) + 2 \cdot \frac{\overline{Y}_{i,j}{}^0}{\overline{Z}_{i,j}{}^0}\,[1 + (-\,a^{02} - b^{02} + c^{02})/4]$$

Based on the observational equations (3) the program proceeds in principle straight forward to partially reduced normal equations, which contain only the unknown orientation parameters, the unknown terrain coordinates having been eliminated.

With the mathematical approach (3) and proper weighting of the observations the adjustment proceeds mathematically according to the method of least squares, approaching the final results by iterations.

3. Some basic attributes of the system

3.1 Initial values

Linearization of the observational equations (1) needs initial values for each aerial station (X_o, Y_o, Z_o), for the ground coordinates of the terrain points and also for the tilts (a, b, c). On those initial values it depends how fast we will get convergence of the iteration process. It might even happen that with poor approximations the iteration process will diverge. Therefore we tried to find a safe way to obtain easily rather good initial values and to accelerate also the iteration process of final bundle adjustment.

Creating something like a photo mosaic with all photos, we are computing a horizontal blockadjustment (Anblock) with the photos treated as independent units like in PAT–M, [1]. This enables us not only to generate ground coordinates for each image point, whose agreement with the final adjusted coordinates depends only on the magnitude of the height differences within the block and on the tilts, it also allows to locate and eliminate gross errors, such as misidentified or misnumbered points, before entering into the bundle adjustment. This preparatory program does not require initial values to be read in at all. The method of obtaining initial values is also entirely independent from the size and overlap conditions of the block, and also from ground control distribution. In particular there is no need for preliminary strip formation.

The displacements ΔX, ΔY of the plate-coordinate systems, as resulting from the Anblock-adjustment, are directly introduced as the horizontal coordinates of the aerial stations. The resulting scale factor of each image is multiplied with the principal distance to generate the flying height, and also the rotation parameters from the Anblock-adjustment are interpreted as rather good approximations of the final kappa values of the plates. Later on their influence on covergence will be shown to be considerable. The initial Z coordinate for the ground coordinates is taken as the root mean square value of the heights of all vertical control points.

A recent proposal is being investigated according to which the computation of initial values might be abandoned completely.

3.2 Normal equations

The elaborate process of sorting of the photos in the optimum sequence is taken over by the computer. Provided that identical points in different photos have been given the same point number by searching for identical numbers reordering of the photos can be obtained: Starting with a read in first "image-group" the computer will find all connections between them and between the second image group consisting of those images, which have points in common with image group one. Those newly sorted images determine the structure of the noımal equations. The normal equations are generated submatrix by submatrix and row by row and are externally stored. They are forming a banded matrix whose structure and whose band width depend on the size of submatrices and on the number of images forming one image-group. The size of such submatrices can be chosen by the program and can therefore be adapted to the size of the block and to the core capacity of the computer. The subprogram HYCHOL for the solution of symmetrical matrices – based on a Cholesky algorithm for submatrices – is the backbone of the PAT–M and PAT–B adjustment programs for aerial triangulation at our institute [1]. The program was written by H. KLEIN two years ago. With it we can estimate the computing time for the solution of normal equations with the CDC 6600 computer by the following formulae (6) very accurately:

$$SS\,[sec] = 3{,}8 \cdot 10^{-6}\, n_M + 2{,}5 \cdot 10^{-3}\, n_T + 0{,}1 n_T \cdot (12000 + 3\, t^2)/131072 \tag{6}$$

$$n_M = \sum_{1}^{u} ((s^2/2 - s/2 + 1/3)\, t^3 + 2\, st^2) \tag{6a}$$

$$n_T = \sum_{1}^{u} (3\, s^2/2 + 15\, s/2) \tag{6b}$$

SS = System time in seconds
n_M = Number of multiplications
n_T = Number of submatrix-transports between core and disc
u = Number of hyperrows
s = Number of submatrices per hyperrow
t = Number of columns in a submatrix

Obviously the image group read in first will be decisive for the resulting band width of the system. Skillful choice can considerably reduce the time needed for the solution of normal equations, as is demonstrated in the following example.

For a block of 17 photos two different first image groups are chosen: In figure 1 the photo number 2, in figure 2 the photos with the numbers 4 and 11. The roman numbers show the number of the respective image group.

It follows from equations (4) that the system time of case 1 will be about double that of case 2, due to different band width.

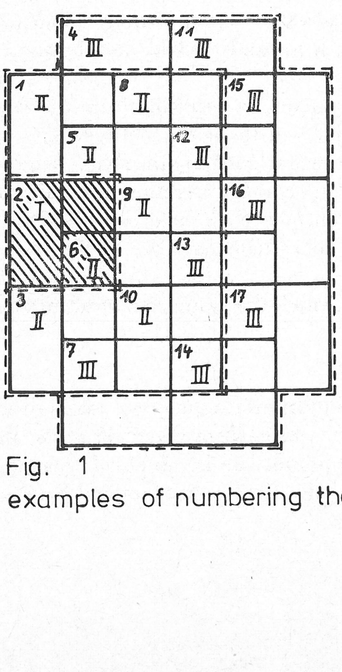

Fig. 1

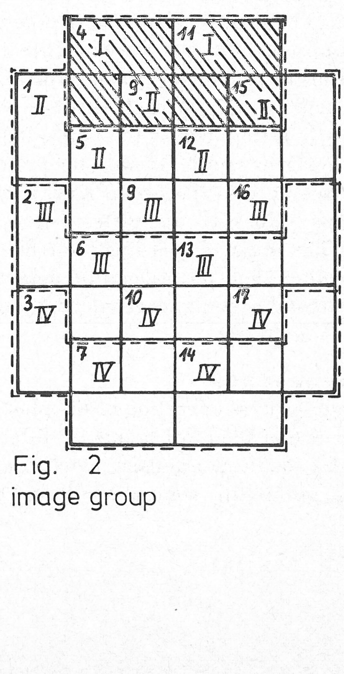

Fig. 2

examples of numbering the first image group

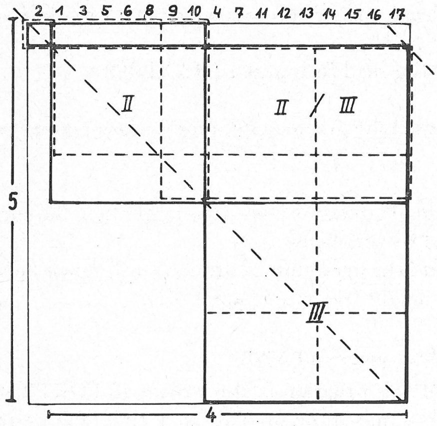

Fig. 3

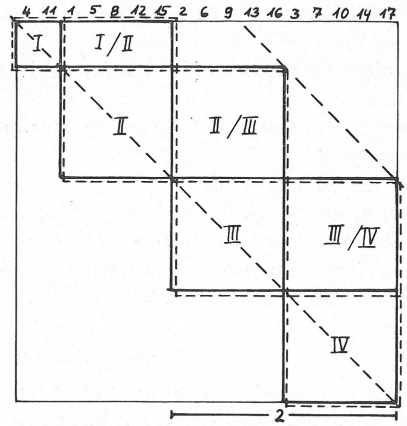

Fig. 4

6 × 6 matrix
submatrix

bandwidth: 16 × 6 = 96
4 hypercolumns
5 hyperrows

bandwidth: 10 × 6 = 60
2 hypercolumns
4 hyperrows

structure of reduced normal equations

4. A general description of the program

4.1 Input data requirements

a) updated image coordinates (in optional sequence)
b) horizontal coordinates, elevations and associated standard deviations of the ground control points
c) principal distance
d) relinearization criteria

By reading in a constant in part one the sign of all y image coordinates can be changed. This avoids a break-down of the Rodriguez-Matrix in case of kappa $\approx 200^g$.

4.2 Segmentation of the program

a) Reading in of all photo and ground control coordinates, and storage on disk; image numbers of first image group read in.
b) Sorting of all images and points according to the read in first image group. The computing time in this part is increasing with the square of the number of photos.
c) Utilizing the horizontal ground control coordinates the normal equations for the Anblock-adjustment (for initial values) are generated and solved. The transformed image coordinates are externally stored.
d) In part 4 the adjusted planimetric coordinates of the transformed image points and their residuals are computed.
e) The Anblock-transformation parameters for each photo are used to generate the initial values for the final bundle adjustment.
f) Bundle-adjustment
The iteration process of the bundle-blockadjustment starts with the initial values of part five. After each iteration the increments for the aerial stations and rotations are added to the values with whom the adjustment was performed. With these transformation parameters the photos will be rotated followed by the relinearization and generation of the new coefficients of elements of reduced normal equations.

The iteration process is carrying on until at least one of the following four criteria is reached:

- ITER = MAXIMO
  The number of iterations (ITER) has reached the read in number (MAXIMO)
- MAX $\leqq$ FS
  The maximum alteration of the coordinates against the previous iteration is less or equal to the read in criteria (FS).
- QMV (i) $\geq$ QMV (i — 1)
  The root mean square value of residuals after the i^{th} iteration QMV (i) is larger than the corresponding value of the iteration before QMV (i — 1) (divergence).
- The "convergence indicator" is larger than 15 and the maximum coordinate difference between the last two iterations is less than 0,5 $^0/_{00}$ of flying height (see also chapter 6.1).

5. Special features of the program PAT-B

Being based on the adjustment program PAT–M this program is also written in FORTRAN (ASA-norm) and is therefore easily transferable. It needs a maximum of 40 K at a time for storage. The intricacy of the adjustment system with a very great number of numerical operations require that at least a medium scale computer is applied. Because of the built-in Anblock determination of initial values it is no longer necessary to read in initial values for the transformation parameters, they will be found automatically. There is no limitation on number and location of pass points as long as the system is defined. Virtually there is no limitation neither on the shape nor on the size of the block as long as large computers are available. The possibility of choosing different weight matrices for the ground control coordinates (all image coordinate measurements have unit weight without correlation between x and y) allows to adjust simultaneously for example APR-generated, trigonometric, and/or levelled vertical control, as well as aerodist, astronomic and trigonometric horizontal control points with proper weights for each group. Moreover weights of control points can be set zero. All ground control coordinates with weight 0 will not take part effectively in the adjustment. They will be treated as check points and will receive residuals. The possibility of choosing a different principal distance for each photograph as well as the independency from a special image format should give the program a universal applicability for various projects. The program will also converge with photos tilted up to 50^g.

Until now it is not possible to introduce parameters for corrections of inner orientation [3].

6. Practical experiences with the program

6.1 Convergence

In this complex system of bundle adjustment the rate of convergence is of great importance. Here the results of two different cases are presented.

a) Artificial test block of 10 photos, height differences of up to 75 % of flying height, $\varphi \cong \omega \cong \pm 20^g$ and $\varkappa \cong \pm 40^g$.

b) A block of 208 photos (photoscale 1 : 28000, 2700 km², 8 strips).

The root mean square value of residuals (QMV) before adjustment was

case a) $\approx$ 10000 μm ($\approx$ 40000 μm with initial value $\varkappa = 0^g$)
After five iterations (6 iterations with $\varkappa = 0^g$) the maximum coordinate correction was less than 0.01 $^0/_{00}$ of flying height.

case b) QMV $\cong$ 600 μm and 2 iterations.

Different weights and different numbers of control points turned out to hardly influence the convergence. For reasonable height-differences (see also 3.1) and φ, $\omega \leqq \pm 6^g$ two iterations were enough in most cases. The computational evidence which we have up to now seems to point to something like a "convergence indicator" which is nearly independent from initial values. We noticed that if the maximum coordinate differences between two iterations changed by more than a factor

15 the next iteration brought only the confirmation of having reached the final point. The following diagrams 5, 6, and 7 are illustrating the relationship. In all three figures the ordinate scale is logarithmic.

I. Strip of 8 photos (initial value for kappa $= 0^g$) (superwide angle, s. w.)
II. Test-block of 10 photos (wide angle, w.)
III. Block of 208 photos (w.)
IV. Block of 26 photos (w.)
V. Diverging case
VI. Strip of "semi-images" (20 photos) (s. w.)

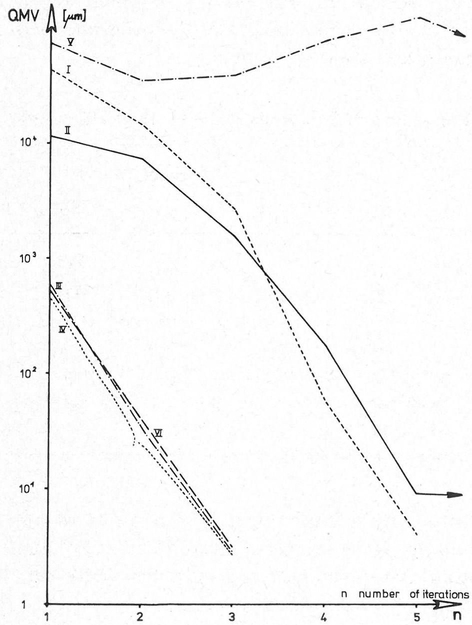

Fig. 5 Root mean square value (QMV) of residuals against number n of iterations.

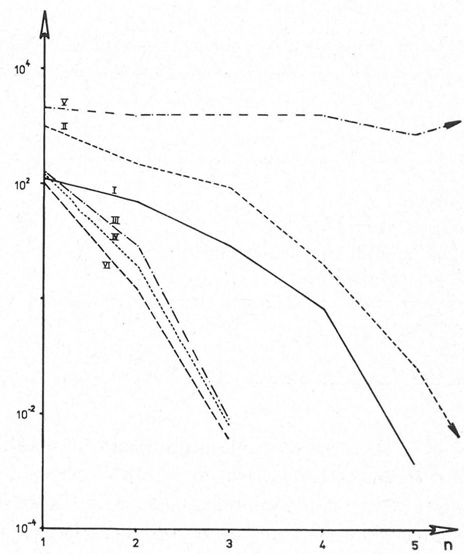

Fig. 6 Maximum coordinate differences MAX [in m] between successive iterations.

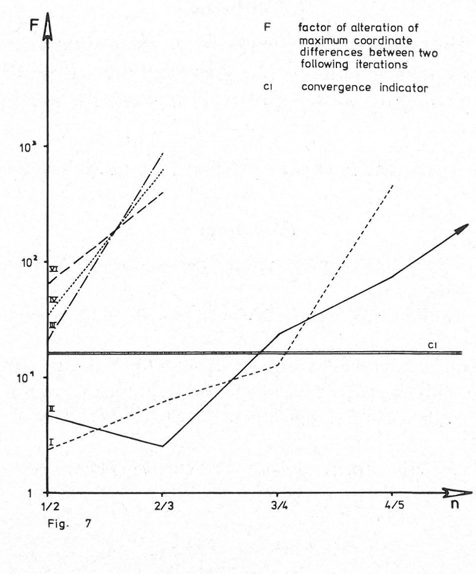

Fig. 7 Convergence indicatoı

Figure 6 shows the decrease of maximum coordinate differences for convergence and one example (V) with divergence. The slope ot the curve of decreasing maximum coordinate differences between two iterations could give an adequate convergence indicator (ci) which is illustrated in figure 7. This indicator seems to be nearly independent from photoscale, principal distance and quality of initial values.

6.2 Computing time

The following table 1 will show the computing time with CDC 6600 for the total adjustment of a strip of 26 photos and a block of 208 photos, split up in Central Processor (CP)- and Input-Output (IO)-time. Also the computing times required by the three main parts of the programs (sorting, determination of initial values and bundle adjustment) are listed separately.

Table 1. Empirical computing times (in sec. with CDC 6600) for a strip of 26 photos and a block af 208 photos

	strip			block		
	CP	IO	SS	CP	IO	SS
Sorting	2	10	5	50	190	105
Initial values (Anblock)	17	30	26	128	68	153
Bundle adjustment (iteration)	45	61	70	716	104	800
Σ (= sum) [sec]	64	101	101	894	362	1058
SS/photo			3.9 sec			5.1 sec
R			1 : 3.0			1 : 6.7

R is the time-ratio of computing initial values against the bundle adjustment. The computing time for the block turned out to be 1058 SS or 5,1 sec per photo. Here I want to mention the very good agreement of the practical results with the predicted computing time by H. EBNER [2]. Those predicted values can be furthermore used as rather accurate estimations of computing time.

7. Conclusion

The computing time and the convergence seems to be encouraging and the universality of this program makes it feasible to adjust also large blocks with even more than 1000 photos within reasonable computing times. With the program we are undertaking some further investigations on theoretical and practical results.

We will also report on comparisons between independent model-and bundle-adjustment soon.

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